Superstring amplitudes in genus 0 and 1

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String Math
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Overview

Superstring perturbation

- Expansion in series of genus g world-sheets. Integrate over world-sheet moduli space.
- Expansion in $\alpha' = \ell_S^2$

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Strategy: first integrate over configuration space of points on a Riemann surface of genus g. Then integrate over the moduli of the Riemann surface.

The last step is redundant in the case $g = 0$.

$\mathcal{M}_{0,4}$: Veneziano and Virasoro-Shapiro

Beta function is an integral on $\mathcal{M}_{0,4} = \mathbb{P}^1 \backslash \{0, 1, \infty\}$

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$$= \frac{s+t}{st\alpha'} \exp\Big(\sum_{n=2}^{\infty} (-1)^{n-1} \frac{\zeta(n)}{n} \sigma_n\Big)$$

where $\sigma_n = (\alpha')^n((s+t)^n - s^n - t^n)$. Involves all zeta values.

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Closed string gives complex beta function:

$$\int_{\mathbb{P}^1(\mathbb{C})} |x|^{-2\alpha' s - 2} |1-x|^{-2\alpha' t - 2} d^2 x = \frac{\Gamma(\alpha' s)\Gamma(\alpha' t)\Gamma(1 - \alpha' s - \alpha' t)}{\Gamma(s\alpha' + t\alpha')\Gamma(1 - \alpha' s)\Gamma(1 - \alpha' t)}$$

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$$= \frac{s+t}{st\alpha'} \exp\Big(\sum_{n=1}^{\infty} \frac{2\zeta(2n+1)}{2n+1} \sigma_{2n+1}\Big)$$

where $d^2 x = (2\pi i)^{-1}\, dx \wedge \overline{dx}$. Only involves odd zeta values.

Open vs closed amplitudes in genus 0

Distinct points $z_0, \ldots, z_{n+2}$ on a Riemann sphere. By $\mathrm{PSL}_2(\mathbb{C})$ action, can place $z_0 = 0, z_{n+1} = 1, z_{n+2} = \infty$.

For a permutation $\pi \in \Sigma_{n+3}$, let

$$z_\pi = \prod_{i \in \mathbb{Z}/(n+3)\mathbb{Z}} (z_{\pi(i)} - z_{\pi(i+1)})$$

omitting term $z_{n+2} = \infty$.

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Open string amplitudes reduce to $n!$ integrals:

$$A^{\mathrm{open}}(\pi) = \int_{0<z_1<\ldots<z_n<1} \prod_{i<j} (z_i - z_j)^{\alpha' s_{ij}} \, \frac{dz_1 \ldots dz_n}{z_\pi}$$

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Closed string amplitudes reduce to complex integrals:

$$A^{\mathrm{closed}}(\pi, \pi') = \int_{\mathbb{C}^n} \prod_{i<j} |z_i - z_j|^{2\alpha' s_{ij}} \frac{dz_1 \ldots dz_n}{z_\pi} \wedge \frac{d\overline{z}_1 \ldots d\overline{z}_n}{\overline{z}_{\pi'}}$$

Kawai-Lewellen-Tye formula (1986)

Expresses closed tree-level ($g = 0$) amplitudes as quadratic expression in open amplitudes: approximately

$$A^{\mathrm{closed}}(\rho, \sigma) = \sum_{\rho,\sigma} A^{\mathrm{open}}(\rho) S(\rho; \sigma) A^{\mathrm{open}}(\sigma)$$

for certain factors $S(\rho; \sigma)$ in the Mandelstam variables s_{ij}.

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Slogan:

'Multiply then integrate = integrate then multiply'

Plan

- Mathematics behind generalised KLT formulae
- Single-valued projections
- (Cosmic Galois group)
- New theory of modular forms from genus 1 string amplitudes

Single-valued integration

The usual theory of integration pairs a differential form ω with a domain of integration σ

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We shall discuss how to pair certain differential forms ω with a 'dual differential form' ν

$$I^{\mathsf{sv}} = \mathop{\int\!\!\!\!\!\!\mathsf{S}}_{\nu} \omega \quad \in \quad \mathbb{R}$$

Single-valued integration (joint with C. Dupont)

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It can be interpreted as a 'p-adic period at the infinite prime $p = \infty$'. First some examples.

