Topological Strings, Resurgence and Quantum Mechanics

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Introduction

In recent years it has become clear that, in some backgrounds, topological string theory is closely related to quantum-mechanical models and quantum integrable systems.

This relation has led to interesting developments, e.g. non-perturbative definitions of some topological string theories.

The quantum systems involved in this correspondence are not always elementary. In this talk, I will discuss applications of topological string theory to basic models in Quantum Mechanics. This turns out to shed new light on old problems.

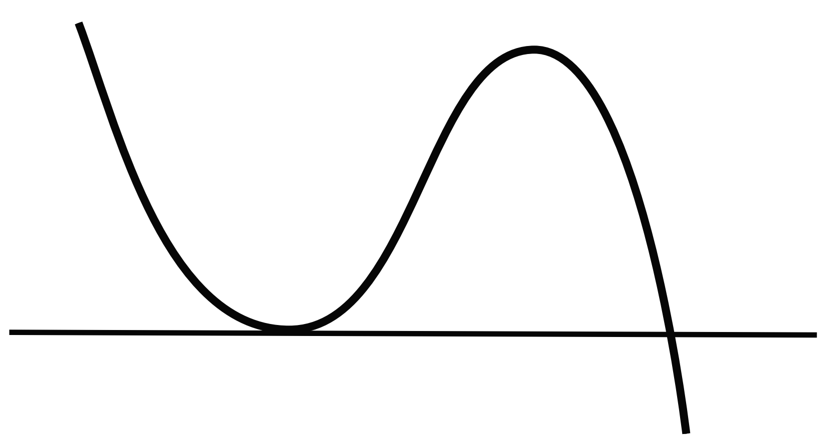
Back to basics: anharmonic oscillators

Arguably, the simplest models in QM are described by Hamiltonians of the form

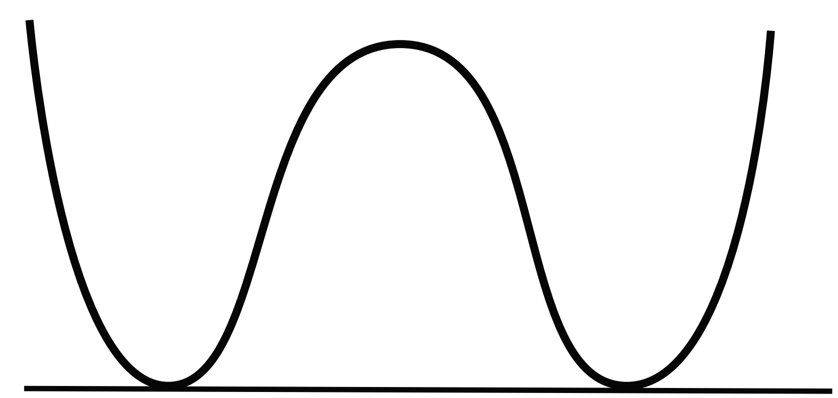
$$[\mathsf{x}, \mathsf{p}] = \mathrm{i}\hbar \qquad \mathsf{H} = \mathsf{p}^2 + V_N(\mathsf{x}) \longleftarrow$$

degree *N* polynomial

However, if the degree of the polynomial *N* is ≥ 3 , there is no explicit solution



cubic potential: the basic example in Heisenberg's seminal paper of 1925



quartic potential: symmetric double-well

Almost an exact solution: the exact WKB method, i.e. *all-orders WKB* + *Borel-Ecalle resummation*

[Voros, Silverstone, Zinn-Justin, Delabaere, Pham, Alvarez…]

All-orders WKB gives a "quantum" Liouville one-form, which is a *formal* power series in $\hbar^2$

$$P(x,\hbar)\mathrm{d}x = p(x)\mathrm{d}x + \sum_{n\geq 1} \hbar^{2n} p_n(x)\mathrm{d}x$$

on the curve Σ_{WKB} $\qquad H(p,x) = E$

wave function $$\psi(x,\hbar) = \frac{1}{\sqrt{P(x,\hbar)}} \exp\left[\frac{\mathrm{i}}{\hbar}\int^x P(x',\hbar)\mathrm{d}x'\right]$$

