Teichmüller TQFT, Complex Chern-Simons Theory and Duality

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V.M., arXiv:1710.04354;
D. Gang, V.M., M. Yamazaki, “to appear”.

- The subject of my talk will be Chern-Simons-like TQFTs in three dimensions that appear by reduction of the 6d $(2,0)$ theory on a lens space, or equivalently, that appear on the topological side of the 3d-3d correspondence.
- A lot is already known about them. There are explicit derivations by reduction from six dimensions, and there are state-integral models that compute partition functions of these theories on large classes of three-manifolds.
- However, a number of properties of these theories seem puzzling. In this talk, I will argue that some of these puzzles can be explained by S-duality and properties of the Kapustin-Witten PDEs.

Review: The 3d-3d Correspondence

- We start with a $(2,0)$ superconformal theory in six dimensions. These theories are labeled by ADE Dynkin diagrams, and for simplicity I will restrict to the case of A_1. This theory describes the low energy physics on two coincident M5-branes.

- The theory is put on a product six-manifold $M_3 \times L$ with a particular supergravity background that preserves at least two supercharges. Here M_3 is an arbitrary (smooth and oriented) three-manifold, and L is a three-manifold that must admit some extra geometric structure for the background to exist.

- I will restrict to the case when L is a degree k circle bundle over S^2. It will be denoted L_k, or sometimes L_k^b, to specify also a complex parameter b of the supergravity background.

- The two factors M_3 and L_k^b in the product are not on equal footing. Along M_3, the theory is topologically twisted, and supersymmetric observables are expected to be independent of the metric on M_3. Their dependence on the geometry of L_k^b is via the parameter b.

Review: The 3d-3d Correspondence

$$\begin{tikzcd} & \text{6d (2,0) superconformal theory on } M_3 \times L^b_k \arrow[dl, "\text{reduce on } M_3"'] \arrow[dr, "\text{reduce on } L^b_k"] & \\ \text{3d } \mathcal{N}=2 \text{ SCFT T}[M_3] \text{ on } L^b_k & & \begin{array}{c}\text{Chern-Simons-like TQFT on } M_3 \\ \text{with coupling constants determined by } k \text{ and } b\end{array} \end{tikzcd}$$

- Reducing the 6d theory on M_3, we expect to get a 3d $\mathcal{N}=2$ superconformal theory $\mathrm{T}[M_3]$. It is a topological invariant of M_3, valued in quantum field theories. To explicitly identify this theory is not an easy matter.

- For M_3 with an ideal triangulation, an algorithm for building a candidate theory $\mathrm{T}^{\mathrm{DGG}}[M_3]$ was proposed by [T. Dimofte, D. Gaiotto, S. Gukov, arXiv:1108.4389]. It is known that in general, $\mathrm{T}^{\mathrm{DGG}}[M_3]$ describes only a subsector of the full $\mathrm{T}[M_3]$. (Whether the full $\mathrm{T}[M_3]$ must exist as a superconformal 3d theory for an arbitrary M_3 is not entirely clear, see *e.g.* [D. Gang, K. Yonekura, arXiv:1803.04009].) Other proposals for $\mathrm{T}[M_3]$ for different classes of M_3 exist in the literature.

Review: The 3d-3d Correspondence

$$\begin{tikzcd}[column sep=small] & \text{6d (2,0) superconformal theory on } M_3 \times L_k^b \arrow[dl, "\text{reduce on } M_3"'] \arrow[dr, "\text{reduce on } L_k^b"] & \\ \text{3d } \mathcal{N}=2 \text{ SCFT T}[M_3] \text{ on } L_k^b & & \begin{array}{c}\text{Chern-Simons-like TQFT on } M_3 \\ \text{with coupling constants determined by } k \text{ and } b\end{array} \end{tikzcd}$$

- If we instead reduce the 6d theory on L_k^b, the result is a TQFT on M_3. For the 6d theory of type A_1, it is the $\mathrm{SL}(2,\mathbb{C})$ complex Chern-Simons theory,

  $$Z_{\mathrm{CS}} = \int_{\mathcal{C}} \mathrm{D}\mathcal{A}\, \mathrm{D}\widetilde{\mathcal{A}} \exp\left(\frac{i}{2}(k-iv)\mathrm{CS}(\mathcal{A}) + \frac{i}{2}(k+iv)\mathrm{CS}(\widetilde{\mathcal{A}})\right)\,,$$

  where the integer level k is the degree of the circle bundle L_k, and the complex level v also depends on the parameter b of the supergravity background,

  $$v = -ik\frac{b^2-1}{b^2+1}\,.$$

  [C. Cordova, D.L. Jafferis, arXiv:1305.2891; T. Dimofte, arXiv:1409.0857.]

- Part of the statement of the 3d-3d correspondence is equality of the partition functions of $\mathrm{T}[M_3]$ on L_k^b and complex Chern-Simons theory on M_3,

  $$Z_{T[M_3]}[L_{k,b}] = Z_{\mathrm{CS}}[M_3]\,.$$

Review: The 3d-3d Correspondence

- Let us further restrict to $k = 1$, that is, to L_k^b being the squashed three-sphere S_b^3. The TQFT then is a complex Chern-Simons theory at integer level $k = 1$.
- Interestingly, before Cordova and Jafferis derived this Chern-Simons theory from six dimensions by a supergravity computation, the correct TQFT was thought to be $\mathrm{SL}(2,\mathbb{R})$ or "$\mathrm{SL}(2,\mathbb{R})$-like" Chern-Simons theory.
  [Y. Terashima, M. Yamazaki, arXiv:1103.5748.]
- Indeed, consider this TQFT on a product $M_3 \simeq \mathbb{R} \times C$, for a two-manifold C. The space of states that the TQFT associates to C is the space of BPS states of the 6d theory on $\mathbb{R} \times C \times S_b^3$, or equivalently, the space of BPS states on $\mathbb{R} \times S_b^3$ of the 4d $\mathcal{N} = 2$ class S theory $\mathrm{T}[C]$. The usual AGT correspondence identifies this space with Liouville conformal blocks at central charge $c = 13 + 6(b^2 + b^{-2})$. [N. Nekrasov, E. Witten, arXiv:1001.0888.]
- It has been known since long ago that Liouville conformal blocks arise as the Hilbert space of $\mathrm{SL}(2,\mathbb{R})$-like Chern-Simons theory [H. Verlinde, 1990.]
- One goal of my talk will be to explain what this $\mathrm{SL}(2,\mathbb{R})$-like Chern-Simons theory really is, and why it is related to $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons with $k = 1$.

