General relativity and important physical quantities

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July 13, 2016

This talk is based on joint work with Po-Ning Chen and Mu-Tao Wang.

Exactly 100 years ago, Einstein accomplished one of the most spectacular work in physics and radically changed the view of space and time in the history of mankind. The foundation laid by Isaac Newton on the theory of gravity was completely changed by the theory of general relativity .

In the very successful theory of Newton, space is static and time is independent of space . By 1905, when Einstein established special relativity along with Poincar and others, it was realized that space and time are linked and that the very foundation of special relativity, and that information cannot travel faster than light, is in contradiction with Newtonian gravity where action at a distance was used.

Einstein learnt from his teacher in 1908 that special relativity is best described as the geometry of the Minkowski spacetime. He realized gravitational potential cannot be described by a scalar function. It should be described by a tensor. After tremendous helps from his two friends in mathematics : Grossmann and Hilbert, Einstein finally wrote down the famous Einstein equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}.$$

Note that Hilbert was the first one that write down the action principle of gravity, which plays the most important role in any attempts to quantize general relativity. The action is given by the total scalar curvature of the metric tensor which is considered to be the gravitational potential. If gravity is coupled with other matter, we simply add the matter Lagrangian.

For a closed surface S enclosing a spacelike region D, the Lagrangian formulation gave the definition of a surface Hamilton which can be described as follows

$$\int_S N\,{}^2K + N^\mu p_{\mu\nu} r^\nu.$$

Here w^ν is a future time-like unit vector field along S that corresponds to unit translation. We write $w^\mu = N\, n^\mu + N^\mu$ alone the surface S, where n^μ is the timelike unit normal of D restricting to S. N is the lapse and N^μ is the shift. r^μ is the space-like unit normal orthogonal to n^μ along S, and p is the second fundamental form of D, and ${}^2K$ is the mean curvature of S with respect to r^μ.

Here the geometric quantities are determined by the spacelike region D and the gauge choice of timelike vector fields along D .

The field equation was used by Einstein to calculate the perihelion of mercury and the light bending predicted by the Schwarzschild solution of the Einstein equation, which was found shortly. This was of course a great triumph of the Einstein theory of general relativity. However, since the theory is highly nonlinear and the geometry of space time is dynamical, the actual understanding of Einstein equation was very difficult : even to Einstein himself.

Gravitation radiation was predicted by Einstein and later he tried to retract the idea, although the retraction was not successful. Einstein thought that the equation determined gravity completely. But that is actually not true as we cannot tell what is the initial condition for the field equation and we have difficulty to find the boundary condition.

There are many important problem in GR which are not solved. Many of them need deep understanding of geometry and analysis The first major question is the question of the structure of singularities. Black hole appeared in Schwarzschild and Kerr solutions .

Penrose proposed the famous question of cosmic censorship, He claimed that for a generic space time, every singularity is hidden behind some membrane similar to black hole appeared in the above solutions.

The dynamical problem of the Einstein equation is still not solved yet. It is only understood in the very weak field limit case by the work of Christodoulou-Klainerman and Christodoulou.

There are many important physical quantities and questions that were understood in Newtonian mechanics. However, their counterparts are not easy to formulate, let alone to understand! They are largely due to the problem of gauge choice in general relativity . This problem started even before the field equation was written down, when Einstein attempted to use divergence free coordinate choice .

Einstein succeeded to define total mass for an isolated physical system, which was made precise by the famous work of Arnowitt-Deser-Misner. It was an important quantity to measure the whole physical system. Already Einstein found it difficult to know properties of such total mass. It needs to be positive for the system to be physically stable. This was finally proved by Schoen and me in 1979.

Physical quantities are gauge independent and their relation to geometry becomes very interesting. And we shall discuss them in this talk.

As is well known, it is not possible to find mass density of gravity in general relativity. The mass density would have to be first derivative of the metric tensor which is zero in suitable chosen coordinate at a point.

But we still desire to measure the total mass in a space like region bounded by a closed surface.

The mass due to gravity should be computable from the intrinsic and the extrinsic geometry of the surface.

It has been important question to find the right definition.

Penrose gave a talk on this question in my seminar in the Institute for Advances Study in 1979.
The quantity is called quasilocal mass.

Penrose listed it as the first major problem in his list of open problems.

Many people, including Penrose, Hawking-Horowitz, Brown-York and others worked on this problem and various definitions were given.

I thought about this problem and attempted to look at it from point of view of mathematician.

I list properties that the definition should satisfy :

1. It should be nonnegative and zero for any closed surfaces in flat Minkowski spacetime

2. It should converge to the familiar ADM mass for asymptotically flat spacetime if we have a sequence of coordinate spheres that approaches the spatial infinity of an asymptotic flat slice.

3. It should convergent to the Bondi mass when the spheres divergent to the cut at null infinity

4. It should be equivalent to the standard Komar mass in a stationary spacetime .

It turns out that this is not so easy to find such a definition.

In the time symmetric case, my former student Robert Bartnik proposed a definition which satisfies the above properties. But his definition does not allow him to give an effective calculation of the mass.

About 15 years ago, I was interested in how to formulate a criterion for existence of black hole, that Kip Thorne called hoop conjecture.

The statement says that if the quasi-local mass of a closed surface is greater than certain multiple of the diameter of the surface, then the closed surface will collapse to a black hole. (perhaps the length of shortest closed geodesic is a better quantity than diameter)

Hence a good definition of quasi-local mass is needed.

I was visiting Hong Kong at that time and I lectured on related materials.

I suggested to L. F. Tam to look at the recent work that I did on the existence of black hole due to boundary effects of the mean curvature. I suggested the work of Bartnik on the quasi-local mass and his work on construction of three manifolds with zero scalar curvature using foliation by quasi sphere. Bartnik did a remarkable calculation turning part of prescribing scalar curvature into a parabolic equation .

Soon afterwards, Tam told me that he generalized the statement with Shi to sphere that is not necessary round.

When I came back to Harvard, Melissa Liu and I generalized this statement of Shi-Tam to spheres that are the boundary of a three dimensional space like hyper surface in a spacetime which satisfies the dominant energy condition.

The total mean curvature is replaced by the total integral of the spacetime length of the mean curvature vector. This is a quantity independent of the choice of the three manifolds that the surface may bound. The difference between this quantity and the corresponding quantity of the isometric embedding of the surface into Euclidean space is positive.

I thought this should be the quasi local mass of the surface .

Then I found out that in the time symmetric case, this was in fact derived to be the quasi local mass by Brown-York and Hawking-Horowitz based on Hamiltonian formulation. (Their definition actually depend on the choice of the three manifold that the surface bounds)

The definition of Liu-Yau is quite good as it satisfies most of the properties mentioned above

However the mass so defined is too positive and may not be trivial for surfaces in the Minkowski spacetime. This is also true for the mass of Brown-York and Hawking-Horowitz.

Hence Mu-Tao Wang and I changed the definition and considered isometric embedding of the two dimensional surface into a Minkowski spacetime.

Such embedding are not unique and we have to optimize the quantities among all embeddings.

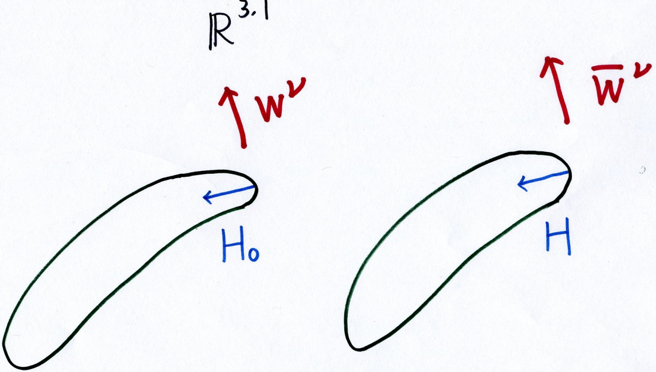
To be precise, the Wang-Yau definition of the quasi local mass can be defined in the following way:

Given a surface S, we assume that its mean curvature vector is spacelike. We embed S isometrically into $\mathbb{R}^{3,1}$.

Given any constant unit future time-like vector w (observer) in $\mathbb{R}^{3,1}$, we can define a future directed time-like vector field $\overline{w}$ along S by requiring

$$\langle H_0, w\rangle \; = \; \langle H, \overline{w}\rangle$$

where H_0 is the mean curvature vector of S in $\mathbb{R}^{3,1}$
and H is the mean curvature vector of S in spacetime.



$$\langle H_0, W \rangle = \langle H, \bar{W} \rangle$$

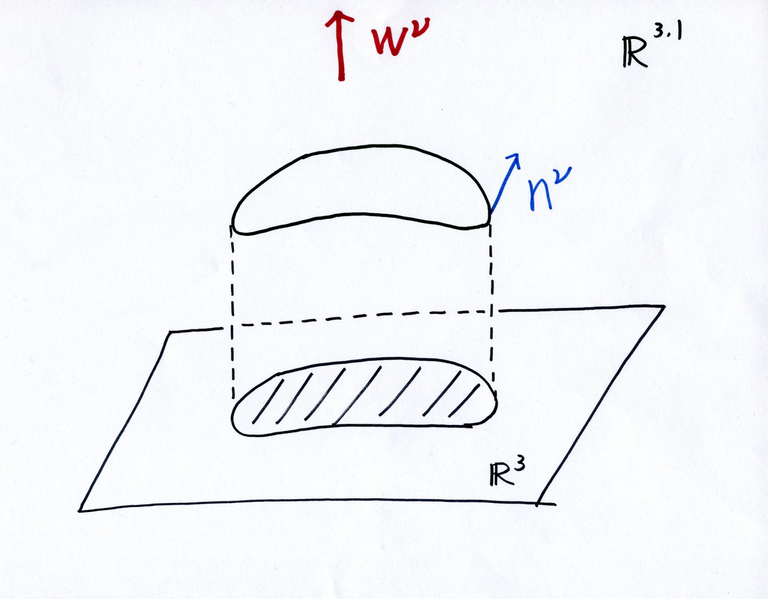
$$W^\nu = N n^\nu + N^\nu \qquad \bar{W}^\nu = N \bar{n}^\nu + N^\nu$$

Note that given any surface S in $\mathbb{R}^{3,1}$ and a constant future time-like unit vector w^ν, there exists a canonical gauge n^μ (future time-like unit normal along S) such that

$$\int_S N^2 K_0 + N^\mu (p_0)_{\mu\nu} r^\nu$$

is equal to the total mean curvature of $\hat{S}$, the projection of S onto the orthogonal complement of w^μ.

In the expression, we write $w^\mu = N\, n^\mu + N^\mu$ alone the surface S. r^μ is the space-like unit normal orthogonal to n^μ, and p_0 is the second fundamental form calculated by the three surface defined by S and r^μ.



W^ν
$\mathbb{R}^{3,1}$
n^ν
$\mathbb{R}^3$

From the matching condition and the correspondence $(w^\mu, n^\mu) \to (\overline{w}^\mu, \overline{n}^\mu)$, we can define a similar quantity from the data in spacetime

$$\int_S N\,{}^2\overline{K} + N^\mu (\overline{p})_{\mu\nu} \overline{r}^\nu \ .$$

We write $E(w)$ to be

$$8\pi E(w) = \int_S N\,{}^2\overline{K} + N^\mu (\overline{p})_{\mu\nu} \overline{r}^\nu - \int_S N\,{}^2K_0 + N^\mu (p_0)_{\mu\nu} r^\nu$$

and define the quasi-local mass to be

$$\inf E(w)$$

where the infimum is taken among all isometric embeddings into $\mathbb{R}^{3,1}$ and timelike unit constant vector $w \in \mathbb{R}^{3,1}$.

The Euler-Lagrange equation (called the optimal embedding equation) for minimizing $E(w)$ is

$$
\begin{aligned}
& div_S\left(\frac{\nabla\tau}{\sqrt{1+|\nabla\tau|^2}}\cosh\theta|H| - \nabla\theta - V\right) \\
& - (\widehat{H}\hat{\sigma}^{ab} - \hat{\sigma}^{ac}\hat{\sigma}^{bd}\hat{h}_{cd})\frac{\nabla_b\nabla_a\tau}{\sqrt{1+|\nabla\tau|^2}} = 0
\end{aligned}
$$

where $\sinh\theta = \frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}}$, V is the tangent vector on Σ that is dual to the connection one-form $\langle\nabla^N_{(\cdot)}\frac{J}{|H|}, \frac{H}{|H|}\rangle$ and $\hat{\sigma}$, $\widehat{H}$ and $\hat{h}$ are the induced metric, mean curvature and second fundamental form of $\hat{S}$ in $\mathbb{R}^3$.

In general, the above equation should have an unique solution τ. We prove that $E(w)$ is non-negative among admissible isometric embedding into Minkowski space.

In summary, given a closed space-like 2-surface in spacetime whose mean curvature vector is space-like, we associate an energy-momentum four-vector to it that depends only on the first fundamental form, the mean curvature vector and the connection of the normal bundle with the properties

1. It is Lorentzian invariant;

2. It is trivial for surfaces sitting in Minkowski spacetime and future time-like for surfaces in spacetime which satisfies the local energy condition.

Our quasi-local mass also satisfies the following important properties:

3. When we consider a sequence of spheres on an asymptotically flat space-like hypersurface, in the limit, the quasi-local mass (energy-momentum) is the same as the well-understood ADM mass (energy-momentum);

4. When we take the limit along a null cone, we obtain the Bondi mass(energy-momentum).

5. When we take the limit approaching a point along null geodesics, we recover the energy-momentum tensor of matter density when matter is present, and the Bel-Robinson tensor in vacuum.

These properties of the quasi-local mass is likely to characterize the definition of quasi-local mass, i.e. any quasi-local mass that satisfies all the above five properties may be equivalent to the one that we have defined.

Strictly speaking, we associate each closed surface not a four-vector, but a function defined on the light cone of the Minkowski spacetime. Note that if this function is linear, the function can be identified as a four-vector.

It is a remarkable fact that for the sequence of spheres converging to spatial infinity, this function becomes linear, and the four-vector is defined and is the ADM four-vector that is commonly used in asymptotically flat spacetime. For a sequence of spheres converging to null infinity in Bondi coordinate, the four vector is the Bondi-Sachs four-vector.

It is a delicate problem to compute the limit of our quasi-local mass at null infinity and spatial infinity. The main difficulties are the following:

(i) The function associated to a closed surface is non-linear in general;

(ii) One has to solve the Euler-Lagrange equation for energy minimization.

For (i), the following observation is useful:

For a family of surfaces Σ_r and a family of isometric embeddings X_r of Σ_r into $\mathbb{R}^{3,1}$, the limit of quasi-local mass is a linear function under the following general assumption that the mean curvature vectors are comparable in the sense

$$\lim_{r\to\infty} \frac{|H_0|}{|H|} = 1$$

where H is the the spacelike mean curvature vector of Σ_r in N and H_0 is that in the image of X_r in $\mathbb{R}^{3,1}$.

Under the comparable assumption of mean curvature, the limit of our quasi-local mass with respect to a constant future time-like vector $T_0 \in \mathbb{R}^{3,1}$ is given by

$$\lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} \Big[-\langle T_0, \frac{J_0}{|H_0|}\rangle(|H_0| - |H|) \\ - \langle \nabla^{\mathbb{R}^{3,1}}_{\nabla\tau} \frac{J_0}{|H_0|}, \frac{H_0}{|H_0|}\rangle + \langle \nabla^{N}_{\nabla\tau} \frac{J}{|H|}, \frac{H}{|H|}\rangle \Big] d\Sigma_r$$

where $\tau = -\langle T_0, X_r\rangle$ is the time function with respect to T_0.

This expression is linear in T_0 and defines an energy-momentum four-vector at infinity.

At the spatial infinity of an asymptotically flat spacetime, the limit of our quasi-local mass is

$$\lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} (|H_0| - |H|)\, d\Sigma_r = M_{ADM}$$

$$\lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} \langle \nabla^N_{-\nabla X_i} \frac{J}{|H|}, \frac{H}{|H|} \rangle d\Sigma_r = P_i$$

where $\begin{pmatrix} M \\ P_i \end{pmatrix}$ is the ADM energy-momentum four-vector, assuming the embeddings X_r into $\mathbb{R}^3$ inside $\mathbb{R}^{3,1}$.

At the null infinity, the limit of quasi-local mass was found by Chen-Wang-Yau to recover the Bondi-Sachs energy-momentum four-vector.

On a null cone $w = c$ as r goes to infinity, the limit of the quasi-local mass is

$$\lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} (|H_0| - |H|) d\Sigma_r = \frac{1}{8\pi} \int_{S^2} 2m \, dS^2$$

$$\lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} \langle \nabla^N_{-\nabla X_i} \frac{J}{|H|}, \frac{H}{|H|} \rangle d\Sigma_r = \frac{1}{8\pi} \int_{S^2} 2m X_i \, dS^2$$

where $(X_1, X_2, X_3) = (\sin\theta \sin\phi, \sin\theta\cos\phi, \cos\theta)$.

The following two properties are important for solving the Euler-Lagrange equation for energy minimization:

(a) The limit of quasi-local mass is stable under $O(1)$ perturbation of the embedding;

(b) The four-vector obtained is equivariant with respect to Lorentzian transformations acting on X_r.

We observe that momentum is an obstruction to solving the Euler-Lagrange equation near a boosted totally geodesics slice in $\mathbb{R}^{3,1}$. Using (b), we find a solution by boosting the isometric embedding according to the energy-momentum at infinity. Then the limit of quasi-local mass is computed using (a) and (b).

In evaluating the small sphere limit of the quasilocal energy, we pick a point p in spacetime and consider C_p the future light cone generated by future null geodesics from p. For any future directed timelike vector e_0 at p, we define the affine parameter r along C_p with respect to e_0. Let S_r be the level set of the affine parameter r on C_p.

We solve the optimal isometric equation and find a family of isometric embedding X_r of S_r which locally minimizes the quasi-local energy.

With respect to X_r, the quasilocal energy is again linearized and is equal to

$$\frac{4\pi}{3} r^3 T(e_0, \cdot) + O(r^4)$$

which is the expected limit.

In the vacuum case, i.e. $T = 0$, the limit is non-linear with the linear term equal to

$$\frac{1}{90} r^5 Q(e_0, e_0, e_0, \cdot) + O(r^6)$$

with an additional positive correction term in quadratic expression of the Weyl curvature.

The linear part consists of the Bel-Robinson tensor and is precisely the small-sphere limit of the Hawking mass which was computed by Horowitz and Schmidt.

The Bel-Robinson tensor satisfies conservation law and is an important tool in studying the dynamics of Einstein's equation, such as the stability of the Minkowski space (Christodoulou-Klainerman) and the formation of trapped surface in vacuum (Christodoulou)

Po-Ning Chen joined in the research about four years ago and we can now defined quasilocal angular momentum and center of gravity

We define quasi-local conserved quantities in general relativity by using the optimal isometric embedding to transplant Killing fields in the Minkowski spacetime back to the 2-surface a physical spacetime.

To each optimal isometric embedding, a dual element of the Lie algebra of the Lorentz group is assigned. Quasi-local angular momentum and quasi-local center of mass correspond to pairing this element with rotation Killing fields and boost Killing fields, respectively.

Consider the following quasi-local energy density ρ

$$\rho = \frac{\sqrt{|H_0|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}} - \sqrt{|H|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}}}{\sqrt{1+|\nabla\tau|^2}}$$

and momentum density j

$$j = \rho\nabla\tau - \nabla[\sinh^{-1}(\frac{\rho\Delta\tau}{|H_0||H|})] - \alpha_{H_0} + \alpha_H.$$

The optimal embedding equation takes a simple form:

$$div(j) = 0.$$

The quasi-local conserved quantity of Σ with respect to an optimal isometric embedding (X, T_0) and a Killing field K is

$$E(\Sigma, X, T_0, K) = \frac{(-1)}{8\pi} \int_\Sigma \left[\langle K, T_0 \rangle \rho + j(K^\top) \right] d\Sigma.$$

Suppose $T_0 = A(\frac{\partial}{\partial X^0})$ for a Lorentz transformation A.

The quasi-local conserved quantities corresponding to $A(X^i \frac{\partial}{\partial X^j} - X^j \frac{\partial}{\partial X^i})$ are called the quasi-local angular momentum and the ones corresponding to $A(X^i \frac{\partial}{\partial X^0} + X^0 \frac{\partial}{\partial X^i})$ are called the quasi-local center of mass integrals.

The quasi-local angular momentum and center of mass satisfy the following important properties:

[1] The quasi-local angular momentum and center of mass vanish for any surfaces in the Minkowski space

[2] They obey classical transformation laws under the action of the Poincaré group.

We further justify these definitions by considering their limits as the total angular momentum J^i and the total center of mass C^i of an isolated system. They satisfies the following important properties:
[1] All total conserved quantities vanish on any spacelike hypersurface in the Minkowski spacetime, regardless of the asymptotic behavior.

[2] The new total angular momentum and total center of mass are always finite on any vacuum asymptotically flat initial data set of order one.

[3] Under the vacuum Einstein evolution of initial data sets, the total center of mass obeys the dynamical formula $\partial_t C^i(t) = \frac{p^i}{p^0}$ where p^ν is the ADM energy-momentum four vector.

Let me say a few more about the angular momentum as this is perhaps of more current interest.

There were several proposals of the definition of quasilocal angular momentum. In the axi-symmetric case, there is the Komar angular momentum.

For a general spacetime, there are definitions proposed by Penrose, Dougan-Mason, Ludvigsen-Vickers etc, which are based on twistor or spinor constructions.

However, there seem to be very few criterion to justify a good definition of angular momentum at the quasilocal level.

For quasilocal mass, essential requirements are (1) $m(\Sigma) \geq 0$, (2) $m(\Sigma) = 0$ for $\Sigma \subset \mathbb{R}^{3,1}$ (3) $m(S_\infty) = ADM$.

There are difficulties even for the definition of total angular momentum of an asymptotically flat initial data set.

Recall that (M, g, k) is an asymptotically flat initial data set if outside a compact subset, there exists an asymptotically flat coordinate system (x^1, x^2, x^3) on each end, such that $g = \delta + O_2(r^{-q})$ and $k = O_1(r^{-p})$, $r = \sqrt{\sum_{i=1}^{3}(x^i)^2}$ for $q > \frac{1}{2}$ and $p > \frac{3}{2}$.

The decay order $(q > \frac{1}{2}, p > \frac{3}{2})$ guarantees that the ADM mass is a valid definition and the positive mass theorem holds.

In addition to the ADM mass (energy-momentum), there is also a companion definition of angular momentum that is attributed to ADM (Ashtekar-Hansen, Christodoulou, Chrusciel, etc.)

$$J = \frac{1}{8\pi} \int_{S_\infty} \pi(x^i \partial_j - x^j \partial_i, \nu), i < j, \text{ where } \pi = k - (tr_g k)g.$$

$x^i \partial_j - x^j \partial_i$ is considered to be an asymptotically rotation Killing field.

Note that, however, the calculation of angular momentum is more subtle, as the expression of J diverges apparently.

There are proposals (Regge-Teitelboim) of parity condition on (g, k) to assure finiteness of the improper integral.

The definitions can be unphysical even when the decay order is within the range with which the ADM mass is well-defined.

(Chen-Huang-Wang-Y.) There exist asymptotically flat spacelike hypersurfaces in $\mathbb{R}^{3,1}$ or $Sch^{3,1}$ with finite, nonzero ADM angular momentum such that $g = \delta + O(r^{-\frac{4}{3}})$ and $k = O(r^{-\frac{5}{3}})$.

(Chrusciel) If $p + q > 3$, then the ADM angular momentum is finite.

To what extent is the ADM angular momentum a valid definition?

All previous known definitions of quasilocal angular momentum satisfy the covariant properties with respect to the Poincare group and the consistency with Komar definition.
But there does not seem be any other effective criterion that is relevant to general spacetime.
Proposal of a criterion: A quasilocal definition of angular momentum is good if it gives good limit in an asymptotically flat spacetime.

Suppose the ADM mass of (M, g, k) is positive, then there is a unique, locally energy-minimizing, optimal isometric embedding of S_r whose image approaches a large round sphere in $\mathbb{R}^3$.

Take the limit as $r \to \infty$ of the quasi-local conserved quantities on S_r, we obtain $(E, P_i, \tilde{J}_i, \tilde{C}_i)$ where (E, P_i) is the same as the ADM energy-momentum.

$\tilde{J}_i$ is the new total angular momentum we defined and it may differ from the ADM angular momentum.

We prove an invariance of angular momentum theorem in Kerr: any strictly spacelike hypersurface in the Kerr spacetime has the same new total angular momentum.

"Strictly spacelike" means, in Boyer-Lindquist coordinates, $t = O(cr)$ for $|c| < 1$. In particular, the new total angular momentum vanishes for hypersurfaces in $\mathbb{R}^{3,1}$ of $Sch^{3,1}$.

The finiteness theorem of the new total angular momentum we proved does not assume any parity condition.

In evaluating the new total angular momentum, the optimal isometric embeddings provides the necessary correction to cancel any unphysical terms.

In addition, the definition also satisfies

$$\partial_t \tilde{J} = 0$$

along the vacuum Einstein equation.

Note that we can take the limit along a family of large spheres to define total angular momentum at null infinity as well.

This was previously studied by Rizzi, in which it is assumed that the spheres converge to a round sphere and a normalization condition that is equivalent to require that the null hypersurface is the null cone from a point.

The optimal isometric embedding allows a valid definition of angular momentum without imposing the above restrictive assumptions.

We recently compute the quasi-local mass of “spheres of unit size” at null infinity to capture the information of gravitational radiation.

The set-up (following Chandrasekhar) is a gravitational perturbation of the Schwarzschild spacetime which is governed by the Regge-Wheeler equation.

We take a sphere of a fixed areal radius and push it all the way to null infinity. The limit of the geometric data is still that of a standard configuration and thus the optimal embedding equation can be solved in a similar manner.

Let me discuss the result of axial perturbation in more detail.

We consider a metric perturbation of the form

$$-(1-\frac{2m}{r})dt^2+\frac{1}{1-\frac{2m}{r}}dr^2+r^2d\theta^2+r^2\sin^2\theta(d\phi-q_2dr-q_3d\theta)^2$$

The linearized vacuum Einstein equation is solved by a separation of variable Ansatz in which q_2 and q_3 are explicitly given by the Teukolsky function and the Legendre function.
In particular,

$$q_3=\sin(\sigma t)\frac{C(\theta)}{\sin^3\theta}\frac{(r^2-2mr)}{\sigma^2r^4}\frac{d}{dr}(rZ^{(-)})$$

for a solution of frequency σ.

After the change of variable $r_* = r + 2m\ln(\frac{r}{2m} - 1)$, $Z^{(-)}$ satisfies the Regge-Wheeler equation:

$$(\frac{d^2}{dr_*^2} + \sigma^2)Z^{(-)} = V^{(-)}Z^{(-)},$$

where

$$V^{(-)} = \frac{r^2 - 2mr}{r^5}[(\mu^2 + 2)r - 6m],$$

and μ is the separation of variable constant.

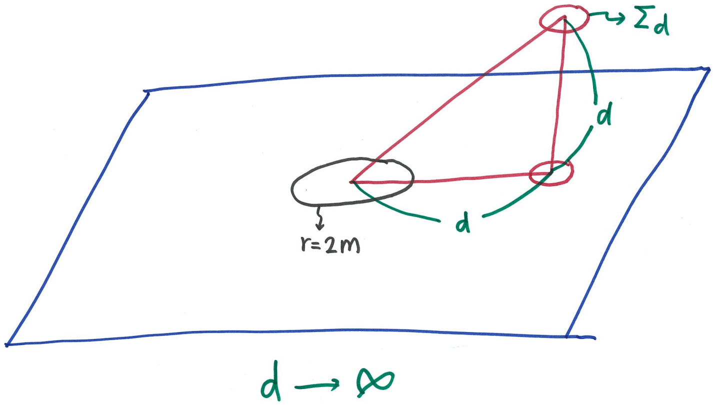
On the Schwarzschild spacetime

$$-(1-\frac{2m}{r})dt^2+\frac{1}{1-\frac{2m}{r}}dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2,$$

we consider an asymptotically flat Cartesian coordinate system (t, y_1, y_2, y_3) with $y_1 = r\sin\theta\sin\phi, y_2 = r\sin\theta\cos\phi, y_3 = r\cos\theta$. Given $(d_1, d_2, d_3) \in \mathbb{R}^3$ with $d^2 = \sum_{i=1}^3 d_i^2$, consider the 2-surface

$$\Sigma_d = \{(d, y_1, y_2, y_3) : \sum_{i=1}^{3}(y_i - d_i)^2 = 1\}.$$

We compute the quasi-local mass of Σ_d as $d \to \infty$.


Σ_d
d
d
$r=2m$
$d \to \infty$

Denote

$$A(r) = \frac{(r^2 - 2mr)}{\sigma^2 r^3} \frac{d}{dr}(rZ^{(-)}),$$

the linearized optimal embedding equation of Σ_d is reduced to two linear elliptic equations on the unit 2-sphere S^2:

$$\begin{aligned}\Delta(\Delta+2)\tau &= [-A''(1-Z_1^2) + 6A'Z_1 + 12A]Z_2Z_3 \\ (\Delta+2)N &= (A'' - 2A'Z_1 + 4A)Z_2Z_3\end{aligned}$$

where τ and N are the respective time and radial components of the solution, and Z_1, Z_2, Z_3 are the three standard first eigenfunctions of S^2. A' and A'' are derivatives with respect to r and r^2 is substituted by $r^2 = d^2 + 2Z_1 + 1$ in the above equations.

The quasi-local mass of Σ_d with respect to the above optimal isometric embedding is then

$$\frac{1}{d^2}\frac{C^2(\theta)}{\sin^6\theta}\{\sin^2(\sigma d)E_1 + \sigma^2\cos^2(\sigma d)E_2\} + O(\frac{1}{d^3})$$

where

$$E_1 = \int_{S^2}(1/2)\left[A^2 Z_2^2(7Z_3^2+1) + 2AA' Z_1 Z_3^2(3Z_2^2-1) - N(\Delta+2)N\right]$$

$$E_2 = \int_{S^2}\left[A^2 Z_2^2 Z_3^2 - \tau\Delta(\Delta+2)\tau\right]$$

In fact, the quasi-local mass density ρ of Σ_d can be computed at the pointwise level. Up to an $O(\frac{1}{d^3})$ term

$$
\begin{aligned}
\rho = & (K - \frac{1}{4}|H|^2) \\
& - \frac{(|H|-2)^2}{4} + \frac{1}{d^2}\{\frac{1}{2}|\nabla^2 N|^2 + ((\Delta + 2)N)^2 - \frac{1}{4}(\Delta N)^2 \\
& - \frac{1}{4}(\Delta \tau)^2 + \frac{1}{2}[\nabla^a \nabla^b (\tau_a \tau_b) - |\nabla \tau|^2 - \Delta |\nabla \tau|^2]\}
\end{aligned}
$$

where K is the Gauss curvature of Σ_d.
The first line, which integrates to zero, is of the order of $\frac{1}{d}$ and is exactly the mass aspect function of the Hawking mass. The $\frac{1}{d^2}$ term of the quasi local mass $\int_{\Sigma_d} \rho \, d\mu_{\Sigma_d}$ has contributions from the second and third lines (of the order of $\frac{1}{d^2}$), the $\frac{1}{d^2}$ term of the first line, and the $\frac{1}{d}$ term of the area element $d\mu_{\Sigma_d}$. The above integral formula is obtained after performing integrations by parts and applying the optimal embedding equation several times.

We can also consider the polar perturbation of the Schwarzschild spacetime in which the metric coefficients g_{tt}, g_{rr}, $g_{\theta\theta}$, and $g_{\phi\phi}$ are perturbed in

$$-(1-\frac{2m}{r})dt^2+\frac{1}{1-\frac{2m}{r}}dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2.$$

The gravitational perturbation is governed by the Zerilli equation

$$(\frac{d^2}{dr_*^2}+\sigma^2)Z^{(+)}=V^{(+)}Z^{(+)},$$

where

$$V^{(+)}=\frac{2(r^2-2mr)}{r^5(nr+3m)^2}[n^2(n+1)r^3+3mn^2r^2+9m^2nr+9m^3],$$

and n is the separation of variable constant.

Again, we can compute the quasilocal mass of spheres of unit-size at null infinity. The calculation is similar to the axial perturbation case but more complicated. The result is somehow different as the leading term is of the order $\frac{1}{d}$ (as opposed to $\frac{1}{d^2}$ for axial-perturbation) with nonzero coefficients.

If such a linear perturbation can be realized as an actual perturbation of the Schwarzschild spacetime, the result would contradict the positivity of the quasilocal mass. From this, we deduce the following conclusion:

There does not exist any gravitational perturbation of the Schwarzschild spacetime that is of purely polar type.

For an actual gravitational perturbation of the Schwarzschild, the vanishing of the $\frac{1}{d}$ gives a limiting integrand that integrates to zero on the limiting 2-sphere at null infinity.

To each closed loop on the limiting 2-sphere at null infinity, we thus associate a non-vanishing arc integral that is of the order of $\frac{1}{d}$, where d is the distance from the source.

We expect the freedom in varying the shape of the loop can increase the detectability of gravitational waves.

We have been using the Minkowski spacetime as the reference spacetime in defining the quasi-local energy. The critical points of the quasilocal energy are optimal isometric embeddings into $\mathbb{R}^{3,1}$.

Recently we are able to take into account of cosmological constants and define quasilocal energy and optimal isometric embeddings in reference to the de-Sitter (dS) or the Anti-de-Sitter (AdS) spacetime.

We recall that for a physical surface Σ with physical data $(\sigma, |H|, \alpha_H)$. Let X be an isometric embedding of σ into $\mathbb{R}^{3,1}$ and $\hat{\Sigma}$ be the projection of $X(\Sigma)$ into the orthogonal complement of T_0. The definition of $E(\Sigma, X, T_0)$ relies on the conservation law relating the geometries of $X(\Sigma)$ and $\hat{\Sigma}$. The optimal isometric embedding equation is the Euler-Lagrange equation of $E(\Sigma, X, T_0)$.

Suppose $\overline{\Sigma}$ is a surface in $\mathbb{R}^{3,1}$ such that the projection along $\overline{T}_0$ is a convex surface. Let $\overline{X}$ be the identity isometric embedding of $\overline{\Sigma}$. Then of course $E(\overline{\Sigma}, \overline{X}, \overline{T}_0) = 0$.

The positivity theorem we proved implies that $E(\overline{\Sigma}, X, T_0) \geq 0$ for any (X, T_0) close to $(\overline{X}, \overline{T}_0)$, and the equality holds if and only if (X, T_0) and $(\overline{X}, \overline{T}_0)$ differ by a Lorentz transformation in $\mathbb{R}^{3,1}$.

This identifies the kernel of the linearized optimal isometric embedding equation and allows us to solve nearby optimal isometric embedding equation using the implicit function theorem.

Thus in any configuration close to the Minkowski spacetime such as the large sphere limit or small sphere limit, the optimal isometric embedding is solvable and the mass and conserved quantities can be evaluated.

In presence of non-zero cosmological constant, we consider the (A)dS spacetimes in static coordinates

$$-\Omega dt^2 + g_{ij} dx^i dx^j$$

where Ω is the static potential corresponding to the time function t, and g is the metric of a space form of constant curvature.

Given a surface in the (A)dS spacetime, we can follow the integral curve of the Killing field $\frac{\partial}{\partial t}$ and obtain a surface in a static slice ($t = constant$), which is the analogue of the projection surface in the Minkowski reference case.

There is also a conservation law relating the geometries of the surface and its "projection" in the static slice.

Thus $E(\Sigma, X, T_0)$ can be defined, where Σ is a physical surface, X is an isometric embedding into the $(A)dS$ spacetime, and T_0 is a translating Killing field of the $(A)dS$ spacetime.

$$
\begin{aligned}
&8\pi E(\Sigma, X, T_0)\\
=&\int \Omega \widehat{H} d\widehat{\Sigma} - \int \Big[\sqrt{(1+\Omega^2|\nabla\tau|^2)|H|^2\Omega^2 + div(\Omega^2\nabla\tau)^2}\\
&\qquad - div(\Omega^2\nabla\tau)\sinh^{-1}\frac{div(\Omega^2\nabla\tau)}{\Omega|H|\sqrt{1+\Omega^2|\nabla\tau|^2}} - \Omega^2\alpha_H(\nabla\tau)\Big]d\Sigma,
\end{aligned}
$$

where τ is the restriction of the time function of T_0 to the image of X, and Ω is the restriction of the corresponding static potential of T_0 to the image of X. $\widehat{\Sigma}$ is the surface in the static slice of T_0, which is obtained by flowing the image of X along the integral curve of T_0.

Set $\Omega = 1$ formally, this recovers the expression of the quasilocal energy with respect to the Minkowski reference.

We can similarly write down the Euler-Lagrange equation as the optimal isometric embedding equation.

The results we obtained are no longer as strong as the Minkowski reference case. Nevertheless, we prove the following:

If $\overline{\Sigma}$ is a convex surface in a static slice of the $(A)dS$ spacetime, then the second variation of the quasilocal energy is non-negative.

We expect this is sufficient to solve the optimal isometric embedding for configurations in a physical spacetime that is a perturbation of the (A)dS spacetime.

Thank you!