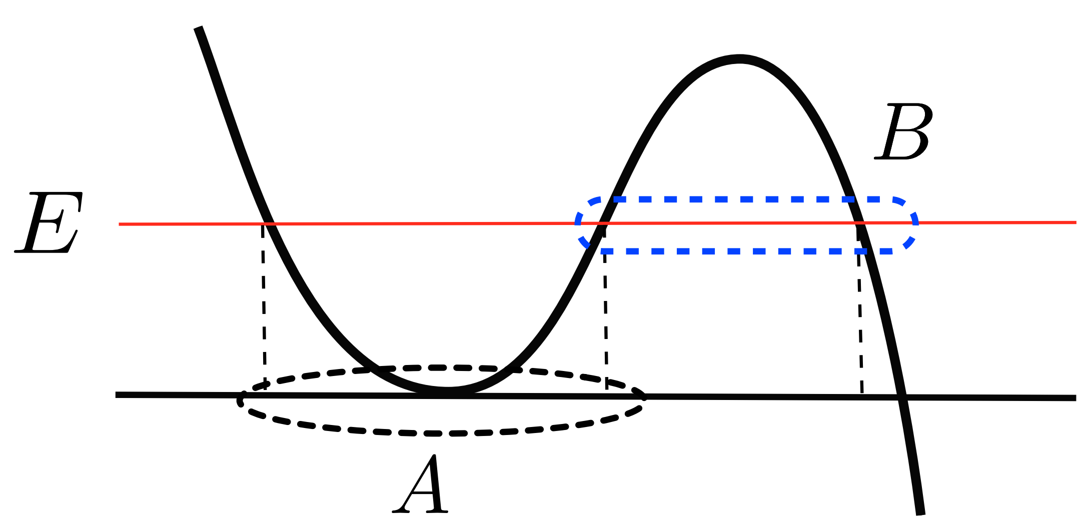
Basic building blocks of the exact WKB method: *quantum periods* and *Voros symbols*

$$\mathcal{V}_\gamma = \exp\left(\frac{\mathrm{i}}{\hbar}\oint_\gamma \underbrace{P(x,\hbar)\mathrm{d}x}_{\text{quantum period}}\right)$$

γ one-cycle in Σ_{WKB}

Quantum periods define a deformed special geometry

[Nekrasov-Shatashvili, Mironov-Morozov, ACDKV]

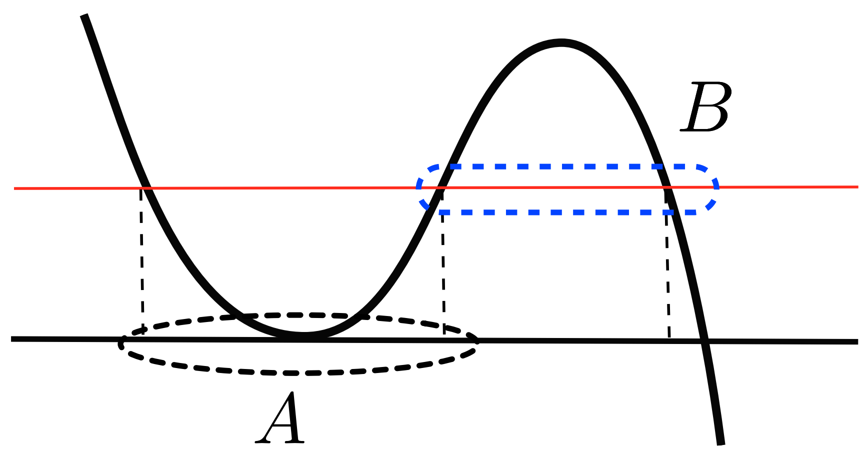


$$a = \oint_A P(x,\hbar)\mathrm{d}x$$

$$\frac{\partial F_{\mathrm{NS}}}{\partial a} = \mathrm{i}\oint_B P(x,\hbar)\mathrm{d}x$$

