Space-time reflection anomalies in (2+1)D topological quantum field theories

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Goal:

- Understand (2+1)D TQFT with global symmetry

- Motivation from Condensed Matter Physics: understand how to characterize distinct gapped quantum many-body phases of matter

  "Symmetry-enriched topological phases of matter" (SET states)

  (e.g. fractional quantum Hall states, quantum spin liquids)

Based on:

- M.B., M. Cheng, arXiv:1706.09464
- M.B., P. Bonderson, M. Cheng, C.-M. Jian, K. Walker, arXiv:1612.07792
- M.B., P. Bonderson, M. Cheng, Z. Wang, arXiv:1410.4540

(2+1)D TQFT is characterized by

Moore-Seiberg, 1989
Witten, 1989
Wen, Read, Turaev, Kitaev,
Walker, Wang,…

- Unitary Modular Tensor Category, $\mathcal{C}$

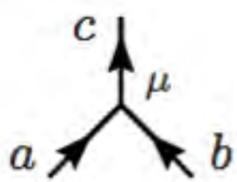
  Topologically distinct classes of quasiparticles (anyons) $\leftarrow\rightarrow$ isomorphism classes of simple objects

  Describes braiding and fusion of topologically non-trivial quasiparticles

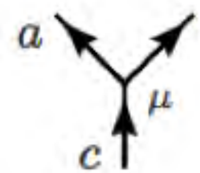
- Chiral central charge, c (c mod 8 determined by $\mathcal{C}$)

Unitary Modular Tensor Category

- Quasiparticles types (simple objects) {a, b, c, …}
- Fusion Rules $a \times b = \sum_c N_{ab}^c c$
- Fusion/Splitting spaces:

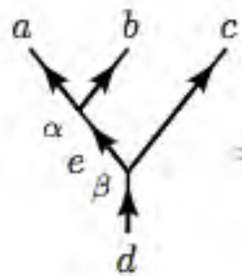


$\propto \langle a, b; c, \mu | \in V_{ab}^c$

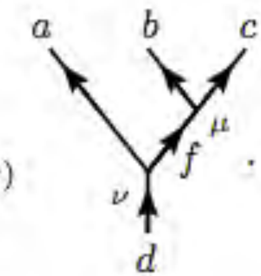


$\propto | a, b; c, \mu \rangle \in V_c^{ab}$

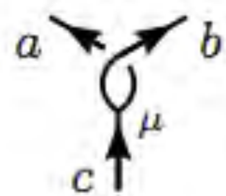
- F-Symbols



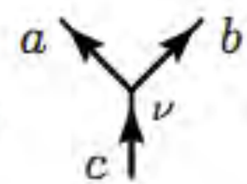
$= \sum_{f,\mu,\nu} \left[F_d^{abc}\right]_{(e,\alpha,\beta)(f,\mu,\nu)}$



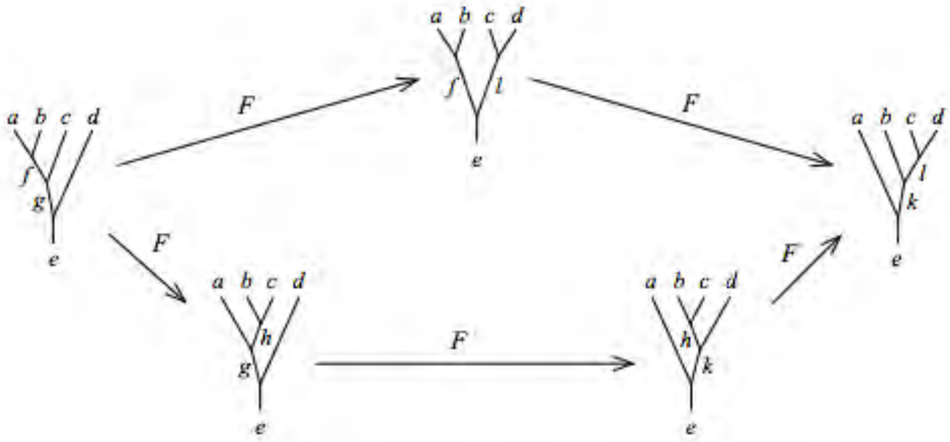
- Braiding (R-Symbols)



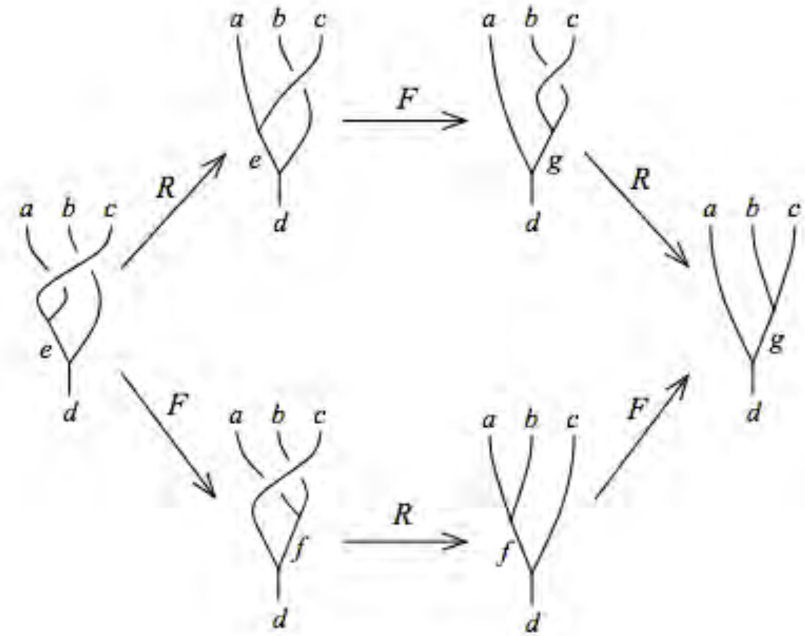
$= \sum_{\nu} \left[R_c^{ab}\right]_{\mu\nu}$



Consistency Conditions:



