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Geometry and Dynamics of Information Spacetime Derived from Entanglement Spectrum

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Related works are arXiv:1310.1831,1407.2667,1408.6633,1409.3908, 1508.02538,1508.04679,1508.06515.

Relative information entropy → Fisher metric

Probability distribution $\sum_n p_n(\theta) = 1$

θ : (model-dependent) internal parameters (vector valued)
→ This parameter set determines a particular physical state.

'Relative' information entropy

$$D(\theta) = -\sum_n p_n(\theta)\log p_n(\theta) + \sum_n p_n(\theta)\log p_n(\theta + d\theta)$$

$$= \boxed{\sum_n \frac{\partial p_n(\theta)}{\partial \theta^\mu}} d\theta^\mu \qquad \frac{\partial}{\partial \theta^\mu}\sum_n p_n(\theta) = 0$$

$$+ \frac{1}{2}\boxed{\sum_n p_n(\theta)\frac{\partial \log p_n(\theta)}{\partial \theta^\mu}\frac{\partial \log p_n(\theta)}{\partial \theta^\nu}} d\theta^\mu d\theta^\nu + \cdots$$

Fisher metric

$$g_{\mu\nu}(\theta) = \sum_n p_n(\theta)\frac{\partial \log p_n(\theta)}{\partial \theta^\mu}\frac{\partial \log p_n(\theta)}{\partial \theta^\nu} = \langle \partial_\mu \gamma\, \partial_\nu \gamma \rangle$$

$$g_{\mu\nu}(\theta) = \langle \partial_\mu \gamma\, \partial_\nu \gamma \rangle = \langle \partial_\mu \partial_\nu \gamma \rangle \qquad \gamma_n(\theta) = -\log p_n(\theta)$$

$$g_{\mu\nu}(\theta) = \sum_n p_n(\theta) \frac{\partial \log p_n(\theta)}{\partial \theta^\mu} \frac{\partial \log p_n(\theta)}{\partial \theta^\nu} = \left\langle \partial_\mu \gamma \, \partial_\nu \gamma \right\rangle$$

$$\left\langle \partial_\mu \gamma \right\rangle = -\sum_n p_n(\theta) \frac{\partial \log p_n(\theta)}{\partial \theta^\mu} = -\sum_n \frac{\partial \, p_n(\theta)}{\partial \theta^\mu} = 0$$

$$0 = \partial_\nu \left\langle \partial_\mu \gamma \right\rangle = -\sum_n \frac{\partial \, p_n(\theta)}{\partial \theta^\nu} \frac{\partial \log p_n(\theta)}{\partial \theta^\mu} + \left\langle \partial_\nu \partial_\mu \gamma \right\rangle$$

$$0 = -\sum_n p_n(\theta) \frac{\partial \log p_n(\theta)}{\partial \theta^\nu} \frac{\partial \log p_n(\theta)}{\partial \theta^\mu} + \left\langle \partial_\nu \partial_\mu \gamma \right\rangle$$

$$g_{\mu\nu}(\theta) = \left\langle \partial_\mu \gamma \, \partial_\nu \gamma \right\rangle = \left\langle \partial_\mu \partial_\nu \gamma \right\rangle$$

Why Fisher-metric approach is powerful for AdS/CFT ?

Fisher metric:

$$g_{\mu\nu}(\theta) = \langle \partial_\mu \gamma \, \partial_\nu \gamma \rangle = \langle \partial_\mu \partial_\nu \gamma \rangle \qquad \gamma_n(\theta) = -\log p_n(\theta)$$

The Fisher metric can be defined for any statistical problem. However, if we define the probability distribution from some information of our target quantum field theory, the metric naturally gives us a way of transformation from quantum data to corresponding classical geometry.

In general relativity, the metric is a solution of the Einstein equation. In the present case, the metric is 'defined' by more elementary information that originates in a microscopic model.

Connection to quantum entanglement

Schmidt decomposition of any pure state ψ

$$|\psi(\theta)\rangle = \sum_n \sqrt{\lambda_n(\theta)}\,|n\rangle_A \otimes |n\rangle_{\bar{A}} \qquad \langle\psi(\theta)|\psi(\theta)\rangle = \sum_n \lambda_n(\theta) = 1$$

Entanglement spectrum

$$\gamma_n(\theta) = -\log \lambda_n(\theta)$$

$$\lambda_n(\theta)$$

Entanglement entropy

$$S(\theta) = -\sum_n \lambda_n(\theta) \log \lambda_n(\theta) = \langle \gamma \rangle$$

Fisher metric

$$g_{\mu\nu}(\theta) = \langle \partial_\mu \gamma\, \partial_\nu \gamma \rangle = \langle \partial_\mu \partial_\nu \gamma \rangle$$

Once we obtain the Schmidt coefficients, we can immediately find both of entanglement entropy and Fisher metric.