quantum free energy

The spectrum of H is determined by an "exact" quantization condition (EQC). Typically it is a vanishing condition involving Voros symbols



$$1 + \mathrm{e}^{\mathrm{i}a/\hbar} + \mathcal{S}\, \mathrm{e}^{-\partial_a F_{\mathrm{NS}}/\hbar} = 0$$

determines tower of *resonances* in the cubic oscillator

Perturbatively, we have

$$1 + \mathrm{e}^{\mathrm{i}a/\hbar} \approx 0 \longrightarrow \oint_A p(x)\mathrm{d}x \approx 2\pi\hbar\left(n + \frac{1}{2}\right)$$

Bohr-Sommerfeld

The EQC is only well-defined once a resummation prescription has been chosen for the formal power series. The coefficients of resummed Voros symbols depend on this prescription ("resurgent" version of Stokes phenomenon)

$$1 + \mathrm{e}^{\mathrm{i}a/\hbar} + \mathcal{S}\,\mathrm{e}^{-\partial_a F_{\mathrm{NS}}/\hbar} = 0$$

$$\mathcal{S} = 0, 1$$

Obvious questions:

perturbative: how does one calculate Voros symbols? What is their structure?

non-perturbative: can we find EQCs involving *actual functions*, and not (resummed) formal power series?

As we will now see, topological string theory has something interesting to say about both questions

Voros symbols and the holomorphic anomaly

In topological string theory, we are often interested in the perturbative series of the free energies, $F_g(t)$. One of the most powerful methods to obtain them is the BCOV holomorphic anomaly equation (HAE)

-Recursive, with initial data $F_0(t), F_1(t)$

-Needs boundary conditions (holomorphic ambiguity), but these can be determined for local CY [Huang-Klemm]

-*Very* efficient if combined with modularity [Huang -Klemm]

-They can be extended to the refined topological string [Huang-Klemm,Krefl-Walcher]

Claim [Codesido-M.M.]:

Voros symbols for anharmonic oscillators are governed by the holomorphic anomaly equation
(in the NS limit of the refined version)

Corollary: Voros symbols are formal power series of *modular forms* on the WKB curve

in genus one:

$$F_{\mathrm{NS}}(\tau,\overline{\tau},\hbar)=\sum_{k\geq 0}F_k(\tau,\overline{\tau})\hbar^{2k}$$

computed recursively

evidence in genus two: [Fischbach-Klemm-Nega]

The HAE can be also used to calculate *exponentially small corrections* to this series [Couso et al, Codesido-M.M.-Schiappa]

From SW theory to QM

Why is there such a connection? It turns out that anharmonic oscillators emerge as a scaling limit of *SU(N)* Seiberg-Witten (SW) theory near the Argyres-Douglas (AD) point [Grassi-Gu, Grassi-M.M., Ito-Shu]