Review: State-Integral Models

- For M_3 with non-empty boundary and an ideal triangulation, it is known how to compute the partition function of complex Chern-Simons theory by a state-integral model. [T. Dimofte, arXiv:1409.0857; J.E. Anderson, R. Kashaev, arXiv:1409.1208; more references in the review: T. Dimofte, arXiv:1608.02961.]

- To each tetrahedron in the decomposition of M_3, one associates a factor of quantum dilogarithm of a local holonomy variable. Gluing corresponds to symplectic reduction in holonomy variables, which is implemented by integration and setting some variables to zero.

- Since Hilbert spaces of complex Chern-Simons theory in general are not finite-dimensional, the partition function for an arbitrary M_3 need not be finite. In fact, a good canonical integration cycle for the state-integral model can be constructed when a triangulation of M_3 admits a positive angle structure.

- The state-integral model is well-defined for $\mathrm{Re}\, b > 0$, or equivalently, for $b^2 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Review: State-Integral Models

- For example, for $k=1$ and $M_3 \simeq S^3 \setminus 4_1$, the partition function is

$$Z[S^3 \setminus 4_1] = \int_{\mathbb{R}+i\epsilon} \mathrm{d}\sigma \exp\left[-i\pi(\sigma - i(b+b^{-1})/2)^2\right] \psi_b^2(\sigma)\,,$$

  where $\psi_b(\sigma)$ is the Faddeev's quantum dilogarithm. (For simplicity, the meridian holonomy parameter was turned off, making the holonomy parabolic.)

- This partition function has interesting perturbative expansions in different regimes:
  - $b^2 \to \infty:$ $\quad 3^{-1/4} \exp\left(-b^2\left(\frac{V}{2\pi} - \frac{i\pi}{12}\right)\right)\left(1+O(b^{-2})\right);$
  - $b^2 \to 1+\epsilon:$ $\quad 3^{-1/2} e^{-3\pi i/4} \exp\left(-\frac{V}{2\pi} - \frac{i\pi}{12}\right) + 3^{-1/2} e^{\pi i/4} \exp\left(\frac{V}{2\pi} - \frac{i\pi}{12}\right) + O(\epsilon);$
  - $b^2 \to -1+i\epsilon:$ $\quad 3^{-1/2} e^{-i\pi/4} \exp\left(-(2i\epsilon^{-1}+1)\frac{V}{2\pi} + \frac{i\pi}{12}\right)(1+O(\epsilon))$
  $+3^{-1/2} e^{-i\pi/4} \exp\left((2i\epsilon^{-1}+1)\frac{V}{2\pi} + \frac{i\pi}{12}\right)(1+O(\epsilon)).$

  Here $V \approx 2.03$ is the hyperbolic volume of M_3 and $\frac{\pi}{12} \pm \frac{iV}{2\pi}$ are the Chern-Simons invariants of the two irreducible $\mathrm{PSL}(2,\mathbb{C})$ flat connections.

- The asymptotics at $b^2 \to -1$ is the perturbative expansion of the Cordova-Jafferis theory. The other two regimes also look as some Chern-Simons perturbative expansions, and I will explain what theories they correspond to.

Chern-Simons Theory from Six Dimensions

- Next I will present a quick derivation of these Chern-Simons theories from six dimensions. This will help to elucidate some of their properties.

$$\begin{tikzcd}[column sep=small] & \text{6d (2,0) superconformal theory on } M_3 \times L_k^b \arrow[dl, "\text{reduce on } M_3"'] \arrow[dr, "\text{reduce on } L_k^b"] & \\ \text{3d } \mathcal{N}=2 \text{ SCFT T}[M_3] \text{ on } L_k^b & & \begin{array}{c}\text{Chern-Simons-like TQFT on } M_3 \\ \text{with coupling constants determined by } k \text{ and } b\end{array} \end{tikzcd}$$

- Optimally, one would like to start with a 6d supergravity background and reduce on L_k^b to three dimensions. But I do not have an explicit description of this background. (A 5d background obtained by reduction on an S^1 fiber, for a particular choice of the metric, can be found in [C. Cordova, D.L. Jafferis, arXiv:1305.2891].)

- Instead, we will first take the size of M_3 to be small, so that the 6d theory can be viewed as a 3d $\mathcal{N} = 2$ theory. Such theory can be put in a known way on a squashed lens space background L_k^b. This background has some nice features, which we will *assume* to hold also when the size of M_3 is not small. With that assumption, it will be easy to see how complex Chern-Simons theory emerges from six dimensions.

Chern-Simons Theory from Six Dimensions

- We view L_k as a two-torus T^2 with coordinates φ_1 and φ_2 fibered over an interval $\mathcal{I}$ with coordinate y. Some one-cycles of T^2 shrink at the ends of the interval. According to [C. Closset *et al*, arXiv:1212.3388], to put a 3d $\mathcal{N}=2$ theory on this manifold, we need to specify a metric and an integrable transversely holomorphic foliation, compatible with the metric.

- For the metric we choose

$$\mathrm{d}s^2 = \mathrm{d}y^2 + \frac{a(y)}{\operatorname{Im}\tau(y)}\left((\mathrm{d}\varphi_1 - \operatorname{Re}\tau(y)\,\mathrm{d}\varphi_2)^2 + (\operatorname{Im}\tau(y)\,\mathrm{d}\varphi_2)^2\right) ,$$

  where the area $a(y)$ and the modular parameter $\tau(y)$ are arbitrary functions of y, required only to satisfy $a > 0$ and $\operatorname{Im}\tau > 0$, as well as some conditions at the ends of $\mathcal{I}$, for the metric to be smooth.