Examples of single-valued functions

The *logarithm* is a multi-valued function on $\mathbb{C}\backslash\{0\}$:

$$\log z = \int_1^z \frac{dx}{x} \ .$$

Changing path of integration results in $\log z \mapsto \log z + 2\pi i\mathbb{Z}$.

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It has a single-valued version, the Bloch-Wigner dilogarithm

$$D(z) = 2i\mathrm{Im}(\mathrm{Li}_2(z) + \log |z| \log(1-z))$$

Multiple zeta values (MZV's)

Defined by Euler (1730's),

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \le k_1 < \ldots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}}$$

where $n_1, \ldots, n_r > 0$ integers, $n_r \geq 2$. They satisfy a plethora of complicated algebraic relations.

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They are values at 1 of multiple polylogarithms (Poincaré, Kummer, Lappo-Danilevskyy):

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These are multi-valued functions on $\mathbb{C} \backslash \{0, 1\}$. Coefficients of the universal KZ equation in genus 0. Iterated integrals on $\mathbb{C} \backslash \{0, 1\}$.

Single-valued multiple zeta values

Theorem (B. 2004)

By taking combinations of products of real and imaginary parts, there is a canonical way to define single-valued versions

$$\mathcal{L}_{n_1,\ldots,n_r}(z) \qquad \text{of} \qquad \mathrm{Li}_{n_1,\ldots,n_r}(z)$$

preserving algebraic and differential (with respect to $\frac{\partial}{\partial z}$) relations.

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The *single-valued multiple zeta values* are defined by

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Example:

$$\zeta_{sv}(2) = D(1) = 0$$

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$$\begin{aligned} \zeta_{\mathrm{sv}}(2n) &= 0 \\ \zeta_{\mathrm{sv}}(2n+1) &= 2\zeta(2n+1) \end{aligned}$$

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The first non-trivial svMZV is at weight 11:

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Warning: Does not exist for general period integrals.

Stieberger's conjecture

Theorem

The open superstring amplitudes for $g = 0$ admit a canonical Laurent expansion in s_{ij} whose coefficients are multiple zeta values

Follows from conjecture of Goncharov-Manin (B. 2006), Terasoma, Schlotterer, Stieberger, Broëdel, ...

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Stieberger and Stieberger-Taylor made the following conjecture:

Theorem (B. and Dupont '18)

$$\mathsf{sv}\, A^{\mathrm{open}}(\pi) = A^{\mathrm{closed}}(\pi; \mathrm{id})$$

Apply sv term by term in the Laurent expansion in s_{ij}.

X a smooth algebraic variety over $\mathbb{Q}$. A *period integral* on X

$$I = \int_\sigma \omega$$

where $\omega \in \Omega^n(X; \mathbb{Q})$ algebraic differential form. Chain $\sigma \subset X(\mathbb{C})$ has boundary $\partial\sigma \subset D(\mathbb{C})$ where $D \subset X$ a divisor.

Periods

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$$\begin{array}{rcl} [\omega] & \in & H^n_{dR}(X, D) \\ [\sigma] & \in & H_n(X(\mathbb{C}), D(\mathbb{C})) = H^n_B(X, D)^\vee \end{array}$$

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It defines a canonical isomorphism (Grothendieck 1964):

$$H^n_{dR}(X, D) \otimes \mathbb{C} \xrightarrow{\sim} H^n_B(X, D) \otimes \mathbb{C}$$

Complex conjugation is continuous

$$(X(\mathbb{C}), D(\mathbb{C})) \xrightarrow{\sim} (X(\mathbb{C}), D(\mathbb{C}))$$

It induces the *real Frobenius*

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It defines a *real comparison isomorphism*

$$\mathrm{sv} : H^n_{dR}(X, D) \otimes \mathbb{R} \xrightarrow{\sim} H^n_{dR}(X, D) \otimes \mathbb{R}$$

This gives a way to pair forms with 'dual forms':

$$\begin{array}{rcl} [\omega] & \in & H^n_{dR}(X, D) \\ [\nu] & \in & H^n_{dR}(X, D)^\vee \end{array}$$

to get a real number, which we denote by

$$\oint_\nu \omega \quad = \quad \langle [\nu], \mathsf{sv}\,[\omega] \rangle \qquad \in \qquad \mathbb{R}$$

It satisfies the usual rules of integration (bilinearity, change of variables, etc). How to make sense of a 'dual form'?