$$\Lambda^N \left(\mathrm{e}^p + \mathrm{e}^{-p}\right) + W_N(x) = 0$$

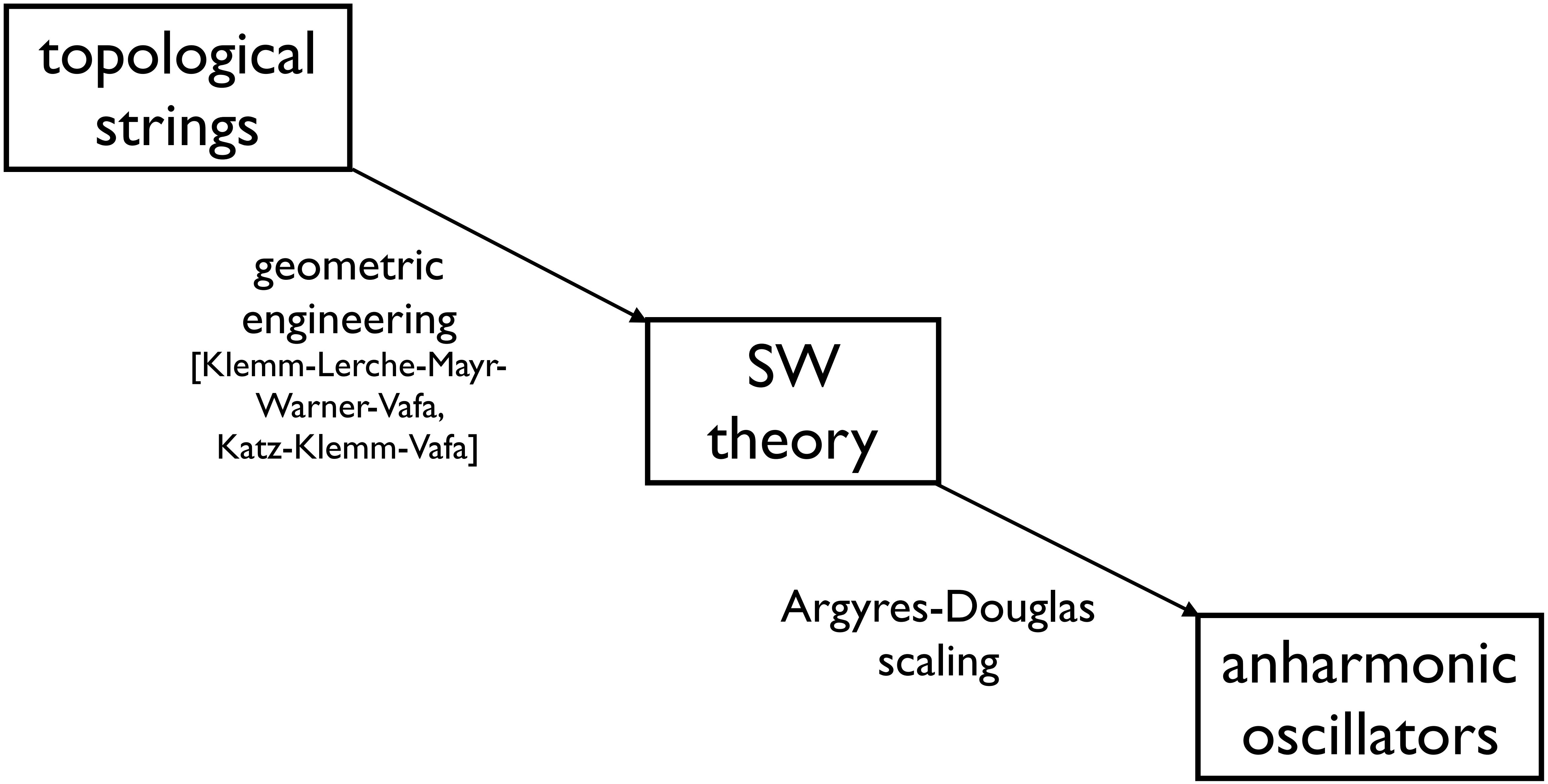
$$W_N(x) = x^N + \sum_{k=2}^{N} x^{N-k} h_k$$

$$\xrightarrow{\alpha \to 0}$$

$$\Lambda^N p^2 + x^N + \sum_{k=2}^{N} \widetilde{h}_k x^{N-k} = 0$$

$$(h_2, \cdots, h_N) = (0, \cdots, 0, -2) + (\alpha^{\Delta_2} \widetilde{h}_2, \cdots, \alpha^{\Delta_N} \widetilde{h}_N)$$

In the quantum version: $\hbar_{\mathrm{SW}} = \alpha \hbar_{\mathrm{QM}}$



The holomorphic anomaly equations governing topological strings are inherited in each limiting procedure