- To specify a THF compatible with the metric, it is enough to pick a unit vector field ξ. We take

$$\xi = i\frac{t-\bar{t}}{1+|t|^2}e_1 + \frac{t+\bar{t}}{1+|t|^2}e_2 + \frac{1-|t|^2}{1+|t|^2}e_3 ,$$

  where $t(y) \in \mathbb{C}\cup\infty$ is a y-dependent projective parameter, and

$$e_1 = \sqrt{\frac{\operatorname{Im}\tau}{a}}\partial_{\varphi_1} , \quad e_2 = \frac{1}{\sqrt{a\operatorname{Im}\tau}}(\operatorname{Re}\tau\partial_{\varphi_1} + \partial_{\varphi_2}) , \quad e_3 = \partial_y$$

  is a basis of unit vector fields.

Chern-Simons Theory from Six Dimensions

- So, our background is determined by the parameters $a(y)$, $\tau(y)$ and $t(y)$. One can show that the THF is integrable if there exists a constant Ψ such that

  $$t^2(y) = -\frac{\Psi - \overline{\tau}(y)}{\Psi - \tau(y)}\,, \qquad (*)$$

  This $\Psi \in \mathbb{C} \cup \infty$ is a new, y-independent parameter. Explicitly, the local holomorphic coordinate for our integrable THF is

  $$z = -\varphi_1 + \Psi\varphi_2 + \int \mathrm{d}y \sqrt{\frac{\mathrm{Im}\,\tau}{a}}\frac{2i}{t + t^{-1}}$$

  The formula $(*)$ fixes $t(y)$ in terms of $\tau(y)$ and Ψ up to a sign. Note that there are two branching points for $t(y)$ located at $\tau = \Psi$ and $\tau = \overline{\Psi}$.

- The 3d $\mathcal{N} = 2$ supergravity background determined by this THF preserves two supercharges of opposite R-charge, which anticommute to a translation by a complex Killing vector

  $$\{Q, \widetilde{Q}\} = \Psi\partial_{\varphi_1} + \partial_{\varphi_2}\,.$$

- It is known that the partition function of a 3d $\mathcal{N} = 2$ theory on a THF background does not depend on all the details of the geometry. When the parameters are varied, it only holomorphically depends on the Dolbeault cohomology class of the THF deformation [C. Closset *et al*, arXiv:1309.5876]. One can show that in our setting, any changes in $a(y)$, $\tau(y)$ and $t(y)$ for constant Ψ lead to trivial (Q-exact) deformations. The partition function depends on the geometry only through a holomorphic dependence on Ψ.

Chern-Simons Theory from Six Dimensions

$$(p_l, q_l) \bullet\!\xrightarrow{\quad\tau(y),\ \Psi\quad}\!\bullet (p_r, q_r)$$

- Assuming that in the 6d lift of this supergravity background we can also freely change $a(y)$ and $\tau(y)$, we take the size of the torus fiber to be small. Reducing the 6d $(2,0)$ theory on the torus fiber, we get a 4d $\mathcal{N}=4$ super Yang-Mills theory on $M_3 \times \mathcal{I}$ with the Kapustin-Witten twist.

- The modular parameter $\tau(y)$ becomes the gauge coupling. Since it depends on y, in general we have a Janus configuration. A (p, q) shrinking cycle at an end of the interval gives rise to a (p, q) fivebrane boundary condition. [D. Gaiotto, E. Witten, arXiv:0804.2902.]

- It is easy to show that the parameter $t(y)$ of the THF background becomes the Kapustin-Witten twisting parameter. The formula for $t(y)$ that appeared earlier can be rewritten as

$$\Psi = \operatorname{Re}\tau + i\,\frac{t^2-1}{t^2+1}\operatorname{Im}\tau\,,$$

  which means that Ψ becomes the canonical parameter of Kapustin and Witten. [A. Kapustin, E. Witten, hep-th/0604151.]

- Note that both $\tau(y)$ and Ψ change by Möbius transformations under large diffeomorphisms of T^2, or equivalently, the 4d S-duality action,

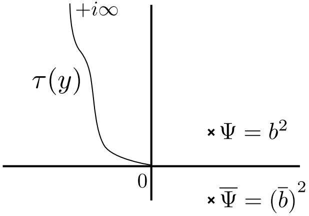
$$\tau \to \frac{a\tau+b}{c\tau+d}\,, \quad \Psi \to \frac{a\Psi+b}{c\Psi+d}\,.$$

Chern-Simons Theory from Six Dimensions

$$(0,1) \xrightarrow{\Psi = b^2} (1,0)$$

- Let us specialize to the case $k = 1$, that is, to the squashed three-sphere S^3_b. We can choose the coordinates on T^2 so that the cycle $(0,1)$ shrinks on the left and $(1,0)$ shrinks on the right.

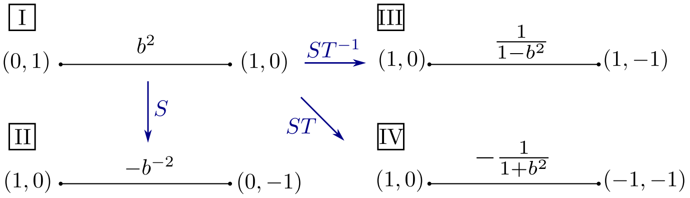
- With this choice of basis, one can show that $\Psi \equiv b^2$, where b is the usual parameter used for the squashed three-sphere background.



- The figure shows a possible profile of $\tau(y)$. The curve can be moved arbitrarily in the upper half-plane, with one end at $+i\infty$ and the other at 0, but it cannot cross the points Ψ and $\overline{\Psi}$, because these are the branching points for the function $t(y)$.

- We see that the background is well-defined for $b^2 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. This is precisely where the Chern-Simons state-integral model is defined.