Duality

Suppose X smooth projective of dimension n, $A \cup B \subset X$ normal crossing divisor. Then Poincaré-Verdier:

$$H^k_{dR}(X \backslash A, B)^\vee \quad \cong \quad H^{2n-k}_{dR}(X \backslash B, A)(n)$$

Use to replace 'dual forms' with actual forms.

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Theorem (B.-Dupont 2018)

Let $\omega \in \Omega^n_X(\log A)$, $\nu \in \Omega^n_X(\log B)$ meromorphic, log. sings. Then

$$\oint_\nu \omega = \frac{1}{(2\pi i)^n} \int_{X(\mathbb{C})} \omega \wedge \overline{\nu} \ ,$$

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Warning. LHS defined in general. RHS badly defined for non-logarithmic ω, ν. Kazhdan-Felder. Very subtle issues.

'KLT-formula' for algebraic varieties

Theorem (B.-Dupont 2018)

In the same setting,

$$\int_{X(\mathbb{C})} \omega \wedge \overline{\nu} = \sum_{\sigma,\tau} \langle \sigma, \tau \rangle \int_{\sigma} \omega \int_{\overline{\tau}} \nu$$

sum over σ a relative homology basis of $H_n(X \setminus A, B)$ and τ a relative homology basis of $H_n(X \setminus B, A)$.

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‘KLT’ slogan revisited:

“Multiply then integrate = integrate then multiply”

Example: logarithm

Recall

$$\log x = \int_1^x \frac{dz}{z}$$

period of

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$$\begin{aligned} \log |x|^2 \quad &= \quad \frac{1}{2\pi i} \int_{\mathbb{P}^1(\mathbb{C})} \frac{dz}{z} \wedge \frac{(\overline{x}-1)d\overline{z}}{(\overline{z}-1)(\overline{z}-\overline{x})} \\ &= \quad \int_1^x \frac{dz}{z} \,.\, 1 + 1 \,.\, \int_1^{\overline{x}} \frac{dz}{z} \end{aligned}$$

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Moduli spaces $\mathcal{M}_{0,n}$

Moduli of Riemann spheres with n ordered marked points

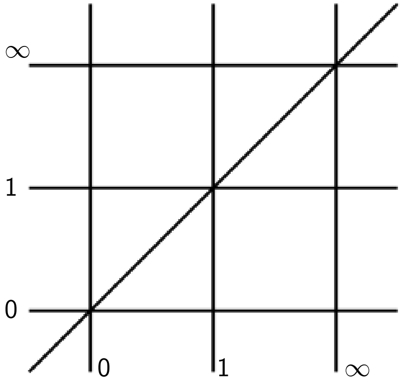
Let $n \geq 4$.

$$\mathcal{M}_{0,n} = \{(z_1, \ldots, z_n) \in \mathbb{P}^1 : z_i \neq z_j\}/\mathrm{PGL}_2$$

Place $z_1 = 0, z_{n-1} = 1, z_n = \infty$. Then

$$\mathcal{M}_{0,4} \cong \mathbb{P}^1 \backslash \{0, 1, \infty\}$$

$\mathcal{M}_{0,5}$



Dihedral coordinates

Let $\mathcal{M}_{0,S}$ be moduli of R.S. with marked points indexed by a set S. Suppose that S has a dihedral (= cyclic up to reversal) order.

Every chord c in the polygon S determines two consecutive pairs

$$c = \{z_i, z_{i+1}, z_j, z_{j+1}\}$$

Forgetting all other marked points defines a *dihedral coordinate*

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General dihedral coordinates are *not* cluster coordinates.

Renormalised amplitudes

Every open string amplitude can be written in dihedral coordinates:

$$I^{\mathrm{open}} = \int_X \Omega \qquad \text{where} \qquad \Omega = \Big(\prod_c u_c^{s_c}\Big)\omega$$

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Theorem (canonical renormalisation) (B.-Dupont '18)

$$I^{\mathrm{open}} = \sum_J \frac{1}{s_J} \int_{X^J} \Omega_J^{\mathrm{ren}} \qquad \text{where } s_J = \prod_{c \in J} s_c$$

J are sets of non-crossing chords in S-gon. Each integrand Ω_J^{ren} is convergent for $\mathrm{Re}(s_c) > -1$, so has Taylor expansion.

