Space-time reflection anomalies in (2+1)D topological quantum field theories

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Goal:

• Understand (2+1)D TQFT with global symmetry

• Motivation from Condensed Matter Physics: understand how to characterize distinct gapped quantum many-body phases of matter

“Symmetry-enriched topological phases of matter” (SET states)

(e.g. fractional quantum Hall states, quantum spin liquids)
Based on:

(2+1)D TQFT is characterized by

- Unitary Modular Tensor Category, \( \mathcal{C} \)

Topologically distinct classes of quasiparticles (anyons) \( \leftrightarrow \) isomorphism classes of simple objects

Describes **braiding and fusion** of topologically non-trivial quasiparticles

- Chiral central charge, \( c \) (\( c \ mod \ 8 \) determined by \( \mathcal{C} \))
Unitary Modular Tensor Category

- Quasiparticles types (simple objects) \{a, b, c, ...\}
- Fusion Rules \( a \times b = \sum_c N^c_{ab} \)
- Fusion/Splitting spaces:

- F-Symbols
- Braiding (R-Symbols)
Consistency Conditions:

Pentagon Equation

Hexagon Equation

Gauge Transformations:

\[ |a, b; c\rangle \rightarrow \Gamma^a_b \Gamma^b_c |a, b; c\rangle \]

\[ F \rightarrow \Gamma \Gamma F \Gamma^{-1} \Gamma^{-1} \]

\[ R \rightarrow \Gamma R \Gamma^{-1} \]
Modular Tensor Category and Topological states

The consistent data \( \{ N_{ab}^c, F_{d}^{abc}, R_{c}^{ab} \} \) provides skeletonization of a UMTC.

Gauge-invariant quantities = Physical Topological invariants

\[
\begin{align*}
\infty_a & = \theta_a \\
\circlearrowleft_d a & = a \\
S_{ab} & = \frac{1}{D} \circlearrowleft_a b
\end{align*}
\]
Now consider (2+1)D TQFT with global symmetry group $G$

Independently of $G$, each UMTC $\mathcal{C}$ has its own group of intrinsic symmetries: $\text{Aut}(\mathcal{C})$
\[ \varphi : C \rightarrow C \]

\[ \varphi(a) = a' \quad \varphi(|a, b; c\rangle) = u_{c'}^{a'b'} |a', b'; c'\rangle \]

\[ N_{a'b'}^c = N_{ab}^c \]

\[ d_{a'} = d_a \]

If \( \varphi \) is space-time parity preserving:

\[ \theta_{a'} = \theta_a \]

\[ S_{a'b'} = S_{ab} \]

\[ \varphi(R_{c}^{ab}) \simeq R_{c'}^{a'b'} \]

\[ \varphi(F_{d}^{abc}) \simeq F_{d'}^{a'b'c'} \]

If \( \varphi \) is space-time parity reversing:

\[ \theta_{a'} = \theta_a^* \]

\[ S_{a'b'} = S_{ab}^* \]

\[ \varphi(R_{c}^{ab}) \simeq (R_{c'}^{a'b'})^* \]

\[ \varphi(F_{d}^{abc}) \simeq (F_{d'}^{a'b'c'})^* \]
Natural Isomorphism: \[ \gamma(a) = a \]
\[ \gamma(|a, b; c\rangle) = \frac{\gamma_a \gamma_b}{\gamma_c} |a, b; c\rangle \]

Equivalence classes \([\varphi]\) (up to natural isomorphism) form a group:

\[ \text{Aut}(C) \]

= group of intrinsic (emergent) symmetries of the TQFT

Note: space-time parity preserving elements of \( \text{Aut}(C) \) referred to as braided auto-equivalences
Global symmetry $G$

$$[\rho] : G \to \text{Aut}(C)$$

$g_a \equiv \rho_g(a)$

$$\rho_g : V_{ab}^c \to V_{gagbg}^c$$

$$\rho_g(|a, b; c\rangle) = U_g(g_a, g_b; g_c)|g_a, g_b; g_c\rangle$$

$$\rho_g(R) = U_g RU_g^\dagger = R$$
$$\rho_g(F) = U_g U_g F U_g^\dagger U_g^\dagger = F$$

Guarantees that all closed anyon diagrams are $G$-symmetric
If $g$ is space-time parity reversing:

$$\rho_g(F) = U_g U_g F U_g^\dagger U_g^\dagger = F^*$$

$$\rho_g(R) = U_g R U_g^\dagger = R^*$$
Natural Isomorphism

\[ \rho_{gh} = \kappa_{g,h} \rho_g \rho_h \]

\[ \rho \] defines an element \[ [\Omega] \in H^3_\rho(G, \mathcal{A}) \]

Etingof, Nikshych, Ostrik 2010

MB, Bonderson, Cheng, Wang 2014

\[ \mathcal{A} = \{ a \in C | d_a = 1 \} \]

Abelian anyons (1 form symmetry group)

In general, computing \[ [\Omega] \] requires full knowledge of F, R symbols

\[ [\Omega] \] is an obstruction to symmetry localization

It is an obstruction to the theory possessing a global symmetry group G and an action \[ [\rho] \]
To proceed, we need to consider combining the (2+1)D TQFT states with additional local degrees of freedom.

In the absence of any global symmetry, the local degrees of freedom can be completely ignored.

In the presence of G, the local degrees of freedom can have non-trivial interplay with the TQFT and cannot be ignored.

Case in point: In general,

$$\rho_{gh} = \kappa_{g,h} \rho_g \rho_h$$

$$\kappa_{g,h} \neq 1$$
Symmetry Localization

Ground state is symmetric: \( R_g |\Psi_0\rangle = |\Psi_0\rangle \)

Consider state with two quasiparticles:
\[
R_g |\Psi_{a,\bar{a};0}\rangle = U_g^{(1)} U_g^{(2)} \rho_g |\Psi_{a,\bar{a};0}\rangle \\
= U_g^{(1)} U_g^{(2)} U_g (g_a, g_{\bar{a}}; 0) |\Psi_{g_a, g_{\bar{a}};0}\rangle
\]

This is only consistent if \([\mathcal{O}] \in H^3_\rho(G, A)\) is trivial

MB, Bonderson, Cheng, Wang 2014
Symmetry Fractionalization

Quasiparticles can carry projective representations

\[ U_g^{(j)} U_h^{(j)} \neq U_{gh}^{(j)} \]

Even if \( R_g R_h = R_{gh} \)

General Result: Symmetry Fractionalization

Classified by \( H^2_{\rho}(G, \mathcal{A}) \) \( \mathcal{A} \subseteq \mathcal{C} \) Abelian anyons

\( H^2 \) torsor

MB, Bonderson, Cheng, Wang 2014

c.f. Etingof, Nikshych, Ostrik 2010
(2+1)D TQFTs with global symmetry $G$ are partially classified by

$$[\rho] : G \rightarrow \text{Aut}(\mathcal{C})$$

How symmetries permute quasiparticles

If $[\mathcal{O}] \in H^3_\rho(G, \mathcal{A})$ is non-trivial $\rightarrow$ symmetry localization obstruction

Symmetry fractionalization not well-defined

$$[t] \in H^2_{[\rho]}(G, \mathcal{A})$$

Symmetry fractionalization classification $H^2$ torsor

Characterizing the symmetry fractionalization class itself requires extra data
If \([\mathcal{O}] \in H_3^\rho(G, \mathcal{A})\) is trivial, and we pick a symmetry fractionalization class, then the symmetry fractionalization class can be anomalous. **Symmetry fractionalization anomaly (’t Hooft anomaly)**

Senthil-Vishwanath 2013

For unitary, space-time parity preserving symmetries \(G\), ’t Hooft anomalies are classified by

\[ \mathcal{H}^4(G, U(1)) \]

Dijkgraaf-Witten 1990; Chen, Gu, Liu, Wen 2011
Etingof, Nikshych, Ostrik 2010;
Cui, Galindo, Plavnik, Wang 2015

(2+1)D theory must exist at the surface of a (3+1)D invertible TQFT (i.e. an SPT or short-range-entangled state)

How to compute ‘t Hooft anomaly?
If $G$ contains only unitary space-time parity preserving symmetries:

Study properties of symmetry defects associated with $G$

$\rightarrow$ G-crossed braided tensor category

G-crossed braided tensor categories $\leftrightarrow$ TQFTs with $G$ symmetry

Provides explicit formulae and consistency conditions to compute all anomalies and completely characterize (2+1)D TQFT with unitary space-time parity preserving $G$


How to treat space-time reflection symmetries?
In the following, I will focus on space-time reflection symmetries

Develop an understanding of how to

1. Characterize symmetry fractionalization
2. Compute ‘t Hooft anomalies
3. Understand symmetry localization $H^3$ obstruction
For time-reversal $T$, with $T^2 = 1$ 

If $a = Ta$ define $\eta_a^T = \pm 1$

Determines whether $a$ carries local Kramers degeneracy 
i.e. local two-dimensional vector space where $T^2 = -1$ locally

\[
\{\eta_a^T\} \text{ must satisfy various consistency conditions. For example:}
\]

1. $\eta_a^T \eta_b^T = \eta_c^T$ \hspace{1cm} for $N_{ab}^c = 1$

2. $\eta_c^T = \theta_c$ \hspace{1cm} when $Tc = c$ and $N_{aT}^c = 1$
Characterizing symmetry fractionalization for space-time reflection symmetries

For spatial reflection $\mathbf{r}$, with $\mathbf{r}^2 = 1$, referred to as $\mathbb{Z}_2^\mathbf{r}$

If $\mathbf{a} = \mathbf{r}\mathbf{\bar{a}}$ define $\eta_{\mathbf{a}}^\mathbf{r} = \pm 1$ = eigenvalue of reflection

$$R_{\mathbf{r}}|\Psi\rangle = \pm|\Psi\rangle$$

Reflection eigenvalue = topological invariant
Note: In Euclidean field theory, space and time are on equal footing.

I will mainly work with the Euclidean field theory and use $r$

Results for anti-unitary time-reversal, $T$, (i.e. in Lorentzian signature) can be obtained by replacing $r$ with $CT$
\{ \eta^r_a \} \text{ determines Euclidean path integral on non-orientable space-times}

\[ \mathcal{Z}(\Sigma_g \times S^1) = \sum_x S^{2-2g}_{0x} \quad \mathcal{Z}(S^3) = 1/D \]

\[ M_a \equiv \mathcal{Z}_a(\mathbb{RP}^2 \times S^1) = \sum_{r x = \bar{x}} S_{ax} \eta^r_x \]

MB, Bonderson, Cheng, Jian, Walker 2016
Anomaly detection: Dehn twist on punctured $\mathbb{RP}^2$ (mobius band)

If $M_a > 0$, pick a state $|\Psi\rangle \in \mathcal{H}(\mathbb{RP}^2; a)$

$$M_a \equiv \mathcal{Z}_a(\mathbb{RP}^2 \times S^1) = \dim \mathcal{H}(\mathbb{RP}^2; a)$$

$$D_\alpha|\Psi\rangle = \theta_a|\Psi\rangle$$
Dehn twist is isotopic to the identity

Consistency requires:

\[ M_a > 0 \quad \Rightarrow \quad \theta_a = 1 \]

If this is not satisfied \( \rightarrow \) symmetry fractionalization is anomalous!

\( \text{'}t \text{ Hooft anomaly} \)
More systematic ‘t Hooft anomaly calculation

Invertible TQFTs in (3+1)D with $\mathbb{Z}_2^r$ or $\mathbb{Z}_2^T$ symmetry have $\mathbb{Z}_2 \times \mathbb{Z}_2$ classification  
Kapustin 2014

Braided fusion categories $\mathcal{B}$ determine (3+1)D TQFTs 

If $\mathcal{B}$ is modular, the associated (3+1)D TQFT is invertible and the surface (2+1)D theory is described by $\mathcal{B}$

Compute $\mathcal{Z}(\mathbb{R}P^4) = \pm 1 \quad \mathcal{Z}(\mathbb{C}P^2) = \pm 1$

Need to extend previous theories to incorporate action of reflection
(3+1)D TQFTs from (2+1)D TQFTs (i.e. UMTCs)

(3+1)D TQFT: assigns complex number $\mathcal{Z}(W^4)$ to every closed $W^4$

For closed $M^3$: space of boundary conditions $C(M^3)$

For $W^4$ with boundary, $\mathcal{Z}(W^4) : C(\partial W^4) \to \mathbb{C}$

$C(M^3)$ = set of all possible anyon diagrams in $M^3$

$$C(M^3; c^{(2)}) = \{ c \in C(M^3) | c|_{\partial M^3} = c^{(2)} \}$$
(3+1)D TQFTs from (2+1)D TQFTs (i.e. UMTCs)

Assign vector space $\mathcal{V}(M^3)$ to closed $M^3$

$$\mathcal{V}(M^3) = \mathbb{C}[\mathcal{C}(M^3)]/\sim$$

For $M^3$ with boundary:

$$\mathcal{V}(M^3; c^{(2)}) = \mathbb{C}[\mathcal{C}(M^3; c^{(2)})]/\sim$$

Path integrals evaluated using gluing formula:

$$Z(W^4)[c] = \sum_{e_{\alpha}} \frac{Z(W^4_{\text{cut}})(c_{\text{cut}} \cup e_{\alpha} \cup \overline{e_{\alpha}})}{\langle e_{\alpha} | e_{\alpha} \rangle} \mathcal{V}(M^3; c^{(2)}_{\text{cut}})$$
Every manifold has a handle decomposition

\[ \text{d-dimensional p-handle} = D^p \times D^{d-p} \]

\[ \text{glue along } S^{p-1} \times D^{d-p} \]

Mobius band = 0 handle \( \cup \) 1 handle

\[ CP^2 = 0 \text{ handle } \cup 2 \text{ handle } \cup 4, \text{ handle} \]

\[ RP^4 = 0 \text{ handle } \cup 1 \text{ handle } \cup 2 \text{ handle } \cup 3 \text{ handle } \cup 4 \text{ handle} \]
Result of the computation

MB, Bonderson, Cheng, Jian, Walker 2016

\[ Z(\mathbb{CP}^2) = \frac{1}{D} \sum_a d_a^2 \theta_a = e^{2\pi i c / 8} \]

\[ Z(\mathbb{RP}^4) = \frac{1}{D} \sum_{a \mid a = r\bar{a}} d_a \theta_a \eta_a^r \]

Non-trivial identity:
\[ \frac{\sum_a \theta_a M_a^2}{\sum_a M_a^2} = Z(\mathbb{RP}^4) Z(\mathbb{CP}^2) \]

See C. Wang, M. Levin 2016 for related conjectures
See also Tachikawa, Yonekura 2016 for spin theories
The previous formulae (and additional arguments) imply additional consistency conditions:

\[ \mathcal{Z}(\mathbb{RP}^4) = \frac{1}{D} \sum_{a \mid a = r\bar{a}} d_a \theta_a \eta_a^r = \pm 1 \]

1. \[ \sum_a \theta_a M_a^2 \]
   \[ \sum_a M_a^2 \]
   \[ \pm 1 \]
   which implies

   \[ \theta_a = \pm 1 \] is independent of \( a \) if \( M_a > 0 \)

3. \( M_a \) is a non-negative integer for all \( a \)

If any of these conditions cannot be satisfied, the (2+1)D theory cannot exist at the surface of any (3+1)D invertible TQFT.

\[ \mathcal{H}_{\rho}^3(\mathbb{Z}_2, \mathcal{A}) \] obstruction is non-vanishing
Example: $\text{Sp}(4)_2$ Chern-Simons theory

This is a time-reversal invariant TQFT

<table>
<thead>
<tr>
<th>Anyon label, $a$</th>
<th>1</th>
<th>$\varepsilon$</th>
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$\varepsilon \times \varepsilon = 1$, $\varepsilon \times \phi_i = \phi_i$, $\varepsilon \times \psi_+ = \psi_-$

$\phi_i \times \phi_i = 1 + \varepsilon + \phi_{\text{min}}(2i, 5-2i)$

$\phi_1 \times \phi_2 = \phi_1 + \phi_2$

$\psi_+ \times \psi_+ = 1 + \phi_1 + \phi_2$

All correlation functions are time-reversal invariant

But:

$$\mathcal{Z}(\mathbb{R}P^4) = \frac{1}{\sqrt{5}} (1 + \eta^{T}_\varepsilon) \neq \pm 1$$

What happened?
Example: $\text{Sp}(4)_2$ Chern-Simons theory

\[ \mathcal{A} = \mathbb{Z}_2 \quad \mathcal{H}^3(\mathbb{Z}_2^T, \mathbb{Z}_2) = \mathbb{Z}_2 \]

By direct computation,

\[ [\Phi] \in \mathcal{H}^3(\mathbb{Z}_2^T, \mathbb{Z}_2) \] is non-trivial

Previous conditions are violated:

1. $\mathcal{Z}(\mathbb{R}P^4) = \frac{1}{\sqrt{5}}(1 + \eta^T_\varepsilon) \neq \pm 1$

2. $M_a = S_{a1} + S_{ae}\eta^T_\varepsilon$ can only be integer if $\eta^T_\varepsilon = -1$

3. But then $M_{\psi_1} = M_{\psi_2} > 0 \quad \theta_{\psi_1} = \theta^*_{\psi_2} = i \neq \pm 1$
Looking at $\text{Sp}(4)_2$ CS theory more closely

Start with $\text{SU}(5)_1$ CS theory

5 particle types $[j]$, $j = 0, .., 4 \pmod{5}$

Possesses $\mathbb{Z}_4^T$ symmetry:

$T : [j] \to [2j] \quad T^2 = C \quad C^2 = 1$

$C : [j] \leftrightarrow [-j]$

Gauge $T^2 = C$

Consider adding Dijkgraaf-Witten term

$\mathcal{H}^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$

With DW term: Result is $\text{Sp}(4)_2$ CS theory

Without DW term: Result is a new obstruction-free theory $\text{Sp}(4)_2^V$
Sp(4)$_2$ CS theory

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$\varepsilon$ is the $T^2$ gauge charge. Thus, local $T^2$ value is -1:

$$\eta_\varepsilon^T = -1$$

However, we also require $\eta_\varepsilon^T = \theta_\varepsilon = +1$

because $N^\varepsilon_{\psi_+, T} = 1$
Sp(4)$_2^V$ theory

Now, we no longer require $\eta^T_\epsilon = \theta_\epsilon = +1$

because $N_{\psi_+}^\epsilon, T\psi_+ = 0$

There are no conflicting constraints on $\eta^T_\epsilon$

$\eta^T_\epsilon = -1$
The difference between the obstruction-free theory $\text{Sp}(4)_2^\vee$ and the obstructed $\text{Sp}(4)_2$ CS theory was the DW term.

Recall that $\mathbb{Z}_2$ gauge theory with a DW term is equivalent to the “doubled semion” (DS) theory: $U(1)_2 \times U(1)_2$ CS theory

DS theory contains the particles $\{1, s, s', b = s \times s'\}$

$$\text{Sp}(4)_2 = (\text{Sp}(4)_2^\vee \times \text{DS})/(1 \sim b\epsilon)$$

Since $\eta_{\epsilon}^T = -1$ we must have $\eta_{b}^T = -1$

But $\eta_{b}^T = -1$ is inconsistent with the requirement

$$\eta_{b}^T = \theta_{b} = 1$$

which holds because $N_{s}^{b}T_{s} = 1$

This is the heart of the problem.
Thus a general way to obtain a theory with a $\mathbb{Z}_2^T \otimes H^3$ obstruction is to start with a theory with $\mathbb{Z}_4^T$ symmetry, and gauge $T^2$ while adding a DW term for the $T^2$ gauge field.

All examples we have can be obtained this way.

(e.g. also $SO(4)_4$ CS theory, and infinite family of other examples)
1. Enlarge the symmetry from $\mathbb{Z}_2^T$ to $\mathbb{Z}_4^T$
   Can show explicitly that $\mathbb{Z}_4^T$ has no obstruction

2. View it as a spin TQFT with $T^2 = (-1)^{N_F}$

3. View the theory not as a true (2+1)D theory, but “pseudo-realized” at the surface of a (3+1)D non-invertible TQFT. A new type of anomaly inflow

4. The true symmetry of this theory is a 2-group symmetry. The 0-form symmetry $\mathbb{Z}_2^T$ and the 1-form symmetry $\mathcal{A}$ are not independent of each other, but intrinsically connected.
Pseudo-realization and “anomaly inflow”

The heart of the problem was the (non)-existence of a doubled semion theory with $\mathbb{Z}_2^r$ symmetry and $\eta^r_b = -1$

But we can pseudo-realize it at the surface of a non-trivial (3+1)D system

$e$ and $m$ particles on mirror plane carry half $\mathbb{Z}_2^r$ charge

Can gap out all edge modes such that reflection eigenvalue of Wilson string for $b$ becomes $-1$

$b \times e$ is condensed at the junction, so $b$ can leak into the bulk as $e$
Layer construction and bulk $Z_2$ gauge theory

Stack layers of (2+1)D $Z_2$ gauge theories
Condense pairs of e particles from neighboring planes
\[ \rightarrow \text{bulk (3+1)D } Z_2 \text{ gauge theory} \]
Condense b x e at the surface
\[ \rightarrow \text{b can leak into bulk as } Z_2 \text{ gauge charge} \]
\[ \rightarrow \text{s and s’ become bound to endpoints of magnetic flux lines} \]
This means that $\text{Sp}(4)_2$ CS theory can also exist with a global $\mathbb{Z}_2^r$ symmetry, as long as it is pseudo-realized at the surface of a (3+1)D system that contains a dynamical $\mathbb{Z}_2$ gauge theory.

Similar phenomenon found in a discrete gauge theory with gauge group $D_{16}$ and with global $\mathbb{Z}_2$ symmetry (internal, unitary) by Fidkowski and Vishwanath (2015).

There, the phenomenon could be related to symmetry fractionalization of strings (flux loops) in the (3+1)D bulk. This interpretation is not available in the case of space-time reflection symmetry.
Infinite sequence of theories with $H^3$ obstructions

Any theory with a $H^3_{\rho}(\mathbb{Z}^T_2, A)$ obstruction is compatible with $\mathbb{Z}^T_4$ symmetry. Thus we can again gauge $T^2$ while adding a DW term for the $T^2$ gauge field. This gives a new theory with $H^3_{\rho}(\mathbb{Z}^T_2, A)$ obstruction.

This process can be repeated indefinitely.

\[
\begin{align*}
SU(5)_1 & \xrightarrow{\text{Gauge } T^2} Sp(4)_2 & \xrightarrow{\text{Gauge } T^2} & \cdots \\
SU(3)_1 \times SU(3)_1 & \xrightarrow{\cdots} SO(4)_4 & \xrightarrow{\cdots} & \cdots \\
U(1) \times U(1) & \xrightarrow{\cdots} K = \begin{pmatrix} m & n \\ n & -m \end{pmatrix} & \xrightarrow{\cdots} & \cdots
\end{align*}
\]
Summary

• Symmetry fractionalization characterized by \( \{ \eta_a^T \} \) and \( \{ \eta_a^r \} \)

Required to determine path integral on non-orientable spacetimes

• Explicit formulae for ‘t Hooft anomalies for global \( \mathbb{Z}_2^T \) symmetries

• Simple sufficient conditions for diagnosing existence of symmetry localization obstruction for global \( \mathbb{Z}_2^T \)

• General method to produce theories with \( \mathbb{Z}_2^T \) \( H^3 \) obstructions

• Various resolutions of \( H^3 \) obstructions.

An unusual type of anomaly inflow where bulk is a dynamical \( \mathbb{Z}_2 \) gauge theory
(1+1)D time-reversal / reflection SPTs

\[ G = \mathbb{Z}_2 \quad \text{Time-reversal, } T, \text{ or reflection, } r \quad \mathcal{H}^2_*(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2 \]

Topological path integral

\[ \mathcal{Z}(\mathbb{RP}^2) = \begin{cases} 
1 & \text{for a trivial SPT state} \\
-1 & \text{for a nontrivial SPT state} 
\end{cases} \]
(1+1)D time-reversal / reflection SPTs

Time-reversal SPT:

Local Kramers degeneracy at edge
(1+1)D time-reversal / reflection SPTs

Reflection SPT:

\[ R_r \Psi = \pm \Psi \]

Reflection eigenvalue = topological invariant
Symmetry fractionalization in (2+1)D

If \( a = T a \) define \( \eta^T_a = \pm 1 \) Determines whether \( a \) carries local Kramers degeneracy

If \( a = r \bar{a} \) define \( \eta^r_a = \pm 1 \) = eigenvalue of reflection