Quantum SW curve: a deformation of QM

The "engineering" of anharmonic oscillators as a limit of SW theory suggests to look more carefully at the *quantum* SW curve

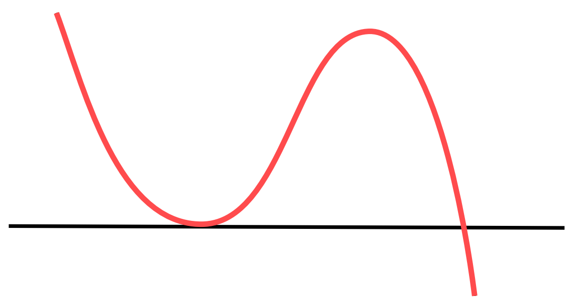
$$[\mathsf{x}, \mathsf{p}] = \mathrm{i}\hbar \qquad \left(\Lambda^N \left(\mathrm{e}^{\mathsf{p}} + \mathrm{e}^{-\mathsf{p}}\right) + W_N(\mathsf{x})\right) |\psi\rangle = 0$$

Equivalently, $\mathsf{H}_N |\psi\rangle = -h_N |\psi\rangle$

$$\mathsf{H}_N = \Lambda^N \left(\mathrm{e}^{\mathsf{p}} + \mathrm{e}^{-\mathsf{p}}\right) + \mathsf{x}^N + \sum_{k=2}^{N-1} \mathsf{x}^{N-k} h_k$$

We are interested in the *actual* spectral problem on $L^2(\mathbb{R})$

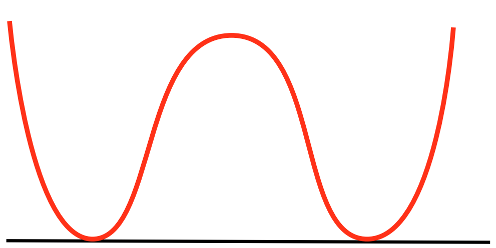
N odd: resonant states



$$\mathsf{H}_3 = \Lambda^3 \left(e^{\mathsf{p}} + e^{-\mathsf{p}} \right) + \mathsf{x}^3 + h_2 \mathsf{x}$$

complex eigenvalues $-h_3^{(n)}(h_2)$ $\quad n = 0, 1, 2, \cdots$

N even: bound states



$$\mathsf{H}_4 = \Lambda^4 \left(e^{\mathsf{p}} + e^{-\mathsf{p}} \right) + \mathsf{x}^4 + h_2 \mathsf{x}^2 + h_3 \mathsf{x}$$