Chern-Simons Theory from Six Dimensions



- In fact, for the S_b^3 background, there are four useful ways to choose the basis of one-cycles on T^2. In the 4d $\mathcal{N}=4$ super Yang-Mills, these choices are related by S-duality transformations. (Expressions above the lines in the figure above are values of Ψ.) In all these duality frames, the TQFT is some kind of Chern-Simons theory, as easily follows from Witten's construction of analytically-continued Chern-Simons via $\mathcal{N}=4$ super Yang-Mills [E. Witten, arXiv:1101.3216].

- More specifically, duality frames I and II lead to $SL(2,\mathbb{R})$-like Chern-Simons theories, which should be more appropriately called Teichmüller TQFT.

- Duality frames III and IV give complex $SL(2,\mathbb{C})$ Chern-Simons theories. The latter is the theory found by Cordova and Jafferis.

- Next let us look at all these theories in more detail, starting with Teichmüller TQFT.

Teichmüller TQFT

$$\underset{(0,1)}{\mathrm{D5}} \xrightarrow{\Psi = b^2} \underset{(1,0)}{\mathrm{NS5}}$$

- In this duality frame, we have the Langlands-twisted $\mathcal{N} = 4$ super Yang-Mills theory on $M_3 \times \mathcal{I}$ with the D5 and the NS5-type boundary conditions at the two ends $y = 0$ and $y = 1$. (We parameterize $\mathcal{I}$ by $y \in [0, 1]$.)

- We will explain, how this configuration leads to an $\mathrm{SL}(2, \mathbb{R})$-like Chern-Simons theory. For this part of the talk, we restrict to $b^2 \in \mathbb{R}^+$. This is the regime where the theory is unitary and can be understood most explicitly.

- The fields in 4d are the usual field content of the Kapustin-Witten twisted theory:
  - a connection A on a principal $\mathrm{SO}(3)$ bundle E;
  - a one-form ϕ valued in $\mathrm{ad}(E)$. It originates from four out of six scalars of the super Yang-Mills theory upon twisting;
  - two $\mathrm{ad}(E)$-valued scalar fields σ and $\overline{\sigma}$, as well as some fermions. For the most part, these fields will be unimportant.

Teichmüller TQFT

- The bulk part of the action is (almost) Q-exact and serves to impose localization on the solutions to the Kapustin-Witten equations on $M_3 \times \mathcal{I}$,

$$F - \phi \wedge \phi - \star_4 \mathrm{d}_A \phi = 0\,, \quad \mathrm{d}_A \star_4 \phi = 0\,,$$

where $\mathrm{d}_A = \mathrm{d} + [A, \cdot]$ is the gauge-covariant de Rham differential. (In general, the equations depend on the twisting parameter t. Since it only appears in Q-exact terms, we can set it to a convenient value, which we have chosen to be $t = -1$.)

- Let us choose the gauge $A_y = 0$. Also, the component ϕ_y of the adjoint one-form vanishes at both ends of the interval as a consequence of the boundary conditions. Then a simple integration by parts argument shows that it vanishes everywhere. The KW equations reduce to flow equations in y for a connection and an adjoint-valued one-form on M_3

$$\partial_y A = - \star_3 (\mathrm{d}_A \phi)\,, \quad \partial_y \phi = - \star_3 (F - \phi \wedge \phi)\,, \qquad (*)$$
$$\mathrm{d}_A \star_3 \phi = 0 \qquad (**) \quad .$$

These are sometimes called the reduced KW equations.

- Let $\Omega_{\mathcal{A}}(M_3)$ be the space of complex $\mathrm{PSL}(2, \mathbb{C})$ gauge fields on M_3. Equations $(*)$ are the downward gradient flow for the functional

$$h = \mathrm{Re}\,(i\mathrm{CS}(\mathcal{A}))$$

on $\Omega_{\mathcal{A}}(M_3)$, while the last equation $(**)$ is the zero moment map condition for the action of the complexified gauge group on $\Omega_{\mathcal{A}}(M_3)$.

$$\underset{(0,1)}{\mathrm{D5}} \xrightarrow{\quad \Psi = b^2 \quad} \underset{(1,0)}{\mathrm{NS5}}$$

- To be more precise, the bulk part of the action is not quite Q-exact, but also contains a Q-closed topological term $\sim i\Psi \int \mathrm{tr}(F \wedge F)$, but we will treat it as a boundary interaction. This topological term combines with the boundary terms at the NS5-type boundary at $y = 1$ into

$$I_{\mathrm{NS5}} = \frac{i\Psi}{4\pi} \int_{y=1} \mathrm{tr}\left(\mathcal{A} \mathrm{d} \mathcal{A} + \frac{2}{3} \mathcal{A}^3 \right) .$$

  Here $\mathcal{A} = A + i\phi$ is a complexified gauge field on $M_3 \times \{y = 1\}$, made out of restrictions of the bulk fields A and ϕ to the boundary $y = 1$.

Teichmüller TQFT

$$\underset{(0,1)}{\overset{\text{D5}}{\bullet}} \xrightarrow{\quad \Psi = b^2 \quad} \underset{(1,0)}{\overset{\text{NS5}}{\bullet}}$$

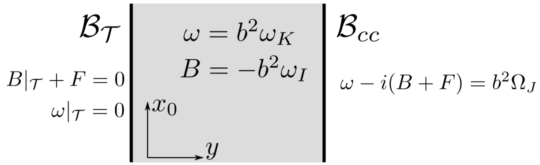
- The D5 boundary condition at $y = 0$ is the Nahm pole. First, it identifies the gauge bundle $\mathrm{ad}(E)|_{y=0}$ with the tangent bundle TM_3. Second, it requires the fields to behave according to a model singularity

$$y \to 0 : \quad A = \omega + \ldots \,, \quad \phi = \frac{e}{y} + \ldots \,,$$

  where ω and e are the Levi-Civita connection and the vielbein of the metric on M_3. Using the identification $\mathrm{ad}(E) \simeq TM_3$, ω and e can be viewed as a connection on $\mathrm{ad}E$ and an adjoint-valued one-form, hence the equations above make sense. The dots stay for higher order terms in $y \to 0$.
- The Nahm pole is a good boundary condition for the KW equations:
  - The Nahm pole solves the KW equations to the order $O(y^{-1})$ as a consequence of the zero-torsion equation $\mathrm{d}_\omega e = 0$ for the Levi-Civita connection;
  - This boundary condition is elliptic in the sense that the linear operator obtained from linearizing the KW equations (supplemented with a suitable gauge fixing condition) at an arbitrary Nahm pole solution is Fredholm [R. Mazzeo, E. Witten, arXiv:1311.3167].