Pentagon Equation



Hexagon Equation

Gauge Transformations:

$$|a, b; c\rangle \to \Gamma^{ab}_{c} |a, b; c\rangle$$

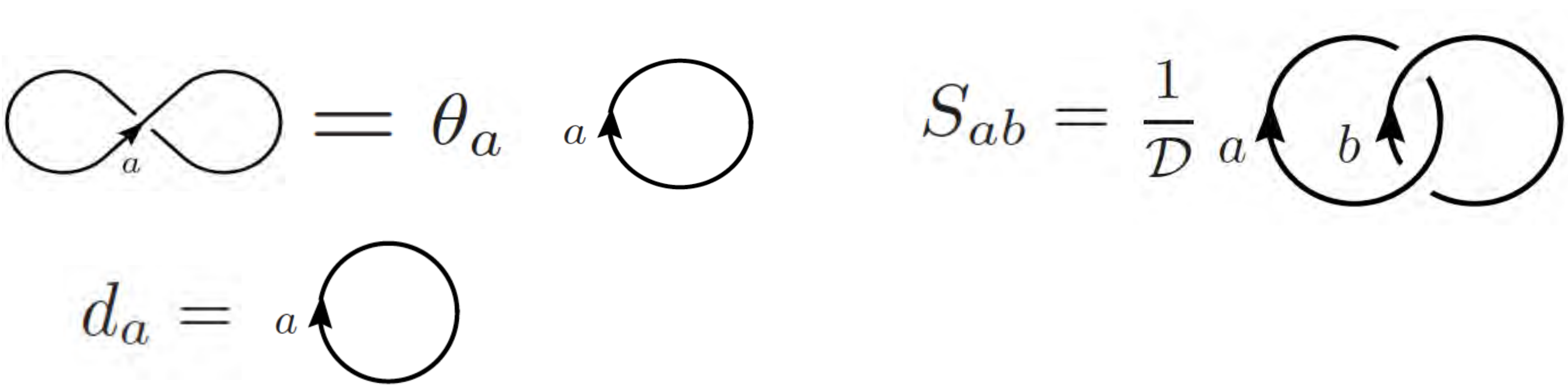
$$F \to \Gamma\Gamma F \Gamma^{-1}\Gamma^{-1}$$

$$R \to \Gamma R \Gamma^{-1}$$

Modular Tensor Category and Topological states

The consistent data $\{N_{ab}^{c}, F_d^{abc}, R_c^{ab}\}$ provides skeletonization of a UMTC.

Gauge-invariant quantities = Physical Topological invariants



Now consider (2+1)D TQFT with global symmetry group G

Independently of G, each UMTC $\mathcal{C}$ has its own group of intrinsic symmetries: $\mathrm{Aut}(\mathcal{C})$

$$\varphi : \mathcal{C} \to \mathcal{C}$$

$$\varphi(a) = a' \qquad \varphi(|a, b; c\rangle) = u_{c'}^{a'b'} |a', b'; c'\rangle$$

$$N_{a'b'}^{c'} = N_{ab}^{c}$$

$$d_{a'} = d_a$$

If φ is space-time parity preserving:

$$\theta_{a'} = \theta_a \qquad \varphi(R_c^{ab}) \simeq R_{c'}^{a'b'}$$

$$S_{a'b'} = S_{ab} \qquad \varphi(F_d^{abc}) \simeq F_{d'}^{a'b'c'}$$

If φ is space-time parity reversing:

$$\theta_{a'} = \theta_a^* \qquad \varphi(R_c^{ab}) \simeq (R_{c'}^{a'b'})^*$$

$$S_{a'b'} = S_{ab}^* \qquad \varphi(F_d^{abc}) \simeq (F_{d'}^{a'b'c'})^*$$

Natural Isomorphism: $\Upsilon(a) = a$

$$\Upsilon(|a, b; c\rangle) = \frac{\gamma_a \gamma_b}{\gamma_c} |a, b; c\rangle$$

Equivalence classes $[\varphi]$ (up to natural isomorphism) form a group :

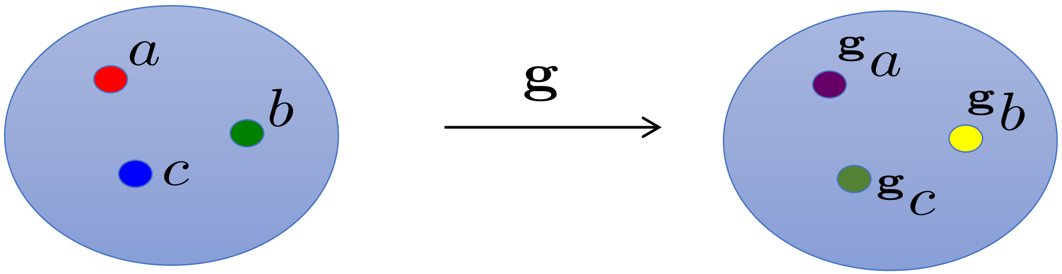
$$\mathrm{Aut}(\mathcal{C})$$

= group of intrinsic (emergent) symmetries of the TQFT

Note: space-time parity preserving elements of $\mathrm{Aut}(\mathcal{C})$ referred to as braided auto-equivalences

Global symmetry G

$$[\rho] : G \to \mathrm{Aut}(\mathcal{C})$$



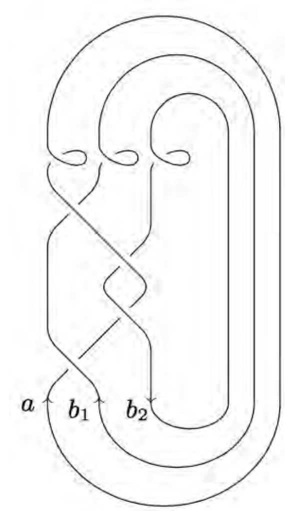
$$^{\mathbf{g}}a \equiv \rho_{\mathbf{g}}(a)$$

$$\rho_{\mathbf{g}} : V_{ab}^{c} \to V_{^{\mathbf{g}}a\,^{\mathbf{g}}b}^{^{\mathbf{g}}c}$$

$$\rho_{\mathbf{g}}(|a,b;c\rangle) = U_{\mathbf{g}}(^{\mathbf{g}}a,^{\mathbf{g}}b;^{\mathbf{g}}c)|^{\mathbf{g}}a,^{\mathbf{g}}b;^{\mathbf{g}}c\rangle$$

$$\rho_{\mathbf{g}}(R) = U_{\mathbf{g}} R U_{\mathbf{g}}^{\dagger} = R \qquad \rho_{\mathbf{g}}(F) = U_{\mathbf{g}} U_{\mathbf{g}} F U_{\mathbf{g}}^{\dagger} U_{\mathbf{g}}^{\dagger} = F$$