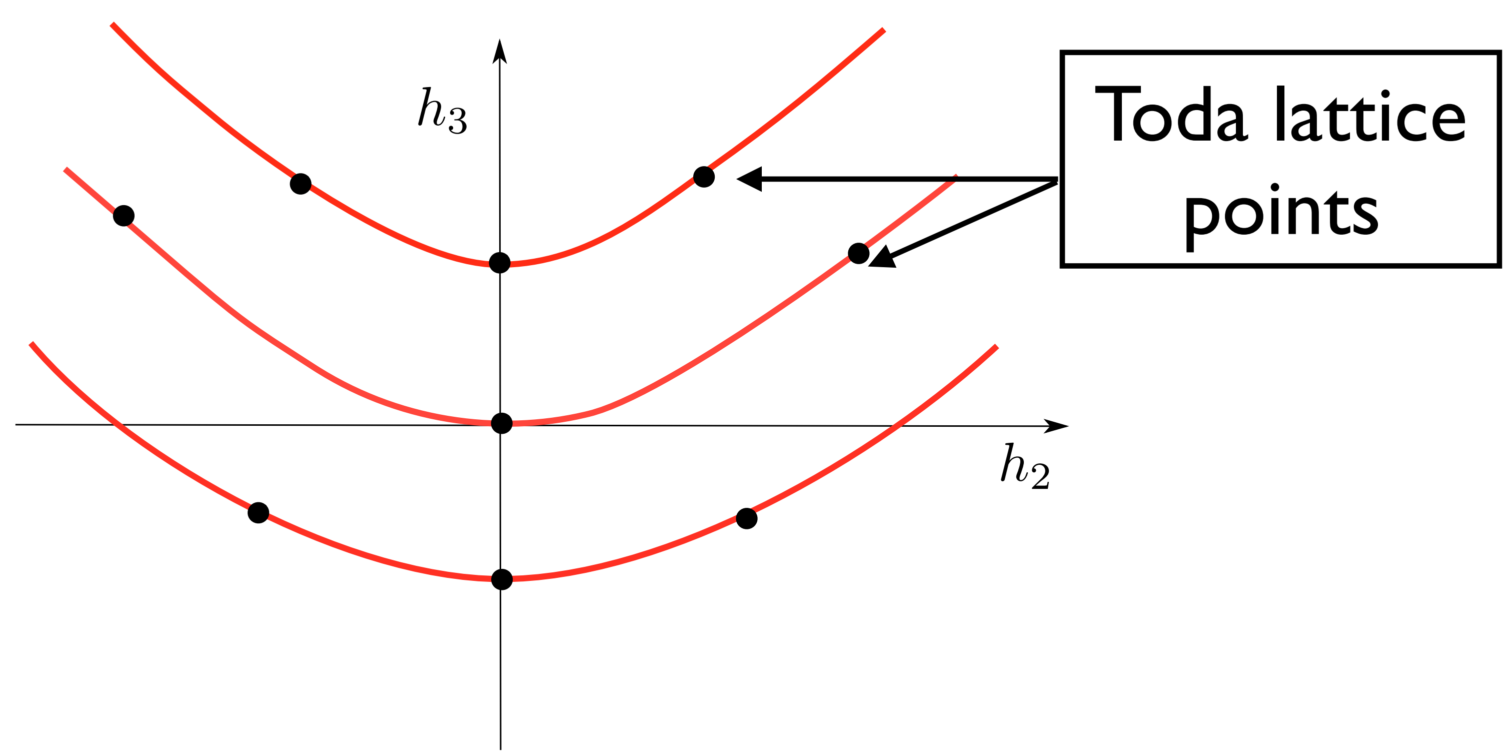
real eigenvalues $-h_4^{(n)}(h_2, h_3)$ $\quad n = 0, 1, 2, \cdots$

Quantization leads to a discrete family of codimension one submanifolds in SW moduli space

Relation to the Toda lattice

A very similar Hamiltonian appears in the Baxter equation of the Toda lattice [Gaudin-Pasquier]. In this case, the boundary conditions one imposes are much more restrictive: solutions only exist for a *discrete set of points* in moduli space, which give the eigenvalues of the commuting Hamiltonians of the quantum Toda lattice.

One can show that these points belong to the submanifolds defined by our quantization problem.



Exact quantization conditions

It turns out that this deformed version of QM is *exactly solvable* [Grassi-M.M.]: one can write down *exact quantization conditions* for *any* potential in terms of actual functions (no resummation needed).

In this problem, the Voros symbols involve the quantum periods of the SW curve:

$$a_i(h_k;\hbar) = \oint_{A_i} \lambda_{\mathrm{SW}}(\hbar) \qquad \lambda_{\mathrm{SW}} = p\,\mathrm{d}x$$

$$\frac{\partial F_{\mathrm{NS}}}{\partial a_i}(h_k;\hbar) = \oint_{B_i} \lambda_{\mathrm{SW}}(\hbar) \qquad i = 1, \cdots, N-1$$

Remarkably, instanton calculus provides a resummation of these formal series in terms of *convergent* expansions:

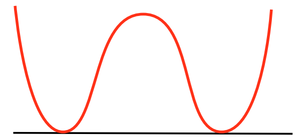
$$F_{\mathrm{NS}}(a_i;\hbar) = \sum_{d\geq 0} f_d(a_i;\hbar)\Lambda^{2dN}$$

$$h_k(a_i;\hbar) = \sum_{d\geq 0} h_{k,d}(a_i;\hbar)\Lambda^{2dN}$$

Alternatively, one can write TBA-like equations determining these functions [Nekrasov-Shatashvili, Kozlowski-Teschner].

