Nonlinear PDE for Future Applications

— Geometry and Inverse Problems —

in cooperation with A3 FORESIGHT PROGRAM

Date	2 October, 2017 – 6 October, 2017
Place	Graduate School of Information Sciences (GSIS), Aobayama Campus, Tohoku University during the period **from October 2nd (Monday) to October 5th (Thursday)**; TOKYO ELECTRON House of Creativity 3F, Lecture Theater, Katahira Campus, Tohoku University **on October 6th(Friday)**
Organizer	Shigeru Sakaguchi (Tohoku University)
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Program

2 October (Mon):	**Lecture hall 206, 2F GSIS**
13:10 – 14:40	Daniel Peralta-Salas (Instituto de Ciencias Matemáticas) Topology of nodal sets of solutions to elliptic PDEs I
15:00 – 16:30	Samuli Siltanen (University of Helsinki) Introduction to linear and nonlinear tomography I
16:45 – 17:45	Minh Mach, Tatiana Bubba, Janne Tamminen, and Zenith Purisha (University of Helsinki) Matlab session I
18:00 – 20:00	Welcome party

3 October (Tue):	**GSIS**
Morning session:	**Lecture room 610, 6F GSIS**
10:40 – 11:30	Jinhae Park (Chungnam National University) Some mathematical questions related to geometry in liquid crystals
Afternoon session:	**Lecture room 207, 2F GSIS**
13:10 – 14:40	Daniel Peralta-Salas (Instituto de Ciencias Matemáticas) Topology of nodal sets of solutions to elliptic PDEs II
15:00 – 16:30	Samuli Siltanen (University of Helsinki) Introduction to linear and nonlinear tomography II
16:45 – 17:45	Minh Mach, Tatiana Bubba, Janne Tamminen, and Zenith Purisha (University of Helsinki) Matlab session II

4 October (Wed):	**Lecture hall 206, 2F GSIS**
9:30 – 10:20	Ruo Li (Peking University) An h-adaptive mesh method for optimal control problem
10:40 – 11:30	Zhiyuan Sun (Peking University) A discontinuous finite element space by patch reconstruction
13:10 – 14:40	Daniel Peralta-Salas (Instituto de Ciencias Matemáticas)) Topology of nodal sets of solutions to elliptic PDEs III
15:00 – 16:30	Samuli Siltanen (University of Helsinki) Introduction to linear and nonlinear tomography III
16:45 – 17:45	Minh Mach, Tatiana Bubba, Janne Tamminen, and Zenith Purisha (University of Helsinki) Matlab session III

5 October (Thu):	**GSIS**
Morning session:	**Lecture room 207, 2F GSIS**
10:40 – 11:30	Free discussion
Afternoon session:	**Lecture hall 206, 2F GSIS**
13:10 – 14:40	Daniel Peralta-Salas (Instituto de Ciencias Matemáticas) Topology of nodal sets of solutions to elliptic PDEs IV
15:00 – 16:30	Samuli Siltanen (University of Helsinki) Introduction to linear and nonlinear tomography IV
16:45 – 17:45	Minh Mach, Tatiana Bubba, Janne Tamminen, and Zenith Purisha (University of Helsinki) Matlab session IV

6 October (Fri):	**TOKYO ELECTRON House of Creativity 3F**
10:00 – 10:50	Masaru Ikehata (Hiroshima University) Detection and range estimation of a hidden object using the time domain enclosure method
11:10 – 12:00	Hiromichi Itou (Tokyo University of Science) On direct and inverse problems involving cracks in elasticity
13:30 – 14:20	Kaoru Maruta (Tohoku University) Near-limit flame pattern formation and regime transition under microgravity, experiments and numerical modelling
14:40 – 15:30	Fernando Charro (University of Coimbra) The Monge-Ampére equation: Classical local applications and recent nonlocal developments
15:50 – 16:40	Yoshihiro Tonegawa (Tokyo Institute of Technology) A time-discrete approximate scheme for multi-phase mean curvature flow
17:30 – 19:30	Banquet

Topology of nodal sets of solutions to elliptic PDEs

Daniel Peralta-Salas

Instituto de Ciencias Matemáticas

In these lectures I will provide an introduction to the study of the topology of the nodal sets (i.e. the zero level sets) of solutions to second-order elliptic PDEs. In the first part, I will introduce a strategy based on two technical tools to address the analysis of these objects: Thom's isotopy theorem and a Runge-type global approximation theorem. This strategy was first presented in [1, 2] to study the topology of the vortex lines of solutions to the Euler equations in fluid mechanics. In the second part, these techniques will allow us to construct solutions to a wide range of elliptic PDEs with nodal sets of complicated (sometimes bizarre) topologies [3]. In particular, the model elliptic equation that I will consider to illustrate the power of these tools is the Helmholtz equation (monochromatic waves). In the last part of the lectures, I will apply these ideas in three seemingly unrelated contexts. First, a 2001 conjecture of Sir Michael Berry about the ex istence of Schrödinger operators in Euclidean space with eigenfunctions having nodal lines of arbitrary knot type [4]; second, the construction of bounded solutions to the Allen-Cahn equation with level sets of any compact topology [5]; third, the study of the nodal sets of high-energy eigenfunctions of the Laplacian on the flat torus [6].

References

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Introduction to linear and nonlinear tomography

Samuli Siltanen

University of Helsinki

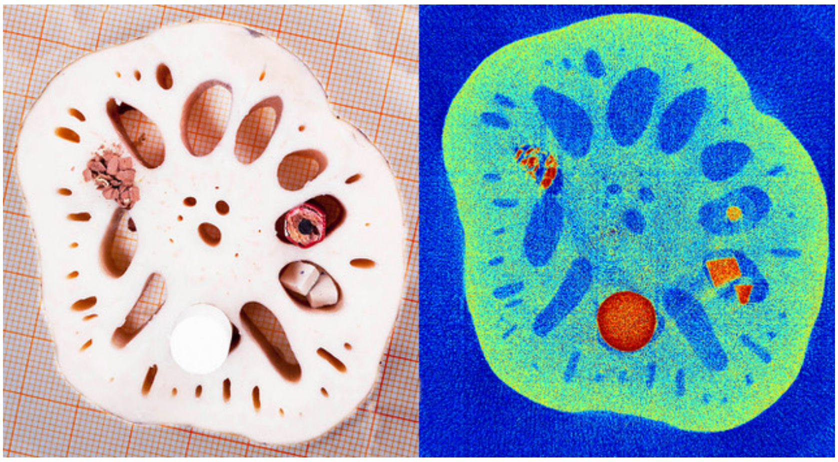

図 1: Example of X-ray Tomography Imaging. Left: phantom containing various chemical elements with different X-ray attenuation properties. Right: reconstruction by the FBP algorithm. See video https://www.youtube.com/watch?v=eWwD_EZuzBI.

X-ray tomography is an imaging method where an unknown physical body is photographed from many directions using X-rays. The X-rays passing through the object lose their intensity exponentially in proportion to the density of the material along the ray according to the Beer-Lambert law. After a calibration step one arrives at the following mathematical problem: can one recover a non-negative, compactly supported function from the knowledge of integrals of that function along lines? Johann Radon showed in his seminal 1917 article how to do that in dimension two, when all possible line integrals are known. Radon's geometric reconstruction formula serves as the foundation of today's Computerized Tomography (CT) scanners in hospitals in the form of the Filtered Back-Projection (FBP) algorithm. FBP is based on inverting the so-called Radon transform.

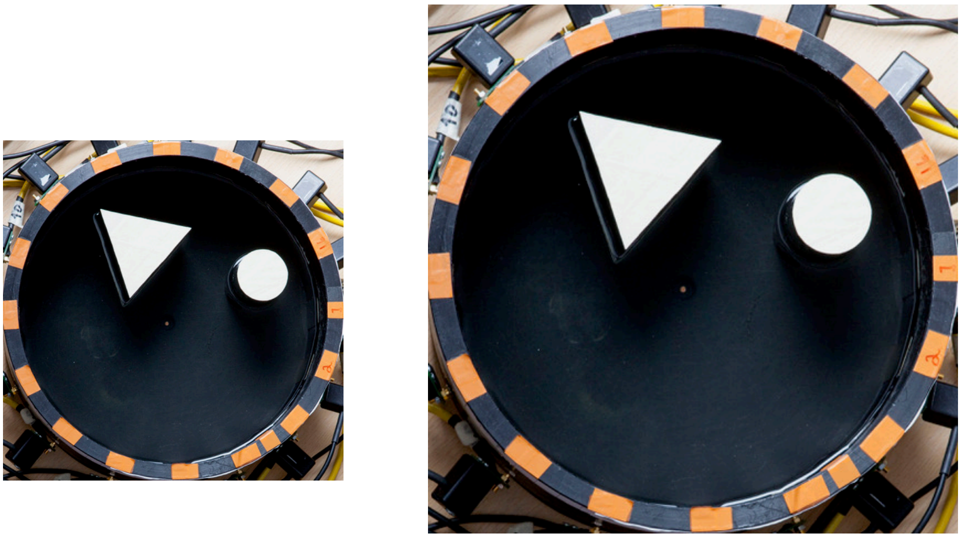

図 2: Example of nonlinear Electrical Impedance Tomography Imaging. Left: phantom with resistive plastic blocks placed in salt water. Right: reconstruction by the D-bar method. See video https://www.youtube.com/watch?v=65Zca_qd1Y8.

Recently, there is growing interest in X-ray tomography imaging based on limited data. The main reason for this is the need to limit the harmful radiation dose to the patient. Mathematically, the problem of recovering a function from an incomplete set of line integrals is an example of a linear ill-posed inverse problem. Ill-posedness means that the reconstruction problem is extremely sensitive to measurement noise and modelling errors. In such situations the FBP algorithm is not optimal. This course discusses variational regularisation methods for limited-data X-ray tomography, including classical Tikhonov regularisation and modern sparsity-promoting algorithms. The core idea behind these methods is complementing the insufficient measurement data by additional information about the unknown function.

The second part of the course focuses on a nonlinear imaging method called Electrical Impedance Tomography (EIT). In EIT one feeds harmless electric currents into a physical body through electrodes attached to the surface of the body. The resulting electrical potentials at the electrodes are measured, and the goal is to recover the internal electrical conductivity distribution from the current-to-voltage measurements. EIT has promising medical applications. Since different organs and tissues have different electric conductivities, an EIT image can show air flowing into lungs, blood being pumped from the heart to the lungs, and potentially distinguish between two types of stroke: bleeding in the brain and blood clot preventing blood flow in the brain.

Mathematically, EIT is modelled by the inverse conductivity problem of Calderon. The goal is to recover the nonnegative coefficient σ of the elliptic partial differential equation (PDE)

$$\nabla \cdot \sigma \nabla u = 0 \tag{0-1}$$

inside a simply connected domain Ω from a set of Cauchy data at the boundary $\partial\Omega$. Often the Cauchy data can be expressed in the form of a so-called Dirichlet-to-Neumann (DN) map Λ_σ. The operator Λ_σ can be thought of an infinite-precision voltage-to-current map. While the PDE (0-1) is linear, the map $\sigma \mapsto \Lambda_\sigma$ is nonlinear with no continuous inverse. The inverse problem of calculating σ from Λ_σ is extremely ill-posed.

The course discusses three types of robust solution methods for the inverse conductivity problem, each with specific strengths and weaknesses. The methods are variational regularisation, Bayesian inversion and the so-called D-bar method. The focus is in the D-bar method. It is based on a nonlinear Fourier transform, which offers a clear frequency-domain understanding of EIT imaging. Presented is also a very recent discovery that connects the D-bar method with the FBP algorithm using complex principal type operators in the sense of Duistermaat and Hörmander.

In the course, students will have an opportunity to experiment with X-ray tomography and EIT imaging using Matlab and the open datasets

`https://www.fips.fi/dataset.php`

and

`https://www.fips.fi/EIT_dataset.php`

The collection of data can be seen in these videos, for example:
https://www.youtube.com/watch?v=eWwD_EZuzBI
and
https://www.youtube.com/watch?v=65Zca_qd1Y8.

Most parts of the course are based on the book *Linear and Nonlinear Inverse Problems with Practical Applications* by Jennifer Mueller and Samuli Siltanen (SIAM 2012).

Some Mathematical Questions related to Geometry in Liquid Crystals

Jinhae Park

Chungnam National University

In the Landau-de Gennes theory, molecules in liquid crystals are described by a traceless symmetric 3×3 matrix Q and the free energy density takes the following form

$$f(Q, \nabla Q) = \frac{1}{2} L_1 Q_{\alpha\beta,\gamma} Q_{\alpha\beta,\gamma} + \frac{1}{2} L_2 Q_{\alpha\beta,\beta} Q_{\alpha\gamma,\gamma} + \frac{1}{2} L_3 Q_{\alpha\beta,\gamma} Q_{\alpha\gamma,\beta},$$
$$+ \frac{1}{2} A \text{tr} Q^2 - \frac{1}{3} B \text{tr} Q^3 + \frac{1}{4} C \left(\text{tr} Q^2 \right)^2 .$$

Minimizers of the Landau-de Gennes energy functional $\int_\Omega f(Q, \nabla Q)\, dx$ for Liquid crystals occupying a domain Ω exhibit many questions of great interest. In this talk, we discuss singuraities of different types which can be observed in minimizers. One of special properties of the Landau-de Gennes energy is that it can describe singularities of half integer degrees which cannot be explained by the classical Oseen-Frank energy. One can show that existence of finite many singular points of $\frac{1}{2}$ degree with topological boundary conditions. We also plan to talk about interface problems between different liquid crystal phases. If time permits, we discuss some mathematical questions which bear resemblance to de Gorge's conjectures.

An h-adaptive Mesh Method for Optimal Control Problem

Ruo Li

Peking University

In this talk, I will introduce some numerical methods for the model optimal control problem with an elliptic constraint equation and a distributed control variable. The optimal control problem is discretized using classical finite element method. We applied the h-adaptive mesh method based on an a posteriori error estimate. Different meshes can be used for control and state variables to achieve even better efficiency.

A Discontinuous Finite Element Space by Patch Reconstruction

Zhiyuan Sun

Peking University

In this talk, I will introduce the discontinuous Galerkin method base on patch reconstruction.The piecewise discontinuous finite element space can be constructed on very flexible meshes by patch reconstruction using least square problems which is adopted to numerically solve elliptic equations using interior penalty discontinuous Galerkin method. The canonical error estimates of the interpolation operator in the new space and the discontinuous Galerkin approximation to the elliptic equations are fully restored by direct numerical analysis. The numerical examples on different usual and unusual meshes demonstrate the numerical efficiency predicted by the error estimations, together with the flexibility of the new method.

keyword: Reconstruction, least square problem, discontinued Galerkin method, elliptic equation

MSC2010: 49N45; 65N21

Detection and range estimation of a hidden object using the time domain enclosure method

Masaru Ikehata

Hiroshima University

Two inverse obstacle problems using waves governed by the wave equations over a finite time interval are considered. The problems are concerned with detection and range estimation of an unknown obstacle embedded in a general rough background medium or placed behind a known impenetrable obstacle. It is shown that the time domain enclosure method enables us to know whether the obstacle exists or not by using a single wave over a finite time interval on an open ball where the wave is generated. Moreover, if the obstacle exists, the method yields also information about the Euclidean distance between the obstacle and the center of the open ball.

keyword: enclosure method, inverse obstacle problems, wave equation

AMS: 35R30

On direct and inverse problems involving cracks in elasticity

Hiromichi Itou

Tokyo University of Science

Crack problems in elasticity have been received great deal of attention in various fields of science and engineering. Cracks form geometrical discontinuities that are the critical sites at which materials fail due to stress concentration at cracks, when the body is subject to loading. One of mathematical difficulties is how to treat such kind of the singularity (cf. [4, 7]).

In this talk, we firstly consider inverse problems for a linear crack in a linearized elasticity [4, 5]. The problem is to extract information about the location and shape of an unknown crack from a single set of the surface displacement field and traction on the boundary of a homogeneous and anisotropic elastic plate. We explain an extraction procedure of an unknown crack by using the enclosure method introduced by Prof. Ikehata; provided that the crack is *linear*, one of two end points of the crack is *known* and located on the boundary of the body, a *well-controlled* surface traction is given on the boundary of the body. This problem can be applied to a nondestructive testing. Also we introduce how to extract information about the location of tips of several cracks located on a line between two electric conductive plates from measured data which are an injecting direct current and the resulted voltage on the accessible side of the plate [6]. The numerical and computational procedures are now on constructing by a collaborative project with Prof. Siltanen, Prof. Ikehata and Dr. Hauptmann.

Next, we deal with a direct problem involving a crack in nonlinear elasticity. Most of the approaches to study fracture problems for brittle elastic solids are based on the linearized elastic constitutive relation or the ad hoc assumptions. Within the context of linearized elasticity, stress concentration at the crack tips leads to the singularity for the strain which contradicts the basic concept within which the approximation is developed, namely that the displacement gradients are sufficiently small which means the infinitesimal strain. In order to resolve this inconsistency, Prof. Rajagopal [15, 16] propose the non-conventional framework of nonlinear elasticity with limiting small strain. The benefit of this model implies strains are bounded uniformly over the cracked body, even while the stresses are concentrated at the crack tip. In this model we analyze a nonlinear crack problem subject to the non-penetration condition [11, 12]. The principal difficulty in analyzing the model concerns the fact that the stresses live only in the non-reflexive L^1-space, therefore the standard existence theorems known for nonlinear elliptic problems (e.g. [3, 14]) are not applicable to this model and it must be properly regularized, see the relevant works by [1, 2] and references therein. Then we introduce the concept of a generalized solution, which is described by generalized variational inequalities and coincides with the weak solution in the smooth case. The well-posedness is proved by the construction of an approximation problem using elliptic regularization and penalization techniques. This research is based on a joint work with Dr. Kovtunenko and Prof. Rajagopal [8–10].

Lastly, we mention about the asymptotic behavior of the stress field around the crack tip. For strain-limiting models, it's an open problem (cf. [13, 17]).

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Near-limit Flame Pattern Formation and Regime Transition under Microgravity -Experiments and Numerical Modelling-

Kaoru Maruta

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Combustion limit including near-limit flame behavior has been studied for more than 150 years. In early period, primary motivation was the safety of coal mine. In 1940's, the first theoretical description on combustion limit was presented [1] and various experiments under microgravity have been conducted since then to date. Numerous combustion experiments have been contributing to the understandings of fundamental physics of near-limit flames. In late 90's, comprehensive and complex natures of the combustion limits for deflagration wave were clarified by low-speed counterflow flame experiments under microgravity [2] which realizes ideal condition for flame extinction experiments where no natural convection induced by gravity. Apart from study on the limit of deflagration wave, another phenomenon termed "flame ball" was first predicted by Zel'dovich in 1940's and its existence was proved through microgravity experiments in drop towers in U.S. [3] and Japan and eventually, space experiments in the Space Shuttle at the beginning of the 2000's. Our final goal is to construct comprehensive combustion limit theory which comprehensively covers the limits of both deflagration wave and flame ball. Currently, preliminary microgravity experiments using airplane are conducted for preparing Space experiments from 2019 and some transitions between deflagration wave and ball-like flame near the limits were successfully observed [4]. Flame ball has essentially pattern formation nature. To interpret observed phenomena, 3D mathematical modelling and analysis have been conducted and some results will be presented.

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Combustion limit: 燃焼限界，Counterflow flame: 対向流火炎，Deflagration wave: 伝播火炎（通常の火炎のこと），Flame ball: 火炎球（特殊燃焼現象，伝播をしない球形の定常火炎），Drop tower: 落下塔（微小重力実験のための施設）.

The Monge-Ampére equation: Classical local applications and recent nonlocal developments

Fernando Charro

University of Coimbra

In this talk we will present the classical local Monge-Ampere equation and some of its applications to optimal transport and differential geometry. We will discuss the degeneracy of the equation and the challenges it poses for regularity of solutions. Finally, we will consider a nonlocal analogue of the Monge-Ampere operator, recently introduced in a joint work with Luis Caffarelli.

A time-discrete approximate scheme for multi-phase mean curvature flow

Yoshihiro Tonegawa

Tokyo Institute of Technology

A family of n-dimensional surfaces $\{\Gamma(t)\}_{t\geq 0}$ in $\mathbb{R}^{n+1}$ is called the mean curvature flow (abbreviated by MCF) if the velocity is equal to its mean curvature at each point and time. Given a smooth surface Γ_0, one can find a smoothly moving MCF starting from Γ_0 until some singularities such as vanishing or pinching occur. The presence of singularities necessitates a weak formulation of MCF, and there have been intensive research dealing with this aspect in the last few decades. Among such attempts, Brakke started a theory of MCF ("Brakke flow") inclusive of singularities in the framework of geometric measure theory in his seminal book [1]. In particular, he developed a general existence theory of MCF: starting from any integral varifold (with a minor technical restriction) of any codimension, he showed that there exists a family of varifolds satisfying the motion law of MCF in a weak sense and existing for all time. One major concern for the validity of his existence theorem is that the proof does not guarantee the non-triviality of the solution when Γ_0 is not a smooth surface. In [3], we rectify this point of non-triviality by introducing a few framework and also modifying Brakke's original argument. The main existence theorem of [3] may be stated roughly as follows.

Theorem 0.1. *Suppose that $\Gamma_0 \subset \mathbb{R}^{n+1}$ is a closed countably n-rectifiable set whose complement $\mathbb{R}^{n+1} \setminus \Gamma_0$ equals $\cup_{i=1}^N E_{0,i}$, where $E_{0,1}, \ldots, E_{0,N} \subset \mathbb{R}^{n+1}$ are mutually disjoint non-empty open sets and $N \geq 2$. Assume that the n-dimensional Hausdorff measure of Γ_0 is finite or grows at most exponentially near infinity. Then, for each $i = 1, \ldots, N$, there exists a family of open sets $\{E_i(t)\}_{t\geq 0}$ with $E_i(0) = E_{0,i}$ such that $E_1(t), \ldots, E_N(t)$ are mutually disjoint for each $t \geq 0$ and $\Gamma(t) := \cup_{i=1}^N \partial E_i(t)$ coincides with the space-time support of a nontrivial Brakke flow starting from Γ_0. Each $E_i(t)$ moves continuously in time with respect to the Lebesgue measure.*

We may regard each $E_i(t) \subset \mathbb{R}^{n+1}$ as a region of "i-th phase" at time t, and $\Gamma(t)$ as the "phase boundaries" which move by the mean curvature in a generalized sense. Some of $E_i(t)$ shrink and vanish, and some may grow and may even occupy the whole $\mathbb{R}^{n+1}$ in finite time. Note that the continuity of $E_i(t)$ guarantees that $\Gamma(t) \neq \emptyset$ at least for a short initial time interval, and $\Gamma(t) \neq \emptyset$ unless $E_i(t) = \mathbb{R}^{n+1}$ for some i.

Notion not stated clearly in Theorem 1 is that of Brakke flow, which is as follows. For simplicity, assume Γ_0 has a finite n-dimensional Hausdorff measure, $\mathcal{H}^n(\Gamma_0) < \infty$.

Definition 0.2. A Brakke flow starting from Γ_0 is a family of n-varifolds $\{V_t\}_{t\geq 0}$ satisfying the following.

(1) $V_0 = |\Gamma_0|$ =unit density varifold induced from Γ_0.

(2) For $\mathcal{L}^1$ a.e. $t \in \mathbb{R}$, V_t is an integral varifold with L^2 generalized mean curvature vector $h(\cdot, V_t)$.

(3) $\|V_t\|(\mathbb{R}^n)$ is decreasing in t and $\int_0^\infty \int_{\mathbb{R}^{n+1}} |h(\cdot, V_t)|^2 \, d\|V_t\| dt \leq \mathcal{H}^n(\Gamma_0)$.

(4) For any $0 \leq t_1 < t_2 < \infty$ and $\phi \in C_c^1(\mathbb{R}^{n+1} \times \mathbb{R}^+; \mathbb{R}^+)$, we have

$$\|V_t\|(\phi(\cdot, t))\Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \int_{\mathbb{R}^{n+1}} \{\nabla\phi(\cdot, t) + \phi(\cdot, t) h(\cdot, V_t)\} \cdot h(\cdot, V_t) + \frac{\partial \phi}{\partial t}(\cdot, t) \, d\|V_t\| dt.$$

Here, $\|V\|$ is the weight measure of V.

The property (4) is a weak form of the motion law of MCF. The property (2) allows a possibility of having a higher (≥ 2) multiplicity representing a "folding" of surfaces (whether it happens or not is not clear). When the multiplicity stays 1 for a.e. $t > 0$, we say that the flow is a unit density flow. Almost everywhere regularity of unit density flow for general Brakke flow has been studied originally by Brakke and is recently completed by [2, 4, 5]. If the flow is a limit of smooth MCF, White's reguralitry theory [6] gives also the almost everywhere regularity. Theorem 1 includes as a part of theorem the existence of $\{V_t\}_{t \geq 0}$ satisfying the Definition 1. We may then define a Radon measure μ on $\mathbb{R}^{n+1} \times \mathbb{R}^+$ by $d\mu = d\|V_t\| dt$. The claim of Theorem 1 is that $\{x \in \mathbb{R}^{n+1} : (x, t) \in \operatorname{spt} \mu\} = \Gamma(t)$ for all $t > 0$.

The proof is divided roughly into two stages, one is a construction of time-discrete approximate flows, and the other is the proof of a suitable compactness theorem of varifolds suited for our purpose. In each time step of the construction, there are two different kinds of motions, one is a locally area-minimizing Lipschitz deformation and the other is a motion by smoothed mean curvature vector. There are a number of estimates measuring the errors of approximations. For the second stage, we prove an analogue of Allard's compactness theorem of integral varifolds. Here the difference is that we only have a control of smoothed mean curvature vectors of converging integral varifolds, not the exact mean curvature vectors. To supplement this point, we have a local area minimizing property in a small length scale. There are roughly three different length scales, grid size for time (very small), smoothing of mean curvature vectors (small) and area-minimizing (not so small). These differing length scales play an important role throughout the analysis.

In my talk, I will mostly concentrate on how one actually constructs the time-discrete approximate MCF. This is a joint work with Lami Kim of Tokyo Institute of Technology.

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