Guarantees that all closed anyon diagrams are G-symmetric



If **g** is space-time parity reversing:

$$\rho_{\mathbf{g}}(F) = U_g U_g F U_g^\dagger U_g^\dagger = F^*$$

$$\rho_{\mathbf{g}}(R) = U_{\mathbf{g}} R U_{\mathbf{g}}^\dagger = R^*$$

Natural Isomorphism

$$\rho_{\mathbf{gh}} = \kappa_{\mathbf{g},\mathbf{h}}\rho_{\mathbf{g}}\rho_{\mathbf{h}}$$

$[\rho]$ defines an element $[\mathfrak{O}] \in H^3_\rho(G, \mathcal{A})$

Etingof, Nikshych, Ostrik 2010

MB, Bonderson, Cheng, Wang 2014

$\mathcal{A} = \{a \in \mathcal{C} | d_a = 1\}$ Abelian anyons (1 form symmetry group)

In general, computing $[\mathfrak{O}]$ requires full knowledge of F, R symbols

$[\mathfrak{O}]$ is an obstruction to symmetry localization

It is an obstruction to the theory possessing a global symmetry group G and an action $[\rho]$

To proceed, we need to consider combining the (2+1)D TQFT states with additional local degrees of freedom.

In the absence of any global symmetry, the local degrees of freedom can be completely ignored.

In the presence of G, the local degrees of freedom can have non-trivial interplay with the TQFT and cannot be ignored.

Case in point: In general,

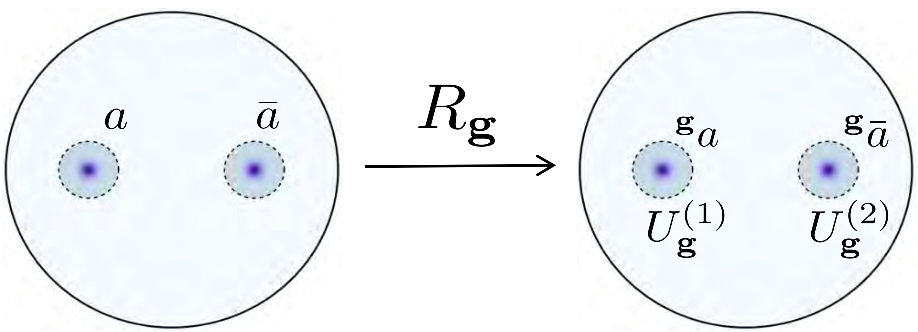
$$\rho_{\mathbf{gh}} = \kappa_{\mathbf{g},\mathbf{h}} \rho_{\mathbf{g}} \rho_{\mathbf{h}} \qquad\qquad \kappa_{\mathbf{g},\mathbf{h}} \neq 1$$

Symmetry Localization

Ground state is symmetric: $$R_{\mathbf{g}}|\Psi_0\rangle = |\Psi_0\rangle$$

Consider state with two quasiparticles:

$$\begin{aligned} R_{\mathbf{g}}|\Psi_{a,\bar{a};0}\rangle &= U_{\mathbf{g}}^{(1)}U_{\mathbf{g}}^{(2)}\ \rho_{\mathbf{g}}|\Psi_{a,\bar{a};0}\rangle \\ &= U_{\mathbf{g}}^{(1)}U_{\mathbf{g}}^{(2)}U_{\mathbf{g}}({}^{\mathbf{g}}a,{}^{\mathbf{g}}\bar{a};0)|\Psi_{{}^{\mathbf{g}}a,{}^{\mathbf{g}}\bar{a};0}\rangle \end{aligned}$$



This is only consistent if $[\mathfrak{O}] \in H^3_\rho(G,\mathcal{A})$ is trivial

MB, Bonderson, Cheng, Wang 2014

Symmetry Fractionalization

Quasiparticles can carry projective representations

$$U^{(j)}_{\mathbf{g}} U^{(j)}_{\mathbf{h}} \neq U^{(j)}_{\mathbf{gh}}$$

Even if $R_{\mathbf{g}} R_{\mathbf{h}} = R_{\mathbf{gh}}$

General Result: Symmetry Fractionalization

Classified by $H^2_\rho(G, \mathcal{A})$

$\mathcal{A} \subseteq \mathcal{C}$

Abelian anyons

H^2 torsor

MB, Bonderson, Cheng, Wang 2014

c.f. Etingof, Nikshych, Ostrik 2010

(2+1)D TQFTs with global symmetry G are partially classified by

$[\rho] : G \to \mathrm{Aut}(\mathcal{C})$ How symmetries permute quasiparticles

If $[\mathfrak{O}] \in H^3_\rho(G, \mathcal{A})$ is non-trivial → symmetry localization obstruction

Symmetry fractionalization not well-defined

$[\mathfrak{t}] \in H^2_{[\rho]}(G, \mathcal{A})$ Symmetry fractionalization classification

H^2 torsor

Characterizing the symmetry fractionalization class itself requires extra data

If $[\mathfrak{O}] \in H^3_\rho(G, \mathcal{A})$ is trivial, and we pick a symmetry fractionalization class, then the symmetry fractionalization class can be anomalous. Symmetry fractionalization anomaly ('t Hooft anomaly)

Senthil-Vishwanath 2013

For unitary, space-time parity preserving symmetries G, 't Hooft anomalies are classified by

$$\mathcal{H}^4(G, U(1))$$

Dijkgraaf-Witten 1990; Chen, Gu, Liu, Wen 2011
Etingof, Nikshych, Ostrik 2010;
Cui, Galindo, Plavnik, Wang 2015

(2+1)D theory must exist at the surface of a (3+1)D invertible TQFT (i.e. an SPT or short-range-entangled state)

How to compute 't Hooft anomaly?

If G contains only unitary space-time parity preserving symmetries:

Study properties of symmetry defects associated with G

→ G-crossed braided tensor category

G-crossed braided tensor categories ⟷ TQFTs with G symmetry

Provides explicit formulae and consistency conditions to compute all anomalies and completely characterize (2+1)D TQFT with unitary space-time parity preserving G

Etingof, Nikshych, Ostrik 2010; MB, Bonderson, Cheng, Wang 2014;
Cui, Galindo, Plavnik, Wang 2015; Chen, Burnell, Vishwanath, Fidkowski 2015

How to treat space-time reflection symmetries?