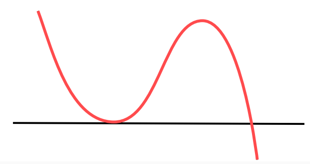
The EQC will involve these resummed quantum periods. It is given by the vanishing of a *single* function in SW moduli space (codimension one!). What is this function?

N even



$$\sum_{\boldsymbol{n}\in\mathcal{W}_N\cdot\boldsymbol{\gamma}} \exp\left(\frac{\mathrm{i}}{\hbar}\frac{\partial F_{\mathrm{NS}}}{\partial\boldsymbol{a}}\cdot\boldsymbol{n}\right)\prod_{\boldsymbol{\alpha}\in\Delta_+}\left(2\sinh\left(\frac{\pi\boldsymbol{a}\cdot\boldsymbol{\alpha}}{\hbar}\right)\right)^{-(\boldsymbol{n}\cdot\boldsymbol{\alpha})^2}=0$$

N odd



$$\sum_{\boldsymbol{n}\in\mathcal{W}_N\cdot\boldsymbol{\gamma}} \exp\left(\frac{\mathrm{i}}{\hbar}\frac{\partial F_{\mathrm{NS}}}{\partial\boldsymbol{a}}\cdot\boldsymbol{n}-\frac{\pi\boldsymbol{a}\cdot\boldsymbol{n}}{\hbar}\right)\prod_{\boldsymbol{\alpha}\in\Delta_+}\left(2\sinh\left(\frac{\pi\boldsymbol{a}\cdot\boldsymbol{\alpha}}{\hbar}\right)\right)^{-(\boldsymbol{n}\cdot\boldsymbol{\alpha})^2}=0$$

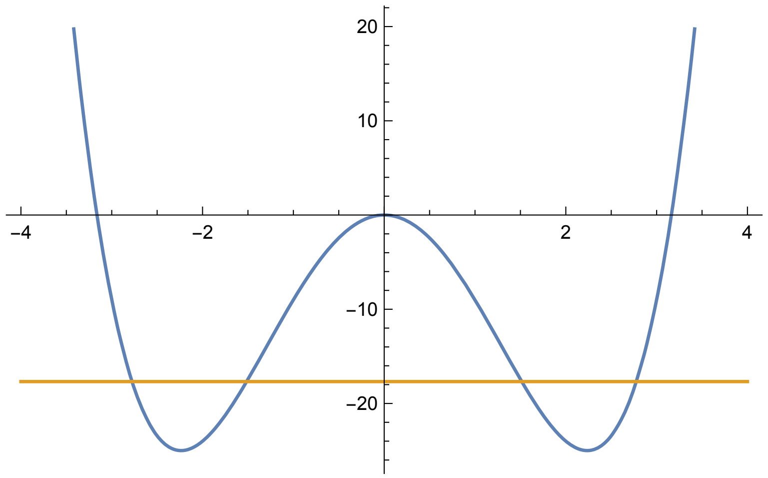
$$\gamma=\sum_{i=0}^{N-1}(-1)^{i-1}\boldsymbol{\lambda}_i \qquad\qquad \mathcal{W}_N\cdot\boldsymbol{\gamma}$$

Weyl orbit

fundamental weights of *SU(N)*

These EQCs can be effectively used to calculate energy levels, in *complete agreement* with numerical calculations of the spectrum

$$\mathsf{H}_4 = \Lambda^4\left(e^{\mathsf{p}} + e^{-\mathsf{p}}\right) + \mathsf{x}^4 - 10\mathsf{x}^2$$