Poles in $s_c \longleftrightarrow$ poles of ω along boundary strata of $\overline{\mathcal{M}_{0,S}}$

Renormalised amplitudes

The identical formalism works for closed string amplitudes:

$$I^{\text{closed}} = \int_{\overline{\mathcal{M}}_{0,n}(\mathbb{C})} \Omega \qquad \text{where} \qquad \Omega = \Big(\prod_c |u_c|^{2s_c} \Big) \omega \wedge \overline{\nu}_X$$

for some ν_X 'dual to' X. We have

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Each term on the right is the single-valued projection of the corresponding term in the renormalisation of the open amplitude, and proves the formula conjectured by Stieberger:

$$\text{sv}\, I^{open} = I^{closed}$$

Different point of view: s_{ij} as complex numbers, not formal variables.

KLT and SV for cohomology with coefficients

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$$\nabla_{\underline{s}} = d - \sum_{i<j} s_{ij} \frac{d(z_i - z_j)}{z_i - z_j}$$

Horizontal sections form a local system

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Apply sv formalism to the *self-dual* object

$$H_{dR} = H^n_{dR}(\mathcal{M}_{0,S}, \nabla_{\underline{s}} \oplus \nabla_{-\underline{s}}) \quad , \quad H_B = H^n(\mathcal{M}_{0,S}, \mathcal{L}_{\underline{s}} \oplus \mathcal{L}_{-\underline{s}})$$

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Immediately deduce KLT-type formula involving intersection numbers on H_B. The latter were computed by K. Matsumoto, Mimachi-Yoshida, . . . , Mizera, and implies KLT formula.

Genus 1

Consider now the *closed* string amplitudes

$$\int_{\overline{\mathcal{M}}_{1,n}(\mathbb{C})} \exp\Big(\sum_{i<j} \alpha' s_{ij} G(z_i - z_j)\Big)$$

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Strategy (Green, Russo, d'Hoker, Vanhove,...) integrate first in the fiber, to obtain functions on

$$\mathcal{M}_{1,1}(\mathbb{C}) \quad \cong \quad \mathrm{SL}_2(\mathbb{Z})\backslash\!\!\backslash\mathbb{H}$$

Obtain $\mathrm{SL}_2(\mathbb{Z})$-invariant functions of the modulus $\tau \in \mathbb{H}$.

Modular graph functions

The fiber integrals can be computed explicitly. To every graph G, associate a nested lattice sum $I_G(\tau)$. It is a modular-invariant

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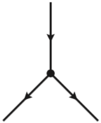
$G =$ 

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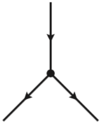
$G =$ 

$$I_G = \pi^{-3} \sideset{}{'}\sum_{m_1,n_1,m_2,n_2} \frac{\mathrm{Im}(\tau)^3}{|m_1\tau+n_1|^2|m_2\tau+n_2|^2|(m_1+m_2)\tau+n_1+n_2|^2}$$

where the sum is over $(m_1,n_1)\in\mathbb{Z}^2$, $(m_2,n_2)\in\mathbb{Z}^2$ such that

$$(m_1,n_1)\neq(0,0),(m_2,n_2)\neq(0,0),(m_1+m_2,n_1+n_2)\neq(0,0)\ .$$

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Zagier proved that $I_G(\tau)$ in the example is a linear combination of $\zeta(3)$ and a real-analytic Eisenstein series.

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Idea of the construction: single-valued machine. Take real and imaginary parts of iterated integrals of Eisenstein series

$$\mathbb{G}_{2k} = -\frac{b_{2k}}{4k} + \sum_{n\geq 1} \sigma_{2k-1}(n) q^n$$

in such a way as to make them modular.

Example

Modular analogue of Bloch-Wigner dilogarithm:

$$\operatorname{Im} \int_\tau^\infty \mathbb{G}_{2a}\mathbb{G}_{2b} - \operatorname{Re}\Big(\int_\tau^\infty \mathbb{G}_{2a}\Big) \times \int_\tau^\infty \underline{\mathbb{G}}_{2b}$$

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Related to

- Universal Mixed elliptic motives
- Mock modular forms
- Weak harmonic Maass forms
- Subtle questions in number theory

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