Teichmüller TQFT: Hilbert Space

To understand what kind of 3d TQFT this setup gives us, let us look at the Hilbert space that it assigns to a two-manifold C. (Assume that the genus of C is $g \geq 2$.)

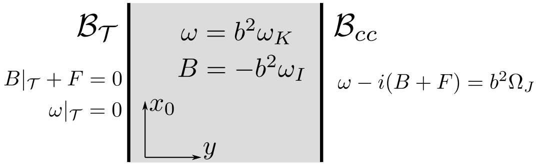


- For that, take $M_3 \simeq \mathbb{R}_{x_0} \times C$ and reduce on C. The result is a sigma-model on $\mathbb{R}_{x_0} \times \mathcal{I}_y$ with the target the SO(3) Hitchin's moduli space $\mathcal{M}_H$. Since we took $t = -1$, it is an A-model in complex structure K. The symplectic form is $\omega = b^2\omega_K$ and the B-field is $B = -b^2\omega_I$.

- The NS5 boundary condition at $y = 1$ reduces to the (B, A, A) canonical coisotropic brane, supported on the whole of $\mathcal{M}_H$, with physics governed by the J-holomorphic symplectic form $b^2\Omega_J = -\frac{ib^2}{4\pi}\int \mathrm{tr}(\delta\mathcal{A} \wedge \delta\mathcal{A})$.

- The D5 boundary condition at $y = 0$ reduces to a (B, A, A) Lagrangian brane supported on the Hitchin's section. (This is a statement about the moduli space of time-independent solutions to the KW equations in a half-space with the Nahm pole boundary condition. It is proved in the papers [D. Gaiotto, E. Witten, arXiv:1106.4789; S. He, R. Mazzeo, arXiv:1710.10645].)

Teichmüller TQFT: Hilbert Space



- Recall that the Hitchin's section is a component of the fixed point set of an I-holomorphic involution σ: $(A,\phi) \to (A,-\phi)$. It is isomorphic to the Teichmüller space $\mathcal{T}$ of C. The imaginary part of the holomorphic symplectic form Ω_J restricted to the Hitchin's section becomes the Weil-Petersson Kähler form ω_{WP} of $\mathcal{T}$ [N. Hitchin, 1986].

- Our brane configuration is an example of the Branes and Quantization setup [S. Gukov, E. Witten, arXiv:0809.0305]. The Hilbert space of our TQFT is the quantization of the Teichmüller space of C with the symplectic form $b^2\omega_{\mathrm{WP}}$. Equivalently, it is the space of Liouville conformal blocks of central charge $c = 13 + 6(b^2 + b^{-2})$. [H. Verlinde, 1990; J. Teschner, arXiv:1005.2846.]

- Hence this theory can be naturally called Teichmüller TQFT.

Teichmüller TQFT: Integration Cycle

$$\underset{\text{Nahm Pole}}{\bullet} \xrightarrow[\text{KW equations}]{} \underset{\exp(ib^2\mathrm{CS}(\mathcal{A}))}{\bullet}$$

- Consider again Teichmüller TQFT on a general three-manifold M_3. To compute the path-integral, we are supposed to solve the reduced KW equations
$$\partial_y A = -\star(\mathrm{d}_A\phi)\,, \quad \partial_y\phi = -\star(F-\phi\wedge\phi)\,, \quad \mathrm{d}_A\star\phi = 0\,.$$
on $M_3\times\mathcal{I}$ with the Nahm pole boundary condition
$$y\to 0: \quad A=\omega+\ldots\,, \quad \phi=\frac{e}{y}+\ldots$$
at $y=0$ and free boundary condition at the other end $y=1$. Evaluating KW solutions at $y=1$, we get a subspace $\mathcal{S}$ in the infinite-dimensional space of complex gauge fields $\Omega_{\mathcal{A}}(M_3)$. This subspace, informally, is middle-dimensional, because the Nahm pole boundary condition is elliptic and leaves free half of the modes. (Here I am being slightly imprecise about gauge symmetry.) Then it makes sense to integrate over $\mathcal{S}$ the top-dimensional holomorphic form $D\mathcal{A}$ on $\Omega_{\mathcal{A}}(M_3)$.

- The non-trivial part of the action is $ib^2\mathrm{CS}(\mathcal{A})$ evaluated at $y=1$, and the 4d path-integral reduces to
$$Z_{\text{Teichm}}[M_3] = \int_{\mathcal{S}} \mathrm{D}\mathcal{A}\exp\left(ib^2\mathrm{CS}(\mathcal{A})\right)\,.$$
It is an analytically-continued Chern-Simons path-integral over $\mathcal{S}$.

Teichmüller TQFT: Integration Cycle

$$Z_{\text{Teichm}}[M_3] = \int_{\mathcal{S}} \mathrm{D}\mathcal{A} \exp\left(ib^2 \mathrm{CS}(\mathcal{A})\right) .$$

- Let us look more closely at this integration cycle. First, for the path-integral to converge, the functional $h = \mathrm{Re}(i\mathrm{CS}(\mathcal{A}))$ should be bounded from above on $\mathcal{S}$. We conjecture that this is true for three-manifolds, for which the Teichmüller TQFT partition function is finite.

- Assuming this to be true, it must be possible to decompose $\mathcal{S}$ in Lefschetz thimbles,

$$\mathcal{S} = \sum_{\mathbb{a} \in \text{irred}} n_{\mathbb{a}} \, \mathcal{C}_{\mathbb{a}} .$$

  Here $\mathbb{a}$ labels *irreducible* flat $\mathrm{PSL}(2,\mathbb{C})$ bundles on M_3 modulo topologically trivial gauge transformations. Reducible critical points do not contribute to this sum (for M_3 of finite volume), as one can see by carefully keeping track of gauge invariance.