Instantons	E_0	E_1
1	$-\underline{17.678}461910066723095$	$-\underline{17.67}800109302647040$
2	$-\underline{17.6784423}60054395243$	$-\underline{17.67798154}085570493$
3	$-\underline{17.6784423707}3924490$	$-\underline{17.6779815510}4316065$
4	$-\underline{17.678442370757}58180$	$-\underline{17.6779815510614}9680$
TBA	-17.67844237075748709	-17.67798155106140212
numerical	-17.67844237075748709	-17.67798155106140212

$$\Lambda = \hbar = 1$$

Derivation from the TS/ST correspondence

These EQCs are a consequence of the TS/ST correspondence, which relates (conjecturally) the spectral properties of quantum mirror curves to the BPS invariants of the toric CY [Grassi-Hatsuda-M.M., Codesido-Grassi-M.M.]

$$\boxed{\text{topological strings}} \xrightarrow[\text{geometric engineering}]{} \boxed{\text{SW theory}}$$

$$\mathrm{e}^{p} + \mathrm{e}^{-p} + \sum_{k=0}^{N} \mathrm{e}^{(N-2k)x} H_k = 0 \longrightarrow \mathrm{e}^{p} + \mathrm{e}^{-p} + \sum_{k=0}^{N} x^{N-k} h_k = 0$$

$$\sum_{\boldsymbol{n}\in\mathcal{W}_N\cdot\boldsymbol{\gamma}} \exp\left(\frac{\mathrm{i}}{\hbar}\frac{\partial F_{\mathrm{NS}}}{\partial \boldsymbol{a}}\cdot\boldsymbol{n}\right)\prod_{\boldsymbol{\alpha}\in\Delta_+}\left(2\sinh\left(\frac{\pi\boldsymbol{a}\cdot\boldsymbol{\alpha}}{\hbar}\right)\right)^{-(\boldsymbol{n}\cdot\boldsymbol{\alpha})^2}=0$$

from perturbative WKB for the quantum mirror curve

from *non-perturbative terms/* GV invariants

In contrast, the quantization conditions determining the Toda lattice points [Nekrasov-Shatashvili] only require perturbative information:

$$\exp\left(\frac{\mathrm{i}}{\hbar}\frac{\partial F_{\mathrm{NS}}}{\partial \boldsymbol{a}}\cdot\boldsymbol{\alpha}_i\right)=-1 \qquad i=1,\cdots,N-1$$

basic roots of *SU(N)*

Why Quantum Mechanics is hard

$$\boxed{\text{SW theory}} \xrightarrow{\text{Argyres-Douglas scaling}} \boxed{\text{anharmonic oscillators}}$$

The instanton sum expressing the quantum periods converges very badly near the AD points. So it is not straightforward to deduce EQCs for QM from our exact result for the quantum SW curve.

We anticipate however that existing EQCs for some QM problems, in terms of integral equations [Dorey-Tateo, Gaiotto], can be deduced from the TBA form of our EQCs.

Instanton calculus/topological vertex resummations have been crucial in obtaining exact solutions to spectral problems: they transform asymptotic series into convergent series (in some cases). However, they are tailored for the semiclassical/large radius regions in moduli spaces.

Are there similar resummations in other regions of moduli space?

Hierarchy of quantum problems

quantum mirror curves	e^{p}, e^{x}	easy
quantum SW curves	e^{p}, x	moderately easy
anharmonic oscillators	p, x	hard

Conclusions

The 100-year old problem of solving quantum anharmonic oscillators can be addressed in the context of topological string theory. For example, Voros symbols can be calculated with the holomorphic anomaly equation.

The intermediate stage between topological strings and ordinary quantum mechanics is the quantum SW curve, which provides a solvable deformation thereof (and a new testing ground for the TS/ST correspondence). Many generalizations are possible (matter content, gauge groups, …).

Solving QM oscillators amounts in this context to an exact determination of the quantum periods of SW theory near AD points.

Thank you for your attention!