In the following, I will focus on space-time reflection symmetries

Develop an understanding of how to

1. Characterize symmetry fractionalization
2. Compute ‘t Hooft anomalies
3. Understand symmetry localization H^3 obstruction

Characterizing symmetry fractionalization for space-time reflection symmetries

For time-reversal **T**, with $\mathbf{T}^2 = 1$ Referrred to as $\mathbb{Z}_2^{\mathbf{T}}$

If $a = {}^{\mathbf{T}}a$ define $\eta_a^{\mathbf{T}} = \pm 1$

Determines whether a carries local Kramers degeneracy
i.e. local two-dimensional vector space where $\mathbf{T}^2 = -1$ locally

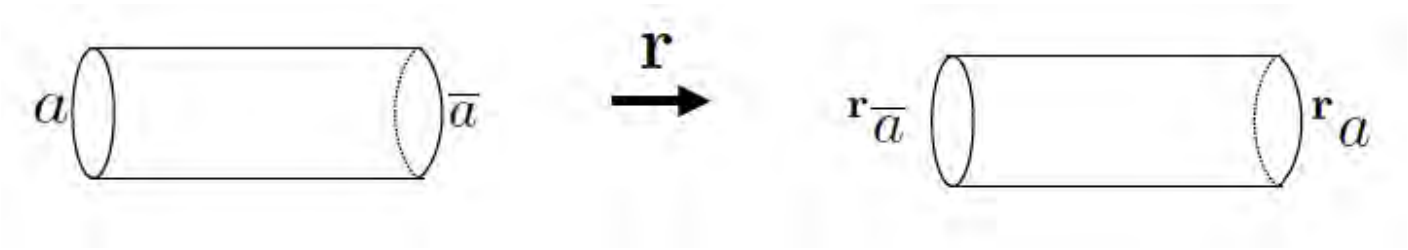
$\{\eta_a^{\mathbf{T}}\}$ must satisfy various consistency conditions. For example:

1. $\eta_a^{\mathbf{T}} \eta_b^{\mathbf{T}} = \eta_c^{\mathbf{T}}$ for $N_{ab}^c = 1$
2. $\eta_c^{\mathbf{T}} = \theta_c$ when ${}^{\mathbf{T}}c = c$ and $N_{a\,{}^{\mathbf{T}}a}^c = 1$

Characterizing symmetry fractionalization for space-time reflection symmetries

For spatial reflection **r** , with $\mathbf{r}^2 = 1$ referred to as $\mathbb{Z}_2^{\mathbf{r}}$

If $a = {}^{\mathbf{r}}\overline{a}$ define $\eta_a^{\mathbf{r}} = \pm 1$ = eigenvalue of reflection



$$R_{\mathbf{r}}|\Psi\rangle = \pm|\Psi\rangle$$

Reflection eigenvalue = topological invariant

Note: In Euclidean field theory, space and time are on equal footing.

I will mainly work with the Euclidean field theory and use **r**

Results for anti-unitary time-reversal, **T,** (i.e. in Lorentzian signature) can be obtained by replacing **r** with **CT**

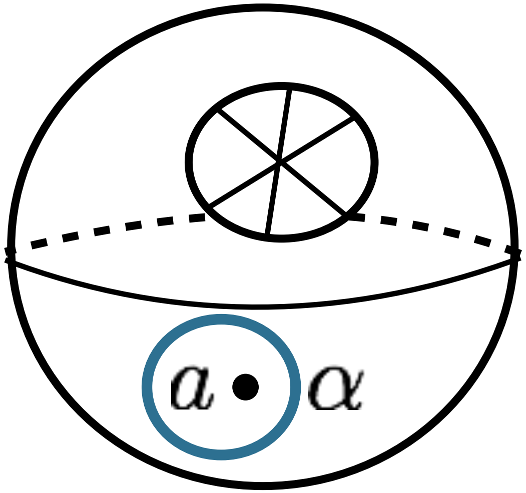
$\{\eta_a^{\mathbf{r}}\}$ determines Euclidean path integral on non-orientable space-times

$$\mathcal{Z}(\Sigma_g \times S^1) = \sum_x S_{0x}^{2-2g} \qquad \mathcal{Z}(S^3) = 1/\mathcal{D}$$

$$M_a \equiv \mathcal{Z}_a(\mathbb{RP}^2 \times S^1) = \sum_{\mathbf{r}\; x=\bar{x}} S_{ax}\eta_x^{\mathbf{r}}$$

MB, Bonderson, Cheng, Jian, Walker 2016

Anomaly detection: Dehn twist on punctured RP^2 (mobius band)

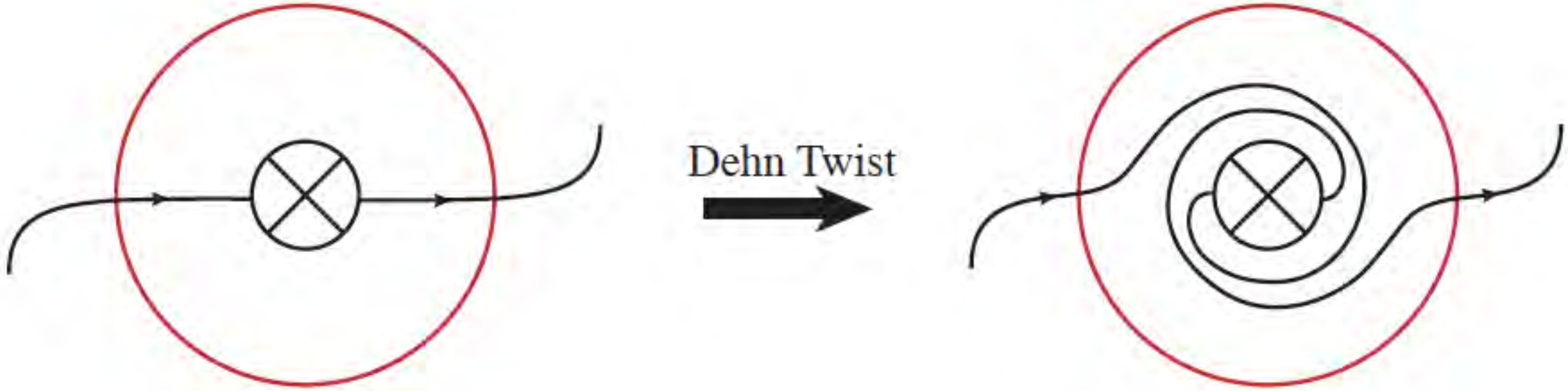


$$M_a \equiv \mathcal{Z}_a(\mathbb{RP}^2 \times S^1)$$

$$= \dim \mathcal{H}(\mathbb{RP}^2; a)$$

If $M_a > 0$, pick a state $|\Psi\rangle \in \mathcal{H}(\mathbb{RP}^2; a)$

$$D_\alpha |\Psi\rangle = \theta_a |\Psi\rangle$$



Dehn twist is isotopic to the identity

Consistency requires:

$$M_a > 0 \quad \Rightarrow \quad \theta_a = 1$$

If this is not satisfied —> symmetry fractionalization is anomalous!

't Hooft anomaly

More systematic 't Hooft anomaly calculation

Invertible TQFTs in (3+1)D with $\mathbb{Z}_2^{\mathbf{r}}$ or $\mathbb{Z}_2^{\mathbf{T}}$ symmetry have $\mathbb{Z}_2 \times \mathbb{Z}_2$ classification   Kapustin 2014

Braided fusion categories $\mathcal{B}$ determine (3+1)D TQFTs

Crane-Yetter 1993, Walker-Wang 2012

If $\mathcal{B}$ is modular, the associated (3+1)D TQFT is invertible and the surface (2+1)D theory is described by $\mathcal{B}$

Compute $\mathcal{Z}(\mathbb{RP}^4) = \pm 1$ $\mathcal{Z}(\mathbb{CP}^2) = \pm 1$

Need to extend previous theories to incorporate action of reflection

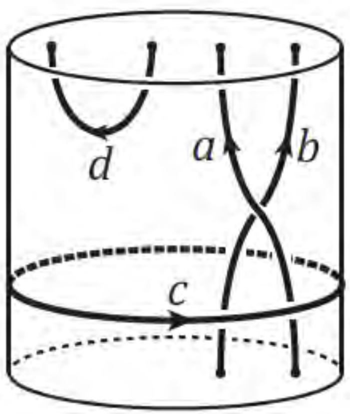
(3+1)D TQFTs from (2+1)D TQFTs (i.e. UMTCs)

(3+1)D TQFT: assigns complex number $\mathcal{Z}(W^4)$ to every closed W^4

For closed M^3: space of boundary conditions $\mathcal{C}(M^3)$

For W^4 with boundary, $\mathcal{Z}(W^4) : \mathcal{C}(\partial W^4) \to \mathbb{C}$

$\mathcal{C}(M^3)$ = set of all possible anyon diagrams in M^3



$$\mathcal{C}(M^3; c^{(2)}) = \left\{ c \in \mathcal{C}(M^3) \middle| \; c|_{\partial M^3} = c^{(2)} \right\}$$

(3+1)D TQFTs from (2+1)D TQFTs (i.e. UMTCs)

Assign vector space $\mathcal{V}(M^3)$ to closed M^3

$$\mathcal{V}(M^3) = \mathbb{C}[\mathcal{C}(M^3)]/\sim$$

For M^3 with boundary:

$$\mathcal{V}(M^3; c^{(2)}) = \mathbb{C}[\mathcal{C}(M^3; c^{(2)})]/\sim$$

Path integrals evaluated using gluing formula:

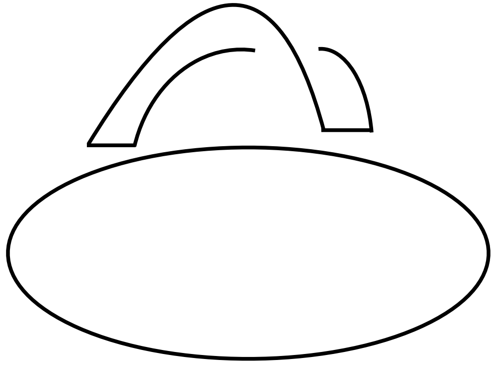
$$\mathcal{Z}(W^4)[c] = \sum_{e_\alpha} \frac{Z(W^4_{\text{cut}})(c_{\text{cut}} \cup e_\alpha \cup \overline{e}_\alpha)}{\langle e_\alpha | e_\alpha \rangle_{\mathcal{V}(M^3; c^{(2)}_{\text{cut}})}}$$

Every manifold has a handle decomposition

$$\text{d-dimensional p-handle} = D^p \times D^{d-p}$$

$$\text{glue along } S^{p-1} \times D^{d-p}$$

$$\text{Mobius band} = 0\,\text{handle} \cup 1\,\text{handle}$$



$$CP^2 = 0\,\text{handle} \cup 2\,\text{handle} \cup 4,\,\text{handle}$$

$$RP^4 = 0\,\text{handle} \cup 1\,\text{handle} \cup 2\,\text{handle} \cup 3\,\text{handle} \cup 4\,\text{handle}$$

Result of the computation

MB, Bonderson, Cheng, Jian, Walker 2016

$$\mathcal{Z}(\mathbb{CP}^2) = \frac{1}{\mathcal{D}} \sum_a d_a^2 \theta_a = e^{2\pi i c/8}$$

$$\mathcal{Z}(\mathbb{RP}^4) = \frac{1}{\mathcal{D}} \sum_{a | a = {}^{\mathbf{r}}\overline{a}} d_a \theta_a \eta_a^{\mathbf{r}}$$

Non-trivial identity:

$$\frac{\sum_a \theta_a M_a^2}{\sum_a M_a^2} = \mathcal{Z}(\mathbb{RP}^4)\mathcal{Z}(\mathbb{CP}^2)$$

See C. Wang, M. Levin 2016 for related conjectures

See also Tachikawa, Yonekura 2016 for spin theories

The previous formulae (and additional arguments) imply additional consistency conditions:

MB, M. Cheng 2017

1. $$\mathcal{Z}(\mathbb{RP}^4) = \frac{1}{\mathcal{D}} \sum_{a|a={}^{\mathbf{r}}\bar{a}} d_a \theta_a \eta_a^{\mathbf{r}} = \pm 1$$

2. $$\frac{\sum_a \theta_a M_a^2}{\sum_a M_a^2} = \pm 1$$ which implies

$\theta_a = \pm 1$ is independent of a if $M_a > 0$

3. M_a is a non-negative integer for all a

If any of these conditions cannot be satisfied, the (2+1)D theory cannot exist at the surface of any (3+1)D invertible TQFT.

$\Rightarrow$ $\mathcal{H}^3_\rho(\mathbb{Z}_2, \mathcal{A})$ obstruction is non-vanishing

Example: Sp(4)$_2$ Chern-Simons theory

This is a time-reversal invariant TQFT

Aharony, Benini, Hsin, Seiberg 2017

Anyon label, a	1	ε	ϕ_1	ϕ_2	ψ_+	ψ_-
Quantum dimension, d_a	1	1	2	2	$\sqrt{5}$	$\sqrt{5}$
Topological twist, θ_a	1	1	$e^{\frac{4\pi i}{5}}$	$e^{-\frac{4\pi i}{5}}$	i	$-i$
Time-reversal action, ${}^{\mathbf{T}}a$	1	ε	ϕ_2	ϕ_1	ψ_-	ψ_+

$$\epsilon \times \epsilon = 1, \; \epsilon \times \phi_i = \phi_i, \; \epsilon \times \psi_+ = \psi_-$$

$$\phi_i \times \phi_i = 1 + \epsilon + \phi_{\min(2i, 5-2i)}$$

$$\phi_1 \times \phi_2 = \phi_1 + \phi_2$$

$$\psi_+ \times \psi_+ = 1 + \phi_1 + \phi_2$$

All correlation functions are time-reversal invariant

But: $$\mathcal{Z}(\mathbb{RP}^4) = \frac{1}{\sqrt{5}}(1 + \eta_\epsilon^{\mathbf{T}}) \neq \pm 1$$

What happened?

Example: $Sp(4)_2$ Chern-Simons theory

$$\mathcal{A} = \mathbb{Z}_2 \qquad\qquad \mathcal{H}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathbb{Z}_2) = \mathbb{Z}_2$$

By direct computation,

$$[\mathfrak{O}] \in \mathcal{H}^3(\mathbb{Z}_2^{\mathbf{T}}, \mathbb{Z}_2)$$ is non-trivial

Previous conditions are violated:

1. $\mathcal{Z}(\mathbb{RP}^4) = \frac{1}{\sqrt{5}}(1 + \eta_\epsilon^{\mathbf{T}}) \neq \pm 1$

2. $M_a = S_{a1} + S_{a\epsilon}\eta_\epsilon^{\mathbf{T}}$ can only be integer if $\eta_\epsilon^{\mathbf{T}} = -1$

3. But then $M_{\psi_1} = M_{\psi_2} > 0 \quad \theta_{\psi_1} = \theta_{\psi_2}^* = i \neq \pm 1$

Looking at Sp(4)$_2$ CS theory more closely

Start with SU(5)$_1$ CS theory

5 particle types [j],   j = 0,..,4 (mod 5)

Possesses $\mathbb{Z}_4^{\mathbf{T}}$ symmetry:

$$\mathbf{T} : [j] \to [2j] \qquad \mathbf{T}^2 = \mathbf{C} \qquad \mathbf{C}^2 = 1$$

$$\mathbf{C} : [j] \leftrightarrow [-j]$$

Gauge $\mathbf{T}^2 = \mathbf{C}$

Consider adding Dijkgraaf-Witten term

$$\mathcal{H}^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$$

With DW term: Result is Sp(4)$_2$ CS theory

Without DW term: Result is a new obstruction-free theory Sp(4)$_2^\vee$

$Sp(4)_2$ CS theory

Anyon label, a	1	ε	ϕ_1	ϕ_2	ψ_+	ψ_-
Quantum dimension, d_a	1	1	2	2	$\sqrt{5}$	$\sqrt{5}$
Topological twist, θ_a	1	1	$e^{\frac{4\pi i}{5}}$	$e^{-\frac{4\pi i}{5}}$	i	$-i$
Time-reversal action, ${}^{\mathbf{T}}a$	1	ε	ϕ_2	ϕ_1	ψ_-	ψ_+

ϵ is the **T**^{2} gauge charge. Thus, local **T**^{2} value is -1:

$$\eta_\epsilon^{\mathbf{T}} = -1$$

However, we also require $\eta_\epsilon^{\mathbf{T}} = \theta_\epsilon = +1$

because $N^{\epsilon}_{\psi_+, {}^{\mathbf{T}}\psi_+} = 1$

$Sp(4)_2^V$ theory

a	1	ϵ	ϕ_1	ϕ_2	ψ_+	ψ_-
θ_a	1	1	$e^{\frac{4\pi i}{5}}$	$e^{-\frac{4\pi i}{5}}$	1	-1
d_a	1	1	2	2	$\sqrt{5}$	$\sqrt{5}$
${}^{\mathbf{T}}a$	1	ϵ	ϕ_2	ϕ_1	ψ_+	ψ_-

Now, we no longer require $\eta_\epsilon^{\mathbf{T}} = \theta_\epsilon = +1$

because $N^{\epsilon}_{\psi_+, {}^{\mathbf{T}}\psi_+} = 0$

There are no conflicting constraints on $\eta_\epsilon^{\mathbf{T}}$

$$\eta_\epsilon^{\mathbf{T}} = -1$$

The difference between the obstruction-free theory $Sp(4)_2^\vee$ and the obstructed $Sp(4)_2$ CS theory was the DW term

Recall that $\mathbb{Z}_2$ gauge theory with a DW term is equivalent to the "doubled semion" (DS) theory: $U(1)_2 \times U(1)_{-2}$ CS theory

DS theory contains the particles {1, s, s', b = s x s'}

$$Sp(4)_2 = (Sp(4)_2^\vee \times \mathrm{DS})/(1 \sim b\epsilon)$$

Since $\eta_\epsilon^{\mathbf{T}} = -1$ we must have $\eta_b^{\mathbf{T}} = -1$

But $\eta_b^{\mathbf{T}} = -1$ is inconsistent with the requirement

$\eta_b^{\mathbf{T}} = \theta_b = 1$ which holds because $N_{s\,{}^{\mathbf{T}}s}^{b} = 1$

This is the heart of the problem

Thus a general way to obtain a theory with a $\mathbb{Z}_2^{\mathbf{T}}$ H^3 obstruction is to start with a theory with $\mathbb{Z}_4^{\mathbf{T}}$ symmetry, and gauge $\mathbf{T}^2$ while adding a DW term for the $\mathbf{T}^2$ gauge field

All examples we have can be obtained this way

(e.g. also $SO(4)_4$ CS theory, and infinite family of other examples)

Resolutions of the H^3 obstruction

MB, M. Cheng, 2017

1. Enlarge the symmetry from $\mathbb{Z}_2^{\mathbf{T}}$ to $\mathbb{Z}_4^{\mathbf{T}}$

   Can show explicitly that $\mathbb{Z}_4^{\mathbf{T}}$ has no obstruction

2. View it as a spin TQFT with $\mathbf{T}^2 = (-1)^{N_F}$

3. View the theory not as a true (2+1)D theory, but "pseudo-realized" at the surface of a (3+1)D non-invertible TQFT. A new type of anomaly inflow

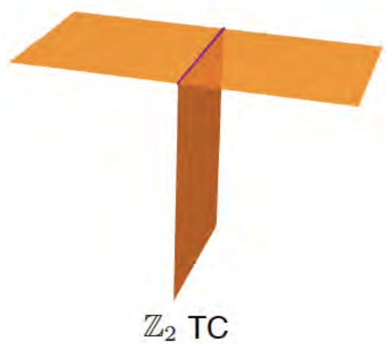
4. The true symmetry of this theory is a 2-group symmetry. The 0-form symmetry $\mathbb{Z}_2^{\mathbf{T}}$ and the 1-form symmetry $\mathcal{A}$ are not independent of each other, but intrinsically connected.

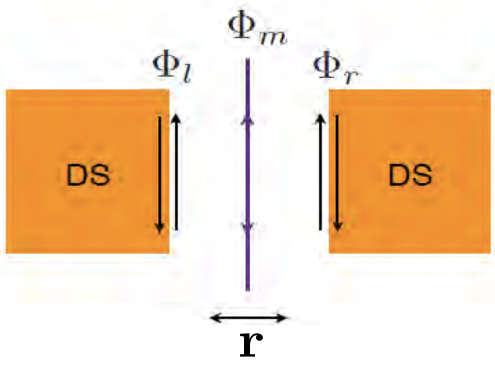
Benini, Cordova, Hsin 2018
Tachikawa 2017

Pseudo-realization and "anomaly inflow"

The heart of the problem was the (non)-existence of a doubled semion theory with $\mathbb{Z}_2^{\mathbf{r}}$ symmetry and $\eta_b^{\mathbf{r}} = -1$

But we can pseudo-realize it at the surface of a non-trivial (3+1)D system





MB, M. Cheng, 2017

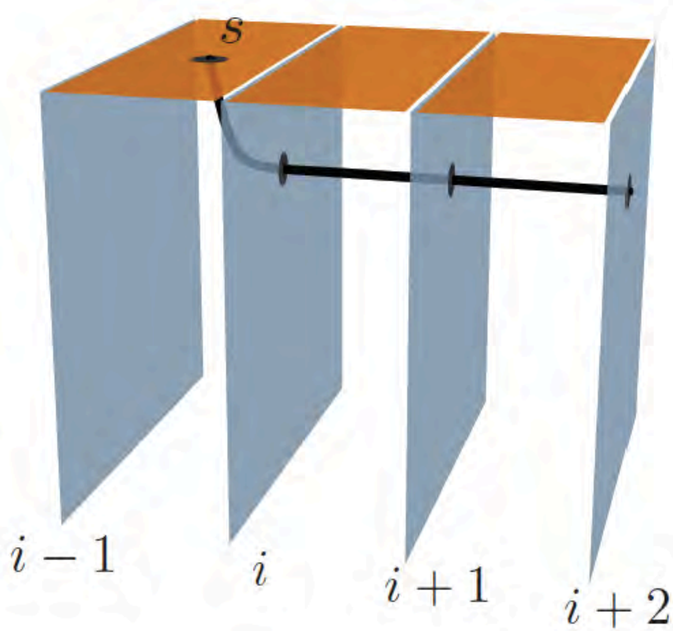
e and m particles on mirror plane carry half $\mathbb{Z}_2^{\mathbf{r}}$ charge

Can gap out all edge modes such that reflection eigenvalue of Wilson string for b becomes -1

b x e is condensed at the junction, so b can leak into the bulk as e

→ pseudo-realization

Layer construction and bulk Z_2 gauge theory



Stack layers of (2+1)D Z_2 gauge theories

Condense pairs of e particles from neighboring planes

→ bulk (3+1)D Z_2 gauge theory

Condense b x e at the surface

→ b can leak into bulk as Z_2 gauge charge

→ s and s' become bound to endpoints of magnetic flux lines

This means that $Sp(4)_2$ CS theory can also exist with a global $\mathbb{Z}_2^{\mathbf{r}}$ symmetry, as long as it is pseudo-realized at the surface of a (3+1)D system that contains a dynamical Z_2 gauge theory

Similar phenomenon found in a discrete gauge theory with gauge group D_{16} and with global Z_2 symmetry (internal, unitary) by Fidkowski and Vishwanath (2015).

There, the phenomenon could be related to symmetry fractionalization of strings (flux loops) in the (3+1)D bulk. This interpretation is not available in the case of space-time reflection symmetry.

Infinite sequence of theories with H^3 obstructions

Any theory with a $\mathcal{H}^3_\rho(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ obstruction is compatible with $\mathbb{Z}_4^{\mathbf{T}}$ symmetry. Thus we can again gauge $\mathbf{T}^2$ while adding a DW term for the $\mathbf{T}^2$ gauge field. This gives a new theory with $\mathcal{H}^3_\rho(\mathbb{Z}_2^{\mathbf{T}}, \mathcal{A})$ obstruction

This process can be repeated indefinitely

$$SU(5)_1 \xrightarrow{\text{Gauge } \mathbf{T}^2} Sp(4)_2 \xrightarrow{\text{Gauge } \mathbf{T}^2} \cdots\cdots$$

$$SU(3)_1 \times SU(3)_1 \longrightarrow SO(4)_4 \longrightarrow \cdots\cdots$$

$$U(1) \times U(1) \quad K = \begin{pmatrix} m & n \\ n & -m \end{pmatrix} \longrightarrow \cdots\cdots$$

Summary

- Symmetry fractionalization characterized by $\{\eta_a^{\mathbf{T}}\}$ and $\{\eta_a^{\mathbf{r}}\}$

  Required to determine path integral on non-orientable spacetimes

- Explicit formulae for 't Hooft anomalies for global $\mathbb{Z}_2^{\mathbf{T}}$ symmetries

- Simple sufficient conditions for diagnosing existence of symmetry localization obstruction for global $\mathbb{Z}_2^{\mathbf{T}}$

- General method to produce theories with $\mathbb{Z}_2^{\mathbf{T}}$ H^3 obstructions

- Various resolutions of H^3 obstructions.

  An unusual type of anomaly inflow where bulk is a dynamical Z_2 gauge theory

(1+1)D time-reversal / reflection SPTs

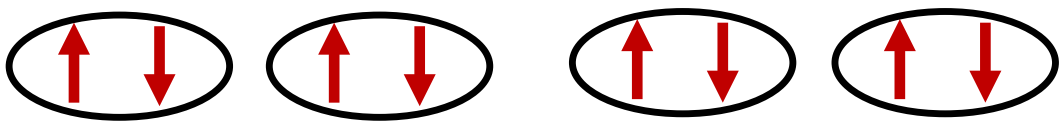
$G = Z_2$ Time-reversal, **T**, or reflection, **r** $\mathcal{H}^2_*(Z_2, U(1)) = Z_2$

Topological path integral

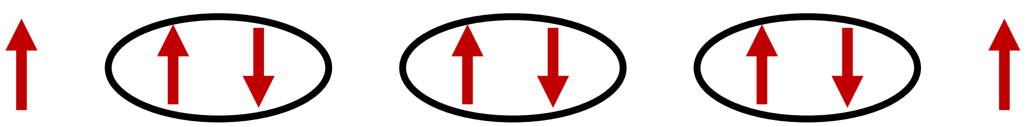
$$\mathcal{Z}(\mathbb{RP}^2) = \begin{cases} 1 & \text{for a trivial SPT state} \\ -1 & \text{for a nontrivial SPT state} \end{cases}$$

(1+1)D time-reversal / reflection SPTs

Time-reversal SPT:



Trivial

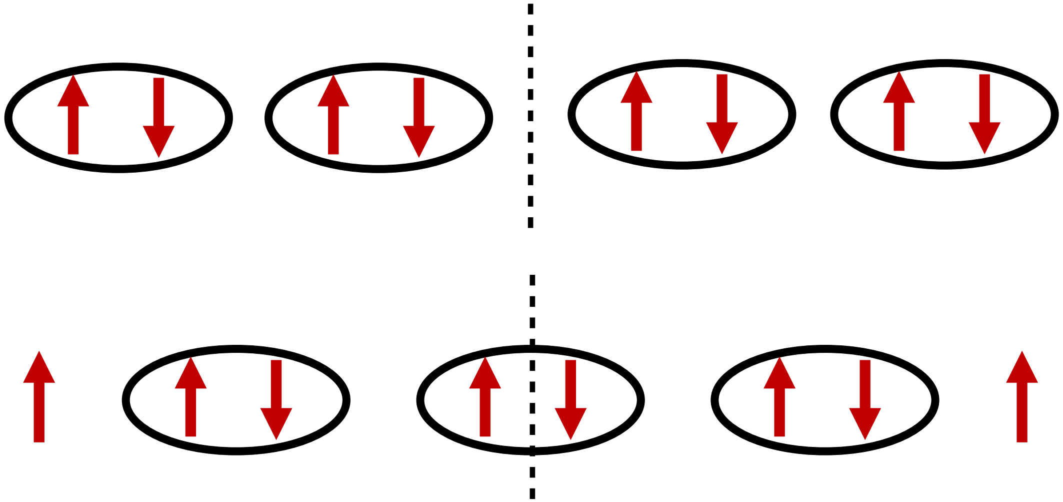


Non-Trivial

Local Kramers degeneracy at edge

(1+1)D time-reversal / reflection SPTs

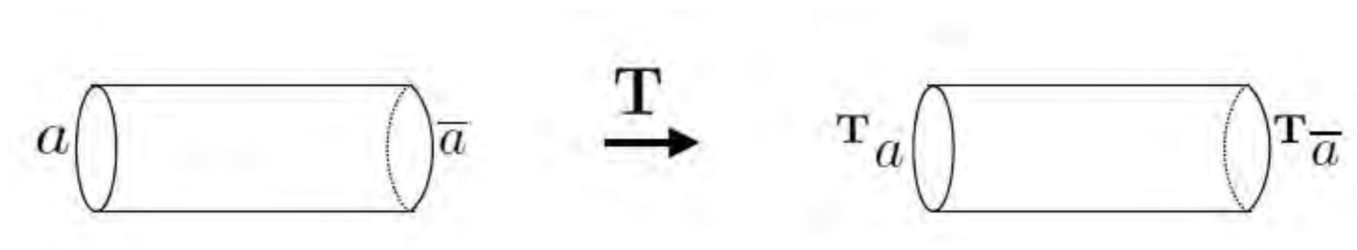
Reflection SPT:



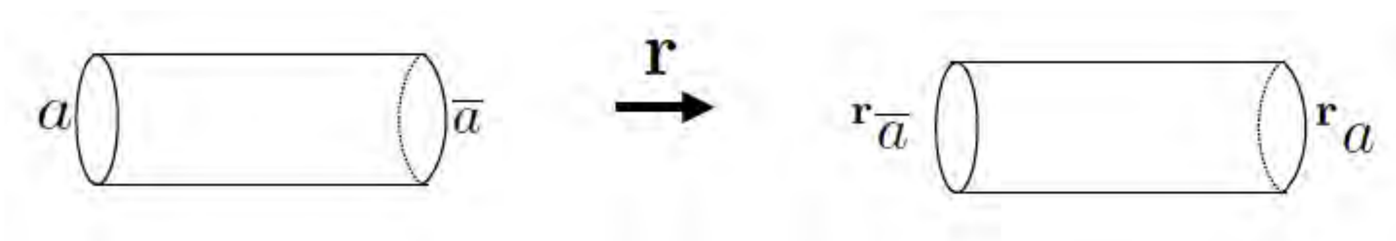
$$R_{\mathbf{r}}|\Psi\rangle = \pm|\Psi\rangle$$

Reflection eigenvalue = topological invariant

Symmetry fractionalization in (2+1)D



If $a = {}^{\mathbf{T}}a$ define $\eta_a^{\mathbf{T}} = \pm 1$ Determines whether a carries local Kramers degeneracy



If $a = {}^{\mathbf{r}}\overline{a}$ define $\eta_a^{\mathbf{r}} = \pm 1$ = eigenvalue of reflection