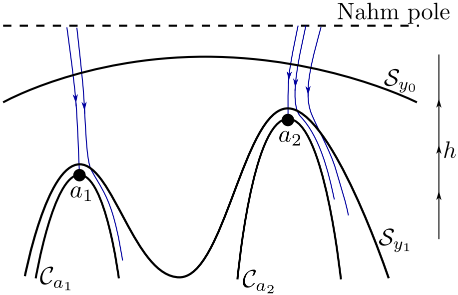
- The partition function decomposes accordingly,

$$Z_{\text{Teichm}} = \sum_{\mathbb{a} \in \text{irred}} n_{\mathbb{a}} \, Z_{\mathbb{a}} ,$$

  where $Z_{\mathbb{a}}$ is a Chern-Simons integral over a Lefschetz thimble. For $b^2 \to +\infty$, such integral has asymptotics

$$Z_{\mathbb{a}} \sim \exp\left(ib^2 \mathrm{CS}(\mathbb{a})\right) .$$

Teichmüller TQFT: Integration Cycle



- To find the decomposition coefficients $n_{\mathbb{a}}$, one takes the cycle $\mathcal{S}$ and applies the gradient flow to it. The integer $n_{\mathbb{a}}$ is the signed count of flows that start at $\mathcal{S}$ and end at a critical point $\mathbb{a}$ at $y \to \infty$. (Assuming that the counting problem can be defined. Also, we only consider isolated critical points, for simplicity.)

- Since $\mathcal{S}$ was itself defined by a gradient flow from the Nahm pole, the numbers $n_{\mathbb{a}}$ count flows in the half-space $y \in [0, \infty)$ from the Nahm pole at $y = 0$ to the critical point $\mathbb{a}$ at infinity.

- We note that this counting problem for the KW equations appears in the conjectural S-dual description of the analytically-continued Chern-Simons invariants [E. Witten, "Fiverbranes and Knots," arXiv:1101.3216].

Integration Cycle for a Hyperbolic Three-Manifold

- Next, suppose that M_3 has a complete hyperbolic metric. Then the integration cycle $\mathcal{S}$ can be identified very explicitly.

- Analyzing in examples the state-integral model partition functions for hyperbolic knot complements, [J.E. Andersen and R. Kashaev, arXiv:1109.6295] conjectured that they always have the asymptotics

$$b^2 \to +\infty : \quad |Z_{\text{Teichm}}[M_3]| \sim \exp\left(-\frac{b^2}{2\pi}\mathrm{V}[M_3]\right) ,$$

  where $V[M_3]$ is the hyperbolic volume. For example, the integral for the 4_1 knot complement is

$$Z[S^3 \setminus 4_1] \approx 3^{-1/4} \exp\left(-\frac{b^2}{2\pi}\left(V - \frac{i\pi^2}{6}\right)\right)\left(1 + O(b^{-2})\right) ,$$

  where $V \approx 2.03$ is the hyperbolic volume of $S^3 \setminus 4_1$. This is similar to the celebrated volume conjecture for the Jones polynomial, with the crucial difference that there is a minus sign in the exponent.

- It is believed that similar asymptotics hold for the partition function on any hyperbolic M_3. An analog of this statement for gauge group $\mathrm{SU}(N)$, $N \to \infty$, is supported by holography. [D. Gang, N. Kim and S. Lee, arXiv:1409.6206].

Teichmüller TQFT on a Hyberbolic Three-Manifold

- Recall that on a hyperbolic three-manifold, there are two special $\mathrm{PSL}(2,\mathbb{C})$ flat connections. If ω and e are the Levi-Civita connection and the vielbein for the hyperbolic metric, they are $\mathcal{A}_{\mathrm{geom}} = \omega + ie$ and $\mathcal{A}_{\overline{\mathrm{geom}}} = \omega - ie$. We have

$$i\mathrm{CS}(\mathcal{A}_{\overline{\mathrm{geom}}}) = -\frac{1}{2\pi}\mathrm{V}[M_3] + i\mathrm{CS}(\omega)\,.$$

  Moreover, the following inequality is true [A. Reznikov, "Rationality of secondary classes" (1996)],

$$-\frac{1}{2\pi}\mathrm{V}[M_3] \leq \mathrm{Re}\left(i\mathrm{CS}(\mathcal{A})\right) \leq \frac{1}{2\pi}\mathrm{V}[M_3]\,.$$

  (We will assume that the equality holds only for $\mathcal{A}_{\mathrm{geom}}$ and $\mathcal{A}_{\overline{\mathrm{geom}}}$, which should be true generically.)

- It means that the exponent that is observed in

$$b^2 \to +\infty : \quad |Z_{\mathrm{Teichm}}| \sim \exp\left(-\frac{b^2}{2\pi}\mathrm{V}[M_3]\right)$$

  is the most subleading one.

- It must be that our integration cycle $\mathcal{S}$ is just the Lefschetz thimble for (some lift of) $\mathcal{A}_{\overline{\mathrm{geom}}}$! (This implication of the Andersen-Kashaev conjecture for the Chern-Simons integration cycle was previously considered in [J. B. Bae, D. Gang and J. Lee, arXiv:1610.09259].)

Teichmüller TQFT on a Hyberbolic Three-Manifold

- Equivalently, it must be that the signed counts $n_{\mathbb{a}}$ of the KW solutions for irreducible $\mathbb{a}$ are almost all zero, except for $\mathbb{a}$ being (some particular lift of) $\mathcal{A}_{\overline{\mathrm{geom}}}$.
- For any hyperbolic M_3, there exists a very simple model solution to the KW equations on $M_3 \times \mathbb{R}^+$ with the Nahm pole, which is

  $$A = \omega\,, \quad \phi = -e \coth y\,.$$

  At $y \to +\infty$, the complex gauge field $\mathcal{A} = A + i\phi$ approaches precisely the flat connection $\mathcal{A}_{\overline{\mathrm{geom}}} = \omega - ie$! We propose the following natural conjecture.
- **Conjecture 1.** For a hyperbolic M_3, the KW equations with $t = -1$ on $\mathbb{R}^+ \times M_3$ with the Nahm pole boundary condition at $y = 0$ have precisely one solution that approaches an irreducible flat connection at $y \to +\infty$. This solution is the model solution above.

