Homological mirror symmetry via families of Lagrangians

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Mirror symmetry

Three facets of mirror symmetry:

1. Enumerative: GW invariants via period computations.
2. Homological: Fukaya categories and coherent sheaves.
3. T-duality: SYZ duality of special Lagrangian fibrations.

We can think of (1) and (2) as “computations” of invariants of symplectic manifolds in terms of an algebraic object.

Principle

A (singular) Lagrangian torus fibration on a sympletic manifold determines a variety which computes all symplectic invariants.

Questions

1. What is not covered by this?
2. How to explicitly compute?

Lagrangian torus fibrations

The theory of integrable systems says that a proper submersion $\pi\colon X \to Q$ with Lagrangian fibres is a torus fibration.
A choice of identification of the fibre with $\mathbb{R}^n/\mathbb{Z}^n$ gives rise to a coordinate chart on the base. The transition functions have derivatives in $GL(n, \mathbb{Z})$, so Q is naturally an *integral affine manifold*, which determines a variety Y.

Examples

1. X is a 2-torus of area A. Q is a circle of length A. Y is the quotient of the algebraic torus by $z \to T^A z$.
2. X is the Thurston manifold ($\mathbb{R}^4/\Gamma$ with Γ nilpotent). Q is a torus whose affine structure has monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in one direction. Y is a Kodaira surface (elliptic fibration with no section).

Mirror Theorem

Theorem (A. arXiv:1703.07898)

If $X \to Q$ admits a Lagrangian section, then the Fukaya category (over the Novikov field) of X embeds fully faithfully in the derived category of coherent sheaves on Y (equivalence if Y algebraic).

If we consider points in Y as *rank*-1 *unitary local systems* on the fibres of $X \to Q$, then the functor assigns to a Lagrangian an object of the derived category with fibres the vector spaces $HF^*(L, X_q)$.
In order to prove that these groups are indeed the fibres of a coherent sheaf, we have to produce a complex of modules over the ring of functions on an open cover, together with compatible equivalences over double intersections. Fukaya constructed the local complexes of modules via a trick that's difficult to generalise.
We implement this local-to-global approach in a slightly different way.

Mirror symmetry locally

Covering Q by polygons gives a cover X by symplectic domains in T^*T^n and of Y by analytic domains of algebraic torus, so the local question is about tori. The new point of view is to introduce objects on the symplectic side, given by the local system corresponding to the regular representation. Since the fibres are tori, the corresponding Floer homology group is

$$\mathbf{k}[\pi_1 T^n] \cong \mathbf{k}[z_1^{\pm}, \ldots, z_n^{\pm}].$$

This works well to recover mirror symmetry between T^*T^n and $(\mathbf{k}^*)^n$, but we have to complete both sides to get the desired rings.

Singular and Immersed Lagrangians

The most interesting examples for mirror symmetry have Lagrangian fibres which are too singular to incorporate in the Fukaya category. The main purpose of this talk is to explain that we now have a clear way of going around this, using only immersed Lagrangians.

Analogy to resolutions of singularities

"Resolve" singular Lagrangians by nearby immersed ones.

Immersed Lagrangian Floer homology goes back to Akaho, and Akaho-Joyce. The main point is that the space of branes supported on such a Lagrangian is given by a subvariety of

$$(\mathbb{C}^*)^n \times \mathbb{C}^{m+1}$$

where n is the rank of H^1, and $m+1$ is the number of self-intersections of degree 1, and the defining equations count holomorphic discs.

Gross-Siebert reconstruction

Recall that integral polytopes in $\mathbb{R}^n$ give rise to toric varieties. Gross and Siebert observed that the boundary Q of a polytope is equipped is a natural *singular* affine structure, i.e. that the natural affine structure on the facets can be extended to all of Q. If the polytope corresponds to a smooth toric variety, they prove that the associated family of Calabi-Yau hypersurfaces can be reconstructed from this structure. Their proof has two ingredients:

1. Local models associated to singularities
2. Gluing procedure (scattering diagrams)

In fact, their approach is much more general, and the reconstruction can be done starting with an abstract polyhedral complex with certain normal information.

Warning

The reconstruction procedure is not completely well-defined on a codimension 2 locus. There will be no indeterminacy on the symplectic side.

Mirror local models

The local models appearing in Gross-Siebert's work arise from the hypersurfaces $X_{m,n} \subset \mathbb{C}^{m+1} \times (\mathbb{C}^*)^n$ given by the equations:

$$\prod_{i=0}^{m} x_i = 1 + \sum_{j=1}^{n} y_j$$

Theorem (in preparation)

The (wrapped) Fukaya category of $X_{m,n}$ is equivalent to the derived category of coherent sheaves of $X_{n,m}$.

Low dimensional examples

We recover mirror symmetry for $(\mathbb{C}^*)^n$ in the most trivial case. The next case is mirror symmetry between the complement of $\prod_{j=0}^{n} x_j = 1$ in $\mathbb{C}^{n+1}$ and the conic bundle over $(\mathbb{C}^*)^n$ with discriminant locus the hypersurface $H_{n-1} = \{\sum_{j=1}^{n} y_j = 1\}$ (c.f. A-Auroux-Katzarkov).

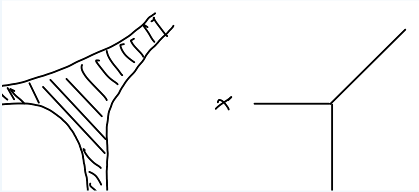
The SYZ fibration

The natural SYZ fibration $X_{m,n} \to \mathbb{R}^{n+m}$ is given by

$$(|x_0|^2 - |x_1|^2, \ldots, |x_{m-1}|^2 - |x_m|^2, |y_1|, \ldots, |y_n|).$$

The discriminant locus of this fibration is the product of the image (ameoba) of H_{n-1} in $\mathbb{R}^n$ with an codimension 1 polyhedral complex in $\mathbb{R}^m$. SYZ mirror symmetry amounts to the "duality" between these two factors.

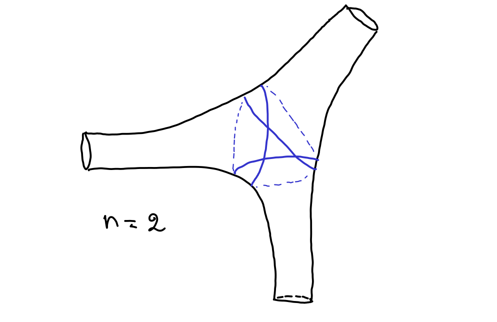
For $(n, m) = (2, 2)$, we get



As alluded to earlier, it is too difficult to study Floer theory for these fibres.

A Landau-Ginzburg model on $X_{m,n}$

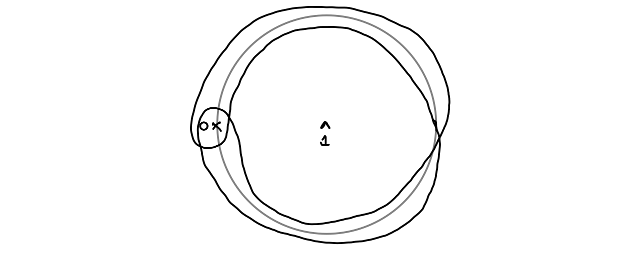
Consider the projection map $\pi \colon X_{m,n} \to \mathbb{C}$ given by $\prod x_i$. The fibre away from 0 and 1 is the product $(\mathbb{C}^*)^m \times H_{n-1}$. Sheridan used an immersed Lagrangian sphere in H_{n-1} to prove mirror symmetry for Calabi-Yau hypersurfaces in $\mathbb{CP}^n$.



Taking the product with a torus in $(\mathbb{C}^*)^m$, we obtain a Lagrangian in the fibre. The parallel transport with respect to curves in the base gives Lagrangians in the total space.

Immersed Lagrangians and singular fibres

The naive choice of circles centered around 1 gives a very singular Lagrangian when the radius is 1. Consider a nearby immersed circle that misses the singularity, and let $L_{m,n}$ denote the parallel transport:



Theorem (in preparation)

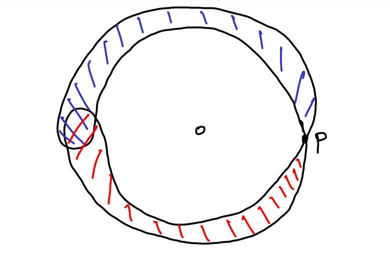
The space of torus objects supported on $L_{m,n}$ is $X_{n,m}$.

This is the immersed implementation of the SYZ conjecture that the mirror is the space of branes supported on fibres.

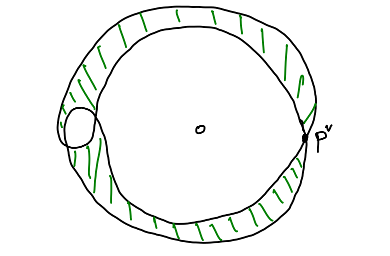
The first (non)-trivial case

Consider $X_{0,1} = \mathbb{C}^*$. The Lagrangian $L_{0,1}$ is an immersed figure 8. There are two degree 1 intersections, so the space of bounding cochains is a subvariety of $\mathbb{C}^2_{x_0,x_1} \times \mathbb{C}^*_z$. We can compute the curvature m_0 and the differential m_1 of the Floer complex:

$$m_0 = (1 - z)(1 - x_0 x_1)p$$



$$m_1 p^\vee = 1 - x_0 x_1$$



All isomorphism classes of torus objects arise by considering $z = 1$. So the space of *unobstructed non-zero* branes supported on $L_{0,1}$ is given by the equation $x_0 x_1 = 1$ in $\mathbb{C}^2$. This is $X_{1,0}$.

From group rings to based loops

Going back to the first part of the talk, HMS without corrections is proved by constructing an object of the Fukaya category with endomorphism algebra the group ring. This is the ring of functions on the space of local systems.

For a general Lagrangians, the group ring has bad formal properties: it is not smooth in the sense of non-commutative geometry. However, the homology of the based loop space $H_* \Omega L$ is always smooth. For a torus, this makes no difference because the universal cover is contractible.

To put the Family Floer approach to HMS in its proper context, we therefore need:

1. A model of $H_* \Omega L$ for embedded Lagrangians in which we can incorporate holomorphic curve corrections.
2. A notion of $H_* \Omega L$ for immersed Lagrangians.

Why do we care?

The usual Floer cohomology of a Lagrangian L produces a *curved A_∞ algebra* on H^*L, with curvature in H^2L, and a set of *bounding cochains* in H^1L which correspond to cancelling the curvature by a change of coordinates.
It is essentially impossible to say anything useful in Floer theory if one cannot find a non-trivial element of the space of bounding cochains.
The point of view I'm advocating is that there is a *curved A_∞ algebra* on $H_{-*}\Omega L$, which controls all branes supported on L, and which may be non-trivial even if there are no "finite" branes supported on L.

Corollary

If $\pi_2 L = 0$, then L supports an unobstructed non-zero object.

Proof.

Curvature always vanishes for a non-positively graded ring, and the degree -1 part vanishes by assumption. □

Floer groups for pairs

Given a Lagrangian L, and a point $x \in L$, denote by $\Omega_x L$ the space of paths from the basepoint to x. We obtain a (derived) local system of chain complexes. Complete to obtain a local system corresponding to a free rank-1 module over $\hat{C}\Omega L$.

For a pair of Lagrangians L_0 and L_1 (which intersect transversely)

$$CF^*(\Omega L_0, \Omega L_1) \equiv \bigoplus_{x \in L_0 \cap L_1} \mathrm{Hom}^c_{\mathbf{k}}(\hat{C}_* \Omega_x L_0, \hat{C}_* \Omega_x L_1)).$$

More generally, we can consider local systems associated to modules over $\hat{C}\Omega L$.

Conjecture

There is an enlargement of the Fukaya category with these objects and morphism spaces.

Computation in the exact case

It is easy to construct this enlargement in the exact case by keeping track of higher dimensional families of holomorphic discs and their boundaries.

Lemma

There is a natural isomorphism

$$C_*(\Omega L) \cong C^*(L, \mathrm{Hom}_{\mathbf{k}}(C_*\Omega_x L, C_*\Omega_x L)).$$

The proof amounts to the statement that L is the classifying space of ΩL, so that a cellular decomposition of L gives rise to a projective resolution of the diagonal bimodule.
In this way, we obtain an embedding of the category of (derived) local systems on L, in this enlargement of the Fukaya category, whenever L is exact.

Pairs of Lagrangians

It is easier to think of a pair of Lagrangians Q_0 and Q_1 meeting transversely at a point x. If X is a symplectic manifold which retracts to $Q_0 \cup Q_1$ (e.g. a neighbourhood), our goal is to express the Floer theory of X in terms of Q_0 and Q_1.

The starting point is the previous loop space construction:

$$\begin{tikzcd} C^*\left(Q_0, \mathrm{Hom}(C_*(\Omega Q_0), C_*(\Omega Q_0))\right) \arrow[d, bend left, "Hom(C_*(\Omega_x Q_0), C_*(\Omega_x Q_1))"] \\ C^*\left(Q_1, \mathrm{Hom}(C_*(\Omega Q_1), C_*(\Omega Q_1))\right) \arrow[u, bend left, "\mathrm{Hom}(C_*(\Omega_x Q_1), C_*(\Omega_x Q_0))"] \end{tikzcd}$$

We have two distinguished modules given by $C_*(\Omega Q_i)$ on one side and 0 on the other. Let $C_*(\Omega Q)$ denote the corresponding algebra.

Theorem (A)

The algebra $C_*(\Omega Q)$ is a completion of the Floer cochains of Lagrangians "dual" to Q_i.