More precisely, this is a moduli-space metric, not real spacetime, since θ are model parameters in the quantum side.

→ We may look at a new class of holographic transformation.

Entanglement spectrum

$$\gamma_n(\theta) = -\log \lambda_n(\theta)$$

$$\lambda_n(\theta)$$

Entanglement entropy

$$S(\theta) = -\sum_n \lambda_n(\theta) \log \lambda_n(\theta) = \langle \gamma \rangle$$

Fisher metric

$$g_{\mu\nu}(\theta) = \langle \partial_\mu \gamma \, \partial_\nu \gamma \rangle = \langle \partial_\mu \partial_\nu \gamma \rangle$$

The Fisher metric is roughly given by the second derivative of the entanglement entropy by the canonical parameters.

$$g_{\mu\nu}(\theta) = \langle \partial_\mu \partial_\nu \gamma \rangle \approx \partial_\mu \partial_\nu S(\theta)$$

This is just an approximation, but this provides us an intuitive Understanding the meaning of the Fisher metric.

$$g_{\mu\nu}(\theta) = \langle \partial_\mu \partial_\nu \gamma \rangle \approx \partial_\mu \partial_\nu S(\theta)$$

One of θ control the energy scale of the entanglement spectrum, and this should be related to L.

Owing to the positivity of the Fisher metric, we require (d=1)

$$S(\theta) \approx -\kappa \log\theta, \theta = \frac{1}{L^\nu} \Rightarrow S = \kappa\nu \log L, g_{\theta\theta} \approx \frac{\kappa}{\theta^2}$$

Then, the warp factor of AdS naturally appears and the entropy coincides with the logarithmic violation formula.

d=2 case → area law scaling can be reproduced

$$S(\theta) \approx -\kappa \log\theta, \theta = e^{-aL/\kappa} \Rightarrow S = aL, g_{\theta\theta} \approx \frac{\kappa}{\theta^2}$$

Truncated quantum state

$$\left|\psi\right\rangle \approx \left|\psi_{\chi}\right\rangle = \sum_{n=1}^{\chi} \sqrt{\lambda_n} \left|n\right\rangle_A \otimes \left|n\right\rangle_{\bar{A}} \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\chi$$

Two scaling relations (area law and finite-entanglement scaling) for the entanglement entropy (ξ : finite-entanglement exponent)

$$S \approx \varsigma \log \chi = a L^{d-1} \qquad \varsigma(c) \log \chi = \frac{c}{6} \log \xi = \frac{c}{6} \log L$$

$$\Rightarrow \chi \approx \exp\left(\frac{a}{\varsigma} L^{d-1}\right) \qquad \Rightarrow \xi = L \Rightarrow \theta \approx \xi^{-1}$$

The parameter χ is related to how many states are necessary to keep numerical accuracy of the optimization of Ψ.

Thus, the inverse of χ is roughly the resolution of the entanglement spectrum.

$$\theta \approx \chi^{-1} \approx \exp\left(-\frac{a}{\varsigma} L^{d-1}\right) \qquad \theta = e^{-aL/\kappa} \left(d = 2\right)$$

$$\varsigma \approx \kappa$$

Entanglement statistical mechanics

θ : Canonical parameters

Microscopic derivation of entanglement thermodynamics

Distribution function in the class of exponential family

$$\lambda_n(\theta) = e^{-\gamma_n(\theta)} = \exp\{\theta^\mu F_{n,\mu} - \psi(\theta)\} = \frac{1}{Z} e^{\theta^\mu F_{n,\mu}} \qquad \psi(\theta) = \log Z$$

Greatly simplify the corresponding geometry (Hessian geometry)

$$\gamma_n(\theta) = \psi(\theta) - \theta^\mu F_{n,\mu} \qquad g_{\mu\nu}(\theta) = \langle \partial_\mu \partial_\nu \gamma \rangle = \partial_\mu \partial_\nu \psi(\theta)$$

Thermodynamic law

$$S(\theta) = \langle \gamma(\theta) \rangle = \psi(\theta) - \theta^\mu \langle F_\mu \rangle = \psi(\theta) - \theta^\mu \partial_\mu \psi(\theta)$$