Teichmüller TQFT on a Hyberbolic Three-Manifold

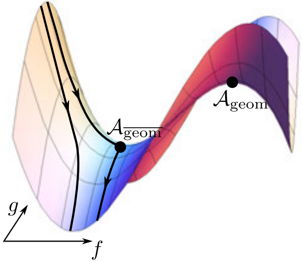
- I do not currently have a proof for Conjecture 1, but I have some further evidence for it. Reduced KW equations describe the gradient flow for the functional
$$h = \mathrm{Re}\,(i\mathrm{CS}(\mathcal{A}))$$
on the space of complex gauge fields $\Omega_{\mathcal{A}}(M_3)$. The function $h_0(y) = (\coth^3 y - 3\coth y)\mathrm{V}[M_3]$ is the evaluation of h on the model solution at a fixed $y \in \mathcal{I}$. Let $h(y)$ be the evaluation of h at y on some other solution.

- **Conjecture 2.** For a hyperbolic M_3, for any Nahm pole solution to the *reduced* KW equations on $M_3 \times \mathbb{R}^+$ other than the model solution, and for any $y > 0$, there is an inequality
$$h(y) < h_0(y)\,.$$
That is, any other solution lies below the model solution at every y.

- If this is true, no other solution can converge to a flat connection at $y \to +\infty$, because no flat connection lies below the conjugate geometric flat connection $\mathcal{A}_{\overline{\mathrm{geom}}} = \omega - ie$.

- Note that Conjecture 2 is formulated for the reduced KW equations. In this way, it prohibits solutions to the full KW equations that for $y \to \infty$ would approach an irreducible $\mathfrak{a}$, but does not prohibit approaching a reducible $\mathfrak{a}$. Also, all these conjectures apply only to the KW equations at $t = -1$ (or $t = 1$, which is equivalent up to $\phi \to -\phi$).

Teichmüller TQFT on a Hyberbolic Three-Manifold



- To illustrate this proposal, let us make a simple ansatz for the fields

$$A = \omega + g(y)e\,, \quad \phi = f(y)e\,. \quad (*)$$

The KW equations reduce to a pair of ODEs for the gradient flow on the plane (f, g) for the polynomial $h = f + g^2 f - \frac{1}{3} f^3$. The profile of this polynomial is shown on the figure. The model solution is the flow that descends along the ridge of the hill on the left. It is easy to show that any other flow that rolls down from the hill (which is equivalent to the Nahm pole boundary condition) does so faster. Thus, our claim is true for the simple reduction $(*)$.

- For the full system of KW PDEs, I tested Conjecture 2 by solving the equations perturbatively in y. The inequality $h(y) - h_0(y) < 0$ holds at least at the first two non-trivial orders.

Teichmüller TQFT: Quick Remarks

- Before we move to other topics, let me make two quick remarks about Teichmüller TQFT to put it into some broader physics context.

- First, it is sometimes said that the 3d TQFT with the space of Liouville conformal blocks as the Hilbert space is $\mathrm{PSL}(2,\mathbb{R})$ Chern-Simons. The reason is that the moduli space of $\mathrm{PSL}(2,\mathbb{R})$ flat connections on a genus $g \geq 2$ surface C decomposes according to the Euler number as

$$\mathcal{M}_{\mathrm{PSL}(2,\mathbb{R})} \simeq \bigcup_{d=-(2g-2)}^{2g-2} \mathcal{M}_d \,,$$

  and the top component $\mathcal{M}_{2g-2}$ here are the flat connections that come from the uniformization of C, hence $\mathcal{M}_{2g-2}$ is isomorphic to the Teichmüller space $\mathcal{T}$. So, in a geometry like $\mathbb{R} \times C$, Teichmüller TQFT is really a subsector of $\mathrm{PSL}(2,\mathbb{R})$ Chern-Simons. But on a general three-manifold, Teichmüller TQFT is just a different theory. The critical points that contribute to the path-integral are not $\mathrm{PSL}(2,\mathbb{R})$ flat connections but rather (on a hyperbolic M_3) the geometric $\mathrm{PSL}(2,\mathbb{C})$ flat connection.

Teichmüller TQFT: Quick Remarks

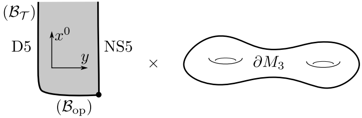
- Second, consider a three-manifold M_3 with a boundary ∂M_3 whose components all have negative Euler characteristic. Suppose that M_3 admits at least one (complete, geometrically finite) hyperbolic structure. Then [D. Sullivan, 1981] for every choice of the conformal structure on ∂M_3, that is, for every point $\mathbf{q}$ in the Teichmüller space $\mathcal{T}$ of ∂M_3, there exists precisely one hyperbolic structure $\mathcal{A}^{\mathbf{q}}_{\overline{\text{geom}}}$ on M_3.

- A natural extension of the Andersen-Kashaev conjecture to this setting would state that the path-integral over the Lefschetz thimble for $\mathcal{A}^{\mathbf{q}}_{\overline{\text{geom}}}$ computes the Teichmüller TQFT wavefunction on M_3 in Kähler quantization,

$$\psi_{M_3}(\mathbf{q}) := \langle \mathbf{q} | M_3 \rangle \,.$$

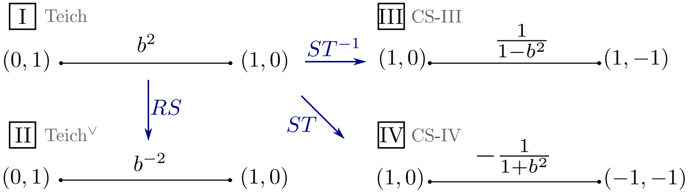
  The reason is that fixing the conformal structure at infinity is equivalent to putting an oper boundary condition for $\mathcal{A}_{\overline{\text{geom}}}$. It is easy to see that this creates the holomorphic eigenstate $\langle \mathbf{q} |$ in Teichmüller TQFT.

$\langle \mathbf{q} | \quad =$



- One well-known example of this story is the holographic setup for computing Virasoro conformal blocks. In that case, M_3 is a handlebody with a network of monodromy defects.

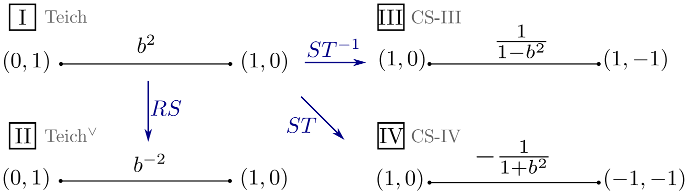
Duality



- Next let us look at the other S-duality frames shown in the figure. In each frame, we indicated the value of the canonical parameter Ψ in terms of the squashing parameter b^2. The D5 and the NS5 boundary conditions in 4d $\mathcal{N} = 4$ SYM are denoted by $(0, 1)$ and $(1, 0)$, respectively.

- The element $S \in \mathrm{SL}(2, \mathbb{Z})$, combined with a reflection, brings the Teichmüller TQFT setup back to itself and changes the level b^2 to $1/b^2$. The state-integral model is manifestly invariant under $b \to b^{-1}$. This basic S-duality was previously studied in [T. Dimofte, S. Gukov, arXiv:1106.4550].

Duality



- In the duality frames III and IV, we have NS5-type boundary conditions at both ends of the interval $\mathcal{I}$. (A $(\pm 1, \pm k)$ boundary condition is the same thing as $(\pm 1, 0)$ with an extra level k Chern-Simons term.) This leads to two complex Chern-Simons theories

$$\int_{\mathcal{C}} \mathrm{D}\mathcal{A}\,\mathrm{D}\widetilde{\mathcal{A}} \exp\left(\frac{i}{2}(k-iv)\mathrm{CS}(\mathcal{A}) + \frac{i}{2}(k+iv)\mathrm{CS}(\widetilde{\mathcal{A}})\right).$$

with coupling constants $\frac{1}{2}(k-iv) = -\Psi$ and $\frac{1}{2}(k+iv) = k+\Psi$, that is,

$$\mathrm{CS}-\mathrm{III}: \quad k=-1,\, iv = -\frac{b^2+1}{b^2-1}\,,$$

$$\mathrm{CS}-\mathrm{IV}: \quad k=1,\, iv = \frac{b^2-1}{b^2+1}\,.$$

- CS-IV is the theory of [C. Cordova and D. Jafferis, 1305.2891].
- Let us identify the integration cycles in both of these theories.

Integration Cycle for CS-III

- The path-integral is

$$\int_{\mathcal{C}} \mathrm{D}\mathcal{A}\, \mathrm{D}\widetilde{\mathcal{A}} \exp\left(-i\Psi \mathrm{CS}(\mathcal{A}) + i(\Psi - 1)\mathrm{CS}(\widetilde{\mathcal{A}})\right), \qquad \Psi = \frac{1}{1-b^2}\,.$$

- There seems to be a puzzle here. The squashed three-sphere background and the Teichmüller TQFT all are well-defined at $b^2 = 1$. The state-integral model partition function is analytic in a finite neighborhood of $b^2 = 1$. Yet for CS-III, $b^2 \to 1$ is the zero coupling limit where the level Ψ goes to infinity. Normally, a Chern-Simons partition function is not analytic at infinite level!

- These unusual properties can hold, if the integration cycle $\mathcal{C}$ is not the usual one. Let us decompose it into Lefschetz thimbles for $\mathcal{A}$ and $\widetilde{\mathcal{A}}$,

$$\mathcal{C} = \sum_{\mathbf{a},\mathbf{b}\in \mathrm{irred}} n_{\mathbf{a},\mathbf{b}}\, \mathcal{C}_{\mathbf{a}} \times \mathcal{C}_{\mathbf{b}}$$

  and find, what should be the coefficients $n_{\mathbf{a},\mathbf{b}}$. (For notational simplicity, I will ignore the fact that flat bundles here should be considered together with their lift to flat bundles modulo topologically trivial gauge transformations.)

- A quick remark is that the sum goes over irreducible flat connections only. This is the usual fact about complex Chern-Simons theories, which is due to infinite volumes of isotropy subgroups for reducible flat connections.

Integration Cycle for CS-III

$$Z_{\mathrm{CS-III}}[M_3] = \int_{\mathcal{C}} \mathrm{D}\mathcal{A}\,\mathrm{D}\widetilde{\mathcal{A}} \exp\left(-i\Psi \mathrm{CS}(\mathcal{A}) + i(\Psi - 1)\mathrm{CS}(\widetilde{\mathcal{A}})\right) ,$$

$$\Psi = \frac{1}{1-b^2} , \quad \mathcal{C} = \sum_{\mathbf{a},\mathbf{b}\in \mathrm{irred}} n_{\mathbf{a},\mathbf{b}}\, \mathcal{C}_{\mathbf{a}} \times \mathcal{C}_{\mathbf{b}} .$$

- The absence of essential singularity at $\Psi = \infty$ implies the following conditions,
  - The coefficient in front of Ψ in the action should vanish at every critical point that contributes. (This is not entirely obvious. For a proof, see [E. Witten, arXiv:1001.2933, sec.3.2.1].) Then $n_{\mathbf{a},\mathbf{b}} = 0$ for $\mathbf{a} \neq \mathbf{b}$, assuming no accidental coincidences.
  - This property of $n_{\mathbf{a},\mathbf{b}}$ should not be spoiled by Stokes jumps in going around $\Psi = \infty$. Assuming any two critical points are