

# Nonlinear PDE for Future Applications

## — Hyperbolic and Dispersive PDE —

**Date** 24 July, 2017 – 28 July, 2017  
**Place** TOKYO ELECTRON House of Creativity 3F, Lecture Theater,  
Katahira Campus, Tohoku University  
**Organizer** Takayoshi Ogawa (Tohoku University)  
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### Program

#### 24 July (Mon)

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13:55 – 14:00	Opening
14:00 – 14:50	Jun-ichi Segata (Tohoku Univ.) Modified scattering for the Klein-Gordon equation with critical nonlinearity in two and three dimensions
15:10 – 16:00	Noboru Chikami (Tohoku Univ.) Some functional inequalities in Hat-Sobolev spaces
16:20 – 17:10	Haruya Mizutani (Osaka Univ.) Remarks on endpoint Strichartz estimates for Schrödinger equations with inverse-square potentials
17:30 – 19:30	Welcome party

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#### 25 July (Tue)

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11:00 – 12:00	Thomas C. Sideris (UCSB) The affine motion of ideal fluids surrounded by vacuum I
13:30 – 14:20	Shuji Machihara (Saitama Univ.) Global well-posedness for one dimensional Chern-Simons-Dirac system in $L^p$
14:40 – 15:30	Yusuke Sugiyama (Tokyo Univ. of Science) Blow-up and estimates of the lifespan of solutions to the 1D compressible Euler equation with variable damping coefficient
15:50 – 16:40	Neal Bez (Saitama Univ.) Strichartz estimates for orthonormal systems of initial data

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#### 26 July (Wed)

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11:00 – 12:00	Thomas C. Sideris (UCSB) The affine motion of ideal fluids surrounded by vacuum II
13:30 – 14:20	Satoshi Masaki (Osaka Univ.) Threshold solutions in mass-subcritical nonlinear Schrödinger equation

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14:40 – 15:30	Kota Uriya (Okayama Univ. of Science) Long range scattering for nonlinear Schrödinger equations with critical homogeneous nonlinearity in 3d
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15:50 – 16:50	Yoshiyuki Kagei (Kyushu Univ.) Bifurcation of the compressible Taylor vortex I
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**27 July (Thu)**

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11:00 – 12:00	Thomas C. Sideris (UCSB) The affine motion of ideal fluids surrounded by vacuum III
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13:30 – 14:20	Nicola Visciglia (Univ. Pisa) On the growth of the Sobolev norms for NLS posed on compact settings
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14:40 – 15:30	Kazuyuki Tsuda (Osaka Univ.) Time periodic problem for the compressible Navier-Stokes equation on two-dimensional whole space
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15:50 – 16:50	Yoshiyuki Kagei (Kyushu Univ.) Bifurcation of the compressible Taylor vortex II
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**28 July (Fri)**

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11:10 – 12:00	Kimitoshi Tsutaya (Hirosaki Univ.) On the Cauchy problem for semilinear wave equations in Euclidean Friedmann-Lemaître-Robertson-Walker spacetimes
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13:30 – 14:20	Kunio Hidano (Mie Univ.) Global existence for a system of quasi-linear wave equations in 3D satisfying the weak null condition
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14:20 – 14:25	Closing
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# The affine motion of ideal fluids surrounded by vacuum

Thomas C. Sideris

University of California, Santa Barbara

These lectures will be devoted to the study of the long time motion of ideal fluids surrounded by vacuum, in the compressible and incompressible cases. This motion is modeled by an initial free boundary value problem for the Euler equations.

## Lecture 1: The general problem; reduction to affine motion.

We shall begin with a formulation of the general initial vacuum free boundary value problem for ideal fluids and a review of results on local well-posedness, [1]-[16]. The motion of the fluid is represented by a one-parameter family of diffeomorphisms  $y \mapsto x(t, y)$  taking each point  $y$  in a reference domain  $B \subset \mathbb{R}^n$  to its location  $x(t, y)$  in the moving domain  $\Omega_t \subset \mathbb{R}^n$  occupied by the fluid at time  $t$ . Under the assumption of nonnegative pressure, we will establish the general lower bound  $\text{diam } \Omega_t \gtrsim t$ , for the diameter of the fluid domain  $\Omega_t$ , [18]. For the remainder of the lectures, we will restrict our attention the case of affine motion, that is,  $x(t, y) = A(t)y$ , where  $A(t)$  is an invertible linear transformation on  $\mathbb{R}^n$  for each time  $t \in \mathbb{R}$ , [17], [19]. In this case, the fluid domains  $\Omega_t$  are ellipsoids. We will see that the nonlinear partial differential equations describing the fluid motion do indeed have affine solutions and that the equations of motion reduce to a globally solvable system of ordinary differential equations in  $\mathbb{R}^{n \times n}$  for  $A(t)$ .

## Lecture 2: Compressible affine motion in 3D.

We shall study the asymptotic behavior of solutions to the system of ODEs derived in Lecture 1, in the three-dimensional ( $n = 3$ ) compressible case with the pressure law  $p = \rho^\gamma$ ,  $\gamma > 1$ , [19]. The reduced system is Hamiltonian and globally solvable. First we will show that  $\text{diam } \Omega_t \sim t$ , proving that the general lower bound is sharp in the compressible affine case. Next, for appropriate values of the adiabatic index  $\gamma$ , we shall construct solutions asymptotic to a linear function of the form  $A_\infty(t) \equiv A_1 t + A_0$ , where  $A_\infty(t)$  is invertible for all  $t \gg 1$ , but  $\text{rank } A_1$  may equal 1, 2, or 3. We will also show that when  $4/3 < \gamma < 2$ , there is a scattering theory, i.e. there is a one-to-one correspondence between initial data and asymptotic states  $A_1 t + A_0$  with  $\text{rank } A_1 = 3$ . Finally, we will also mention a recent stability result [20].

## Lecture 3: Incompressible affine motion in 2D.

We shall study the asymptotic behavior of solutions to the reduced system of ODEs derived in Lecture 1, in the incompressible case, [19]. This is a globally solvable constrained Hamiltonian system

in  $SL(n, \mathbb{R})$ , the Lie group of  $n \times n$  matrices with determinant equal to one. The incompressible affine fluid motion is precisely geodesic flow in  $SL(n, \mathbb{R})$  when viewed as an  $n^2 - 1$  dimensional manifold embedded in  $\mathbb{R}^{n \times n}$ . For  $n = 2$ , we can give a complete description of the geodesics. Generically, solutions are unbounded at  $t = \pm\infty$  and asymptotically linear in time, and there are even straight line solutions. But there is also a periodic orbit in the phase space with stable and unstable manifolds consisting of semi-bounded orbits.

## References

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# Modified scattering for the Klein-Gordon equation with critical nonlinearity in two and three dimensions

Jun-ichi Segata

Tohoku University

We consider the final state problem for the Klein-Gordon equation with gauge invariant critical nonlinearity in two and three space dimensions:

$$\begin{cases} (\square + 1)u = \lambda|u|^{\frac{2}{d}}u & t \in \mathbb{R}, x \in \mathbb{R}^d, \quad d = 2, 3, \\ u - u_{\text{ap}} \rightarrow 0 & \text{in } L^2 \text{ as } t \rightarrow +\infty, \end{cases} \quad (\text{NLKG})$$

where  $\square = \partial_t^2 - \Delta$  is d'Alembertian,  $u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  is an unknown function,  $u_{\text{ap}} : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a given function, and  $\lambda$  is a non-zero real constant.

The power  $1 + 2/d$  is a borderline between the short and the long range scattering theories, since point-wise decay of a solution to the linear Klein-Gordon equation is  $O(t^{-d/2})$  as  $t \rightarrow \infty$ . Although there are several results on the scattering problem for the Klein-Gordon equation with gauge invariant critical nonlinearity in one space dimension, similar problem in higher dimensions was out of scope in the previous works due to lack of smoothness of the nonlinear term.

We show that for a given asymptotic profile  $u_{\text{ap}}$ , there exists a solution  $u$  to (NLKG) which converges to given asymptotic profile  $u_{\text{ap}}$  in  $L^2$  as  $t \rightarrow \infty$ . Here the asymptotic profile  $u_{\text{ap}}$  is given by the leading term of the solution to the linear Klein-Gordon equation with a logarithmic phase correction. We construct a solution to (NLKG) by applying the contraction principle to the integral equation of Yang-Feldman type associated with (NLKG) around a suitable approximate solution. For the two dimensional case, construction of an approximate solution is based on Fourier series expansion of the nonlinearity used in our previous paper [2]. For the three dimensional case, we construct an approximate solution by combining the Fourier series expansion for the nonlinearity and smooth modification of phase correction by Ginibre and Ozawa [1]. My talk is based on the joint work with Satoshi Masaki (Osaka University).

## References

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# Some functional inequalities in Hat-Sobolev spaces

Noboru Chikami

Tohoku University

We show a variant of Gagliardo-Nirenberg inequality in Hat-Sobolev spaces, which improves certain classes of classical Sobolev embeddings.

Let  $s \in \mathbb{R}$  and  $0 < p \leq \infty$ . Hat-Sobolev spaces (or occasionally, *Fourier-Sobolev spaces*) are defined by  $\widehat{H}_p^s(\mathbb{R}^d) := \{u \in \mathcal{S}'_h(\mathbb{R}^d) \mid \widehat{u} \in L^1_{loc}(\mathbb{R}^d), \|u\|_{\widehat{H}_p^s} := \|\lvert\xi\rvert^s \widehat{u}\|_{L^{p'}} < \infty\}$ , where  $\widehat{u}$  denotes the Fourier transform of  $u$  and  $p'$  is the Hölder conjugate. When  $s = 0$ , we denote  $\widehat{L}^p(\mathbb{R}^d) := \widehat{H}_p^0(\mathbb{R}^d)$  with its norm changed accordingly. This type of function space is frequently used in the study of dispersive equations.

Our main result reads as follows: Let  $0 \leq \sigma < s < \infty$  and  $1 \leq q, r \leq \infty$ . Suppose  $p$  satisfies the relation

$$\frac{d}{p} - \sigma = \theta \frac{d}{q} + (1 - \theta) \left( \frac{d}{r} - s \right)$$

where  $0 < \theta < 1 - \frac{\sigma}{s}$ . Then for all  $u \in \widehat{L}^q(\mathbb{R}^d) \cap \widehat{H}_r^s(\mathbb{R}^d)$ ,  $u$  belongs to  $\widehat{H}_p^\sigma(\mathbb{R}^d)$ . Moreover, there exists a positive constant  $C$  such that

$$\|u\|_{\widehat{H}_p^\sigma} \leq C \|u\|_{\widehat{L}^q}^\theta \|u\|_{\widehat{H}_r^s}^{1-\theta} \quad (0-1)$$

holds for all  $u \in \widehat{L}^q(\mathbb{R}^d) \cap \widehat{H}_r^s(\mathbb{R}^d)$ .

Some continuation criterion for the incompressible Navier-Stokes system is established as an application.

# Remarks on endpoint Strichartz estimates for Schrödinger equations with inverse-square potentials

Haruya Mizutani

Osaka University

This talk is partly based on joint work with Jean-Marc Bouclet (Toulouse III). We discuss a recent progress [1, 3, 4] on global-in-time Strichartz and smoothing estimates for the Schrödinger equation with the inverse-square potential on  $\mathbb{R}^n$ ,  $n \geq 3$ :

$$(i\partial_t - H)u = F; \quad u|_{t=0} = u_0; \quad H = -\Delta - a|x|^{-2},$$

where we assume  $a \leq (n-2)^2/4$  to ensure that  $H$  is non-negative. It is known that the inverse-square potential  $|x|^{-2}$  represents a borderline case of the validity of several dispersive estimates for the Schrödinger equation in view of its local singularity at the origin and decay rate at spatial infinity. The coupling constant  $(n-2)^2/4$  is also critical in the sense that if  $a > (n-2)^2/4$ ,  $H$  is not bounded from below and may have negative eigenvalues which prevent any kind of global-in-time dispersive estimates.

In this talk we give a complete picture, in terms of Lorentz spaces, on a validity of Strichartz estimates for both subcritical and critical cases. The main result is summarized as follows:

- In the subcritical case ( $a < (n-2)^2/4$ ), the full set of global-in-time Strichartz estimates (including the double endpoint inhomogeneous case) and local smoothing estimates hold. In fact, we can consider a large class of scaling-critical potentials which particularly includes the inverse-square potential with subcritical coupling constants. This complements an earlier result by [2] in which only homogeneous estimates were studied.
- In the critical case ( $a = (n-2)^2/4$ ), splitting the solution into the “radial” and “non-radial” parts, we show that (i) the radial part satisfies a weak-type endpoint Strichartz estimate; (ii) other endpoint Strichartz estimates in Lorentz spaces (and hence usual endpoint Strichartz estimates) for the radial part fail in general; (iii) the non-radial part satisfies the full set of Strichartz estimates.

## References

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# Global well-posedness for one dimensional Chern-Simons-Dirac system in $L^p$

Shuji Machihara

Saitama University

This is a joint work with Takayoshi Ogawa [5]. We consider the following Cauchy problem of the Chern-Simons-Dirac equation in one spatial dimension:

$$\begin{cases} i\gamma^\mu(\partial_\mu - iA_\mu)\psi = m\psi, & t > 0, x \in \mathbb{R}, \\ \partial_t A_1 - \partial_x A_0 = \psi^* \gamma^0 \psi, & t > 0, x \in \mathbb{R}, \\ \partial_t A_0 - \partial_x A_1 = 0, & t > 0, x \in \mathbb{R}, \\ \psi(0, x) = \psi_0(x), \quad \mathbb{A}(0, x) = \mathbb{A}_0(x), & x \in \mathbb{R}, \end{cases} \quad (0-2)$$

where  $\psi = {}^t(\psi_1(t, x), \psi_2(t, x)) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}^2$  and  $\mathbb{A} = {}^t(A_0(t, x), A_1(t, x)) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2$  be the unknown functions to this problem.  $m > 0$  is a constant,  $\mu = 0, 1$  and  $\gamma^\mu$  are Dirac matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (0-3)$$

The initial data  $\psi_0 = {}^t(\psi_{10}(x), \psi_{20}(x))$  and  $\mathbb{A}_0 = {}^t(A_{00}(t, x), A_{10}(t, x))$  are given functions. We denote by  $\psi^*$  the complex conjugate for each component as  $\psi^* = {}^t\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$  for  $\psi = {}^t(\psi_1, \psi_2)$ .

If we consider the problem in the Sobolev spaces,  $(\psi, \mathbb{A}) \in H^s \times H^r$ , there are some results [1], [6], [7]. There it is known the problem is ill-posedness at the critical case  $H^{-1/2} \times H^{-1/2}$ .

We take a different approach to consider the time global well-posedness of (0-2) than in the Sobolev spaces. We use the Lebesgue spaces  $L^p(\mathbb{R})$ ,  $1 \leq p \leq \infty$ . Our main result is the following.

**Theorem 0.1.** *Let  $1 \leq p \leq \infty$ . Then for any  $\psi_0 \in L^p(\mathbb{R})$  and  $\mathbb{A}_0 \in L^p(\mathbb{R})$ , there exists a time global solution  $(\psi, \mathbb{A}) \in C([0, \infty); L^p(\mathbb{R})) \times C([0, \infty); L^p(\mathbb{R}))$  to (0-2) such that*

(i) *the solution is unique in*

$$\begin{aligned} \psi &\in C([0, \infty); L^p(\mathbb{R})), \quad \mathbb{A} \in C([0, \infty); L^p(\mathbb{R})), \\ \psi^* \alpha \psi &\in L^p_{loc}([0, \infty); L^p(\mathbb{R})), \quad \mathbb{A} \psi \in L^p_{loc}([0, \infty); L^p(\mathbb{R})). \end{aligned}$$

(ii) *The map from the data  $(\psi_0, \mathbb{A}_0)$  to the solution  $(\psi, \mathbb{A})$  is Lipschitz continuous from  $L^p \times L^p$  to  $C([0, \infty); L^p(\mathbb{R})) \times C([0, \infty); L^p(\mathbb{R}))$ .*

The main interest in the above Theorem 0.1 is the critical case  $p = 1$ . The main concern of our proof is devoted to avoid the mass concentration phenomena of the solution. This phenomena is expressed as follows: There exists a constant  $\varepsilon > 0$  such that for any  $\delta > 0$ ,

$$\liminf_{t \rightarrow T} \left( \int_{B_\delta(x)} |\psi(t, x)| dx + \int_{B_\delta(x)} |\mathbb{A}(t, x)| dx \right) \geq \varepsilon,$$

where  $T > 0$  be the maximal existence time.

## References

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# Blow-up and estimates of the lifespan of solutions to the 1D compressible Euler equation with variable damping coefficient

Yusuke Sugiyama

Tokyo University of Science

In this talk, we mainly consider the following Cauchy problem of the compressible Euler equation with time dependent damping

$$\begin{cases} u_t - v_x = 0, \\ v_t + p(u)_x = -\frac{\lambda v}{(1+t)^\mu}, \\ (u(0, x), v(0, x)) = (1 + \varepsilon\varphi(x), \varepsilon\psi(x)). \end{cases}$$

where  $u = u(t, x)$  and  $v = v(t, x)$  are the real valued unknown functions for  $(t, x) \in [0, T] \times \mathbb{R}$ ,  $\lambda \geq 0$ ,  $\mu \geq 0$ ,  $\varepsilon > 0$  and  $p(u) = u^{-\gamma}/\gamma$  with  $\gamma > 1$ .

In [1–3], Pan has found thresholds of  $\mu$  and  $\lambda$  separating the existence and the nonexistence of global solution in small data regime. Namely, in the case with  $0 \leq \mu < 1$  and  $\lambda > 0$  or  $\mu = 1$  and  $\lambda > 2$ , Pan [2] has proved that solutions exist globally in time, if initial data are small and compact perturbations of constant states. While, in the case with  $\mu > 1$  and  $\lambda > 0$  or  $\mu = 1$  and  $0 \leq \lambda \leq 2$ , Pan has proved that solutions can blow up under some conditions on initial data in [1, 3]. However, in these papers, the estimate of the lifespan is  $T^* \leq \exp(C/\varepsilon^2)$  and it is not determined how solutions blow up ( $L^\infty$  blow-up, the derivative blow-up or the blow up of  $p'$ ). The aim of this talk is to give a sufficient condition that the derivative blow-up occurs with the solution itself and the pressure bounded and to estimate the lifespan precisely. Our proof is based on the method of characteristic.

## References

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# Strichartz estimates for orthonormal systems of initial data

Neal Bez

Saitama University

The classical Strichartz estimates for the free Schrödinger propagator  $e^{it\Delta}$  may be stated as

$$\| |e^{it\Delta} f|^2 \|_{L_t^p L_x^q} \lesssim 1 \quad (0-4)$$

for normalised initial data  $\|f\|_{L^2(\mathbb{R}^d)} = 1$ . Here,  $d \geq 1$  and  $p, q \geq 1$  are such that  $\frac{2}{p} + \frac{d}{q} = d$  and  $(p, q, d) \neq (1, \infty, 2)$ . The endpoint case is  $(p, q) = (1, \frac{d}{d-2})$ , for  $d \geq 3$ , and this was proved by Keel and Tao.

If we take any system of initial data  $(f_j)$  in  $L^2(\mathbb{R}^d)$ , then the triangle inequality and the above classical Strichartz estimate immediately imply

$$\left\| \sum_j \lambda_j |e^{it\Delta} f_j|^2 \right\|_{L_t^p L_x^q} \lesssim \|(\lambda_j)\|_{\ell^\alpha} \quad (0-5)$$

with  $\alpha = 1$ . Recently it has been observed that if the system  $(f_j)$  is *orthonormal* then this can be improved in the sense that (0-5) holds for certain  $\alpha \geq 1$  (depending on  $p$  and  $q$ ). This line of investigation was initiated by Frank–Lewin–Lieb–Seiringer and, along with subsequent progress by Frank–Sabin, resulted in an understanding of the optimal  $\alpha$ . In general the optimal  $\alpha$  is somewhere between 1 and 2, but interestingly, at the Keel–Tao endpoint, the optimal  $\alpha$  is equal to 1.

In this talk, we discuss estimates of the form (0-5) for orthonormal systems of data in the Sobolev space  $\dot{H}^s$ . Unlike the classical case, the extension to smooth orthonormal systems of data does not appear to follow trivially (e.g. via Sobolev embedding) from the  $L^2$  case. This is based on joint work with Younghun Hong, Sanghyuk Lee, Shohei Nakamura and Yoshihiro Sawano.

# Threshold solutions in mass-subcritical nonlinear Schrödinger equation

Satoshi Masaki

Osaka University

We consider global behavior of solutions to the nonlinear Schrödinger equations  $i\partial_t u + \Delta u = -|u|^{p-1}u$ . In particular, we want to give a classification of solutions according to their large time behavior. It is known that when  $p > 1 + \frac{2}{d}$  small solutions behaves like a free solution (scattering). As a second step, we would like to find a threshold solution which lies on the boundary between small scattering solutions and solution with other behavior.

There are many studies in this direction in the mass-critical case  $p = 1 + \frac{4}{d}$  or the mass-supercritical case  $p > 1 + \frac{4}{d}$ . Here, we want to consider the mass-subcritical case  $p < 1 + \frac{4}{d}$ .

To studies mass-subcritical case, we encounter at least two problems. First one is to find a right sense of “smallness”. For example, smallness in  $L^2$  or  $H^1$  do not yield scattering because we can construct small in  $H^1$  non-scattering solution from the scaling of ground state solution. The argument suggests that solving the equation in a scale critical space, or in a space embedded into a scale critical space, is necessary to start our study. Remark that, however, the critical Sobolev space has negative regularity, and so well-posedness is already a tough problem. So far, small data scattering is known in radial Sobolev space, weighted space, and Fourier Lebesgue space, for instance.

Second one is characterization of a threshold. The well-known conserved quantity for NLS are mass, moment, and energy. But they are adopted to  $L^2$ -scaling,  $\dot{H}^{1/2}$ -scaling, and  $\dot{H}^1$ -scaling, respectively. Hence, due to the above scaling argument, it seems difficult to characterize a threshold by a combination of these quantities. Hence, we regard the norm of a solution in a “good” function space (of space variable only) as a function of time, and then introduce a global quantity of the time function. The global quantity can be regarded as a kind of *conserved quantity*. Then, we shall find a threshold solution as a function which minimizes the global quantity *among all non-scattering solutions*.

In this talk, I want to give an overview on the existence of threshold solutions in the weighted space framework [1–3], and in the Fourier-Lebesgue framework [4]. Similar results hold in generalized KdV equation [5, 6].

## References

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# Long range scattering for nonlinear Schrödinger equations with critical homogeneous nonlinearity in 3d

Kota Uriya

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In this talk, we consider large time behavior of solutions to nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = F(u). \quad (\text{NLS})$$

Here,  $(t, x) \in \mathbb{R}^{1+3}$  and  $u = u(t, x)$  is a complex-valued unknown function. We suppose that the nonlinearity  $F$  is homogeneous of degree  $1 + 2/3$ , that is,  $F$  satisfies

$$F(\lambda u) = \lambda^{1+\frac{2}{3}} F(u)$$

for any  $u \in \mathbb{C}$  and  $\lambda > 0$ . A typical example is  $F(u) = \mu|u|^{p-1}u$  ( $\mu \in \mathbb{R} \setminus \{0\}$ ). The exponent  $p = 1 + 2/d$  is known as the critical in the sense of long time behavior of the solution in  $\mathbb{R}^d$ . In [2], Masaki and Miyazaki considered the problem in one and two dimensions and gave a sufficient condition on the nonlinearity for that the corresponding equation admits a solution that behaves like a free solution with or without a logarithmic phase correction. Their argument is based on Fourier series expansion of the nonlinearity into a sum of a resonant part and a harmless non-resonant part. We here focus on the study of the three-dimensional case, in which it is required that a solution converges to a given asymptotic profile with a faster rate than in the lower dimensional cases. To obtain the necessary convergence rate, we employ the end-point Strichartz estimate and modify a time-dependent regularizing operator which is introduced in [2]. This regularizing operator is a development of one firstly used in Hayashi-Wang-Naumkin [1]. Moreover, we present a candidate of the second asymptotic profile to the solution, which is originally found in Moriyama-Tonegawa-Tsutsumi [4].

This is a joint work with S. Masaki (Osaka University) <sup>1</sup> and H. Miyazaki (National Institute of Technology, Tsuyama College) <sup>2</sup>.

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# Bifurcation of the compressible Taylor vortex

Yoshiyuki Kagei

Kyushu University

In this talk I will consider the Couette-Taylor problem, a flow between two concentric cylinders, whose inner cylinder is rotating with uniform speed and the outer one is at rest. If the rotating speed is sufficiently small, the Couette flow (laminar flow) is stable. When the rotating speed increases, beyond a certain value of the rotating speed, a vortex flow pattern (Taylor vortex) appears. This phenomenon has been widely investigated as a good subject of the study of pattern formations and transition to turbulent. Mathematically, the occurrence of the Taylor vortex is formulated as a bifurcation problem. The bifurcation of the Taylor vortex is well known for the incompressible Navier-Stokes equations. In this talk, the Couette-Taylor problem is considered for viscous compressible fluids and the bifurcation problem is studied for the compressible Navier-Stokes equations. In the first part of the talk, the spectrum of the linearized operator around the Couette flow is investigated. In the second part of the talk, the bifurcation of the compressible Taylor vortex is proved when the Mach number is sufficiently small. This talk is based on a joint work with Prof. Takaaki Nishida (Kyoto University) and Ms. Yuka Teramoto (Kyushu University).

# On the growth of the Sobolev norms for NLS posed on compact settings

Nicola Visciglia

University of Pisa

We are interested on the large time behavior of solutions to NLS posed on suitable compact settings, namely:

$$\begin{cases} i\partial_t u + \Delta_g u = |u|^2 u, & (t, x) \in \mathbb{R} \times M^d \\ u(0, x) = \varphi \in H^m(M^d) \end{cases} \quad (0-6)$$

where  $\Delta_g$  is the Laplace-Beltrami on the generic compact manifold  $(M^d, g)$ , and cubic NLS perturbed by the harmonic oscillator:

$$\begin{cases} i\partial_t u + \Delta u - |x|^2 u = |u|^2 u, & (t, x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0, x) = \varphi \in H^m(\mathbb{R}^d) \cap L^2_{|x|^m}(\mathbb{R}^d). \end{cases} \quad (0-7)$$

Notice that in both situations we have compact resolvent for the linear part of the equation. There is a huge literature about the Cauchy problems associated with (0-6) and (0-7), relying on the analysis of the corresponding Strichartz estimates.

Our main motivation goes back to the original work of Bourgain on the analysis of the growth of the higher order Sobolev norms, namely the study of the quantity  $\|u(t)\|_{H^m}$  as  $t \rightarrow \infty$ . Notice that for  $m = 1$  this quantity is bounded due to the conservation of the Hamiltonian, however the situation is more involved for  $m > 1$ .

The first result that we present concerns (0-6) for  $d = 2, 3$  (the case  $d = 1$  is not interesting due to the complete integrability of 1D cubic NLS).

**Theorem 0.1.** For every  $u(t, x) \in \mathcal{C}_t(H^m(M^2))$  solution to (0-6) for  $d = 2$ , we get:

$$\sup_{(0, T)} \|u(t, x)\|_{H^m(M^2)} \leq C(\max\{1, T\})^{\frac{m-1}{1-2s_0} + \varepsilon}, \quad (0-8)$$

where  $s_0 > 0$  is such that  $\|e^{it\Delta_g} \varphi\|_{L^4((0,1) \times M^2)} \leq C\|\varphi\|_{H^{s_0}(M^2)}$ .

For every  $u(t, x) \in \mathcal{C}_t(H^m(M^3))$  solution to (0-6) for  $d = 3$ , we get:

$$\sup_{(0, T)} \|u(t, x)\|_{H^m(M^3)} \leq C \exp(CT).$$

Concerning the solutions to (0-7), thanks to a suitable bilinear effect, we obtain

**Theorem 0.2.** For every  $u(t, x) \in \mathcal{C}_t(H^m \cap L^2_{|x|^m})$  solution to (0-7) for  $d = 2$ , we get:

$$\sup_{(0, T)} \|u(t, x)\|_{H^m(\mathbb{R}^2)} + \||x|^m u(t, x)\|_{L^2(\mathbb{R}^2)} \leq CT^{\frac{2}{3}(m-1) + \varepsilon}. \quad (0-9)$$

The talk is based on a series of joint papers with F. Planchon and N. Tzvetkov.

# Time periodic problem for the compressible Navier-Stokes equation on two-dimensional whole space

Kazuyuki Tsuda

Osaka University

We consider time periodic problem and stationary problem for the compressible Navier Stokes equation (CNS) on  $\mathbb{R}^2$  under some symmetry condition.

Ma, Ukai, and Yang [2] studied the time periodic problem on the whole space  $\mathbb{R}^n$ . It was shown in [2] that if  $n \geq 5$ , there exists a time periodic solution around constant state for a sufficiently small time periodic external force. Furthermore, the time periodic solution is stable under sufficiently small initial perturbations and the optimal time decay rate of the perturbation is obtained. Therefore it had been problem to solve the time periodic problem in lower dimensional case.

In [1], we showed that when the space dimension  $n$  is greater than or equal to 3 we have the existence of a time periodic solution for sufficiently small time periodic external force with some symmetry. We also showed the asymptotic stability of the time periodic solution for sufficiently small initial perturbation. Furthermore, we have the optimal time decay estimate of the perturbation.

As for the result without assuming the symmetry condition for CNS, I showed in [5] that there exists a time periodic solution for small time periodic external force on  $\mathbb{R}^n$  for the dimension  $n \geq 3$ . In addition, we obtain the asymptotic stability of the time periodic solution for sufficiently small initial perturbation. The decay of the perturbation in the  $L^\infty$  norm is also showed as time goes to infinity.

Concerning the stationary problem for CNS, Shibata and Tanaka [3] showed that on  $\mathbb{R}^3$  there exists a stationary solution for small external force. Moreover, the asymptotic stability of the stationary solution is obtained for sufficiently small initial perturbation. In [4], the decay rate of the perturbation in the  $L^\infty$  norm is studied.

In this talk it is stated that there exists a time periodic solution to CNS on  $\mathbb{R}^2$  for small time periodic external force satisfying antisymmetry condition. The antisymmetry condition was used in the stationary problem for incompressible Navier-Stokes equation on  $\mathbb{R}^2$  ([6]). Furthermore, the existence of stationary solution to CNS for the stationary problem is also stated under small external force having the antisymmetry.

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# On the Cauchy problem for semilinear wave equations

Kimitoshi Tsutaya

Hirosaki University

The Friedmann-Lemaître-Robertson-Walker spacetimes are the spatially homogeneous and isotropic cosmological models, known as the standard model of modern cosmology. The metric of Euclidean FLRW spacetime is given by

$$(1) \quad ds^2 = -dt^2 + S(t)^2 d\sigma^2,$$

where  $S(t)$  is the scale factor, and  $d\sigma$  is the line element of  $\mathbb{R}^3$ .

In this talk we consider the Cauchy problem for the semilinear wave equation in Euclidean FLRW spacetime

$$(2) \quad \square_g u = F(u)$$

with the initial data given at some positive time  $t_0$ , where  $\square_g$  is the d'Alembertian in the coordinates of (1), and  $F(u)$  behaves like  $|u|^p$  for some  $p > 1$ . For the initial data  $u(t_0, x) = \varphi(x)$ ,  $\partial_t u(t_0, x) = \psi(x)$ , we assume that

$$\varphi(x), \psi(x) = O(|x|^{-1-k}) \quad (|x| \rightarrow \infty).$$

We show the decay estimates of the solutions of  $\square_g u = 0$ , and also the existence of global solutions of (2) for small initial data under some conditions on  $k$  and  $p$ .

# Global existence for a system of quasi-linear wave equations in 3D satisfying the weak null condition

Kunio Hidano

Mie University

We consider the Cauchy problem for a system of quasi-linear wave equations in three space dimensions of the form

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + G_1^{11, \alpha\beta\gamma}(\partial_\gamma u_1)(\partial_{\alpha\beta}^2 u_1) + G_1^{21, \alpha\beta\gamma}(\partial_\gamma u_2)(\partial_{\alpha\beta}^2 u_1) \\ \quad + H_1^{11, \alpha\beta}(\partial_\alpha u_1)(\partial_\beta u_1) + H_1^{12, \alpha\beta}(\partial_\alpha u_1)(\partial_\beta u_2) + H_1^{22, \alpha\beta}(\partial_\alpha u_2)(\partial_\beta u_2) = 0, \\ \partial_t^2 u_2 - \Delta u_2 + G_2^{12, \alpha\beta\gamma}(\partial_\gamma u_1)(\partial_{\alpha\beta}^2 u_2) + G_2^{22, \alpha\beta\gamma}(\partial_\gamma u_2)(\partial_{\alpha\beta}^2 u_2) \\ \quad + H_2^{12, \alpha\beta}(\partial_\alpha u_1)(\partial_\beta u_2) + H_2^{11, \alpha\beta}(\partial_\alpha u_1)(\partial_\beta u_1) + H_2^{22, \alpha\beta}(\partial_\alpha u_2)(\partial_\beta u_2) = 0. \end{cases}$$

Here, summation is assumed from 0 to 3 for  $\alpha, \beta, \gamma$ . Supposing the null condition for the coefficients of the first equation  $\{G_1^{11, \alpha\beta\gamma}\}, \{G_1^{21, \alpha\beta\gamma}\}, \{H_1^{11, \alpha\beta}\}, \{H_1^{12, \alpha\beta}\}$ , and  $\{H_1^{22, \alpha\beta}\}$ , and supposing the null condition only on  $\{G_2^{22, \alpha\beta\gamma}\}$  and  $\{H_2^{22, \alpha\beta}\}$  as for the coefficients of the second equation, we show global existence of small solutions. The feature of this system lies in that it violates the null condition (see [2], [3]) and satisfies the weak null condition (see [4]). Due to the presence of the quadratic terms which do not satisfy the null condition, the standard argument no longer works for the proof of global existence. To get over this difficulty, we extend the ghost weight method of Alinhac [1] so that it works for this system.

This talk is based on a joint work with Kazuyoshi Yokoyama (Hokkaido University of Science).

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