Multiplying entanglement temperature T, we have

$$TS = -F + E$$

Differential form

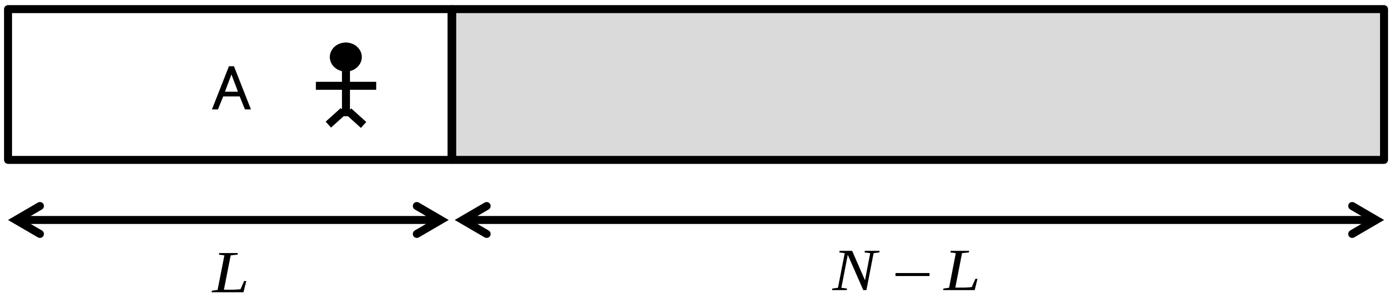
Legendre transform

$$\partial_\nu S(\theta) = -\theta^\mu \partial_\mu \partial_\nu \psi(\theta) = -\theta^\mu g_{\mu\nu}(\theta) = -\theta^\mu \partial_\nu \eta_\mu(\theta)$$

$$dS(\theta) = -\theta^\mu d\eta_\mu(\theta)$$

How to identify the canonical parameters ?

Is the exponential family form really a reasonable assumption → yes!



$a=1$

Tight-binding model on 1D lattice

$$H=-\sum_{i=1}^{N}\left(c_i^{+}c_{i+1}+c_{i+1}^{+}c_i\right)=\sum_k \varepsilon_k c_k^{+}c_k$$

$$\varepsilon_k=-\cos(ka)\approx -1+\frac{1}{2}(ka)^2$$

Carrier density δ

Total carrier number n

$$\delta=\frac{n}{N}$$

(usual CFT: $\delta\to 0$)

The ground-state properties of this model are completely characterized by L and δ

(as well as time t if the state evolves in time after some quench).

→ L, δ, and t are relevant model parameters.

(Be careful that they are 'not' canonical parameters)

Partial density matrix and entanglement spectrum (t=0)

$$\rho_A \propto \exp\left\{-\sum_{l=1}^{L}\varphi_l(L,\delta)n_l\right\}$$

S.-A. Cheong and C. L. Henley, PRB69, 075111 (2004)

Scaling relations for the entanglement spectrum

$$\varphi_l(L,\delta) = Lf(\delta,x)$$

$$x = \frac{l - l_F}{L} \qquad l_F = \delta L + \frac{1}{2}$$

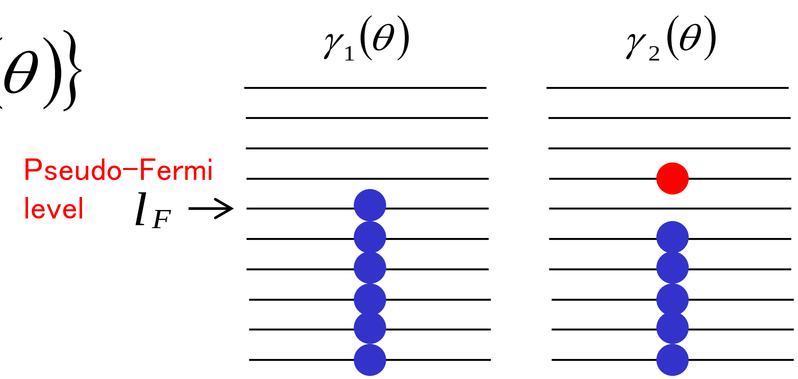
$$\begin{cases} f(\delta,0) = 0 \\ f'(\delta,0) > 0 \\ f(\delta,-x) = -f(1-\delta,x) \end{cases}$$

$$\lambda_n(\theta) = e^{-\gamma_n(\theta)} = \exp\{\theta^\mu F_{n,\mu} - \psi(\theta)\}$$

$$\gamma_n(\theta) = \psi(\theta) - \theta^\mu F_{n,\mu}$$

$$\gamma_1(\theta) \le \gamma_2(\theta) \le \cdots$$

$$\gamma_2(\theta) - \gamma_1(\theta) = \theta^\mu\left(F_{1,\mu} - F_{2,\mu}\right)$$



$$\gamma_2(\theta) - \gamma_1(\theta) = Lf(\delta,1/L) = f'(\delta,0) + \frac{f''(\delta,0)}{2L} + \frac{f'''(\delta,0)}{6L^2} + \cdots$$

Numerical results suggest

very small constant

$$\gamma_2(\theta)-\gamma_1(\theta)=Lf(\delta,1/L)=\boxed{f'(\delta,0)}+\frac{f''(\delta,0)}{2L}+\frac{\boxed{f'''(\delta,0)}}{6L^2}+\cdots$$

only weak δ dependence

We can identify two of canonical parameters as

$$(\theta^1,\theta^2)=\left(\frac{1}{L^2},\frac{f''(\delta,0)}{L}\right)$$

Time evolution → exponential form

$$\theta^3=\frac{t}{L}$$

The canonical parameters are nontrivial functions of the model parameters. The identification of this transformation is quite Important.

Hessian potential

Hessian potential

$$\psi(\theta) \qquad\qquad \psi \Rightarrow \psi + A_\alpha \theta^\alpha + B$$

Entanglement entropy

$$S(\theta) = \psi(\theta) - \theta^\alpha \partial_\alpha \psi(\theta)$$

Fisher metric

$$g_{\mu\nu}(\theta) = \partial_\mu \partial_\nu \psi(\theta)$$

Hessian potential that exactly leads to $\mathrm{AdS_D}$ metric

$$\psi(\theta^1, \ldots, \theta^D) = -\kappa \log\left(\theta^1 - \frac{1}{2}\eta_{ij}\theta^i\theta^j\right) \qquad i, j = 2,3,\ldots, D$$

Coordinate transformation

$$z = \sqrt{\theta^1 - \frac{1}{2}\eta_{ij}\theta^i\theta^j},\ x^i = \frac{1}{2}\theta^i \left(i = 2,3,\ldots, D\right)$$

$$g = g_{\mu\nu} d\theta^\mu d\theta^\nu = 4\kappa \frac{dz^2 + \eta_{ij} dx^i dx^j}{z^2}$$

Entanglement entropy

$$S(\theta)=\psi(\theta)-\theta^{\alpha}\partial_{\alpha}\psi(\theta)$$

$$=-\kappa\log\left(\theta^1-\frac{1}{2}\eta_{ij}\theta^i\theta^j\right)+\kappa-\kappa\frac{\frac{1}{2}\eta_{ij}\theta^i\theta^j}{\theta^1-\frac{1}{2}\eta_{ij}\theta^i\theta^j}$$

$$\approx-\kappa\log\theta^1+\kappa+\kappa\frac{\frac{1}{2}\eta_{ij}\theta^i\theta^j}{\theta^1}+\frac{1}{2}\kappa\left(\frac{\frac{1}{2}\eta_{ij}\theta^i\theta^j}{\theta^1}\right)^2-\kappa\frac{\frac{1}{2}\eta_{ij}\theta^i\theta^j}{\theta^1}\left(1+\frac{\frac{1}{2}\eta_{ij}\theta^i\theta^j}{\theta^1}\right)+\cdots$$

$$\approx-\kappa\log\theta^1+\kappa$$

$$S(\theta)\approx-\kappa\log\theta^1+\kappa\Rightarrow S(L)=2\kappa\log L,\kappa=\frac{c}{6}$$

* Comment on bulk/boundary correspondence

$$z=\sqrt{\theta^1-\frac{1}{2}\eta_{ij}\theta^i\theta^j}\approx\frac{1}{L}$$

If z→0, then L→∞.
Full quantum state is located at the boundary of AdS, and highly-truncated quantum data are stored in the inside of AdS.

Fefferman–Graham–type perturbation

Hessian potential

$$\Psi_h(\theta) = -\kappa \log\left(\theta^1 - \frac{1}{2}\eta_{ij}\theta^i\theta^j - \boxed{h(\theta)}\right)$$

Coordinate transformation

$$z = \sqrt{\theta^1 - \frac{1}{2}\eta_{ij}\theta^i\theta^j - h(\theta)},\, x^i = \frac{1}{2}\theta^i \left(i = 2,3,\ldots,D\right)$$

For small h and x^i

$$G = g + 4\kappa\left[\left(\partial_1\partial_1 h\right)dz^2 + \frac{2\,\partial_1\partial_i h}{z}dz\,dx^i + \frac{\partial_i\partial_j h}{z^2}dx^i\,dx^j\right]$$

$$\partial_i\partial_j h = z^{D-1}H_{ij}, \partial_1\partial_1 h = z^{D-3}H, \partial_1\partial_i h = 0$$

Energy–momentum tensor at the boundary of AdS

$$\partial_i T^{ij} = 0 \Rightarrow \partial_i\partial^i h = 0$$

Hessian potential

$$\Psi_h(\theta) = -\kappa \log\left(\theta^1 - \frac{1}{2}\eta_{ij}\theta^i\theta^j - h(\theta)\right)$$

Entanglement entropy

$$S_h(\theta) = \Psi_h(\theta) - \theta^\alpha \partial_\alpha \Psi_h(\theta)$$

$$= -\kappa \log\left(\theta^1 - \frac{1}{2}\eta_{ij}\theta^i\theta^j - h\right) + \frac{\theta^1 - \eta_{ij}\theta^i\theta^j - \theta^\alpha \partial_\alpha h}{\theta^1 - \frac{1}{2}\eta_{ij}\theta^i\theta^j - h}$$

$$\approx S + \kappa \frac{s_h}{\boxed{z^2}} + \cdots \longrightarrow t^2\text{–time dependence}$$

Entropy gain by the perturbation: $s_h = h - \theta^\alpha \partial_\alpha h$

(This has also the Hessian structure.)

Derivation of Einstein equation from quantum entanglement

'Derivation' of fictitious energy-momentum tensor

→ A kind of inverse problem

Standard general relativity

→ Real matter field determines our spacetime structure.

My strategy:

(1) At first, the Einstein tensor for exp. family is derived.

(2) (1) is transformed into a form similar to energy-momentum tensor.

(3) We can look at what is the source of such tensor.
We would like to know what kind of quantum data behave as this fictitious matter field in the parameter space.

$$g_{\mu\nu}(\theta) = \langle \partial_\mu \gamma \partial_\nu \gamma \rangle \qquad \gamma : \text{Entanglement spectrum}$$

This form looks basically similar to the Lagrangian for free scalar field theory. → the average of the spectrum (entropy) would behave as the scalar field.

Important geometric quantities

Fisher metric (Hessian potential form)

$$g_{\mu\nu} = \partial_\mu \partial_\nu \psi$$

Christoffel symbol and Ricci tensor

$$\Gamma^{\lambda}_{\mu\nu} = -\frac{1}{2} g^{\lambda\tau} T_{\mu\nu\lambda} \qquad T_{\mu\nu\lambda} = \left\langle \partial_\mu \gamma \, \partial_\nu \gamma \, \partial_\tau \gamma \right\rangle = -\partial_\mu \partial_\nu \partial_\tau \psi$$

$$R_{\mu\nu} = \frac{1}{4} g^{\sigma\tau} g^{\rho\varsigma} \left(T_{\varsigma\mu\sigma} T_{\rho\nu\tau} - T_{\rho\sigma\tau} T_{\varsigma\mu\nu} \right)$$

Approximated form of rank-three tensor $T_{\mu\nu\lambda}$

$$T_{\lambda\mu\nu} = \frac{1}{A} \left(g_{\mu\nu} \partial_\lambda S + g_{\mu\lambda} \partial_\nu S + g_{\nu\lambda} \partial_\mu S \right)$$

Derivation of Einstein equation

Entropy (Φ originates in the Fefferman-Graham term)

$$S = S_0 + \phi$$

Pure CFT case ($S=S_0$)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 0$$

Cosmological constant

$$\Lambda = -\kappa \frac{(D-2)(D-1)}{8A^2} < 0$$

Effect of metric perturbation on the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda \approx a T_{\mu\nu}$$

$$a = \frac{1}{2A^2}(D-2)$$

Lagrangian for scalar field Φ

$$L = \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi$$

$$T_{\mu\nu} = g_{\mu\sigma} \frac{\partial L}{\partial(\partial_\sigma \phi)} \partial_\nu \phi - g_{\mu\nu} L$$

Summary and future works

Geometry of information spacetime defined by the Fisher metric for quantum states

- The entanglement spectrum can define the Fisher metric.
- Detailed structure of entanglement spectrum
  → This is crucial to find the canonical parameters.
- Exponential family form and Hessian potential
  → entanglement statistical mechanics
- Hessian potential → entanglement entropy and Fisher metric
- The free fermion model (CFT_{1+1}) naturally leads to AdS_3, but the coordinates are not real spacetime.
  → new kind of quantum/classical correspondence ?

Dynamics of information spacetime

- Einstein eq. → equation of quantum states
- difference of S from its ground-state value is mapped onto a scalar field in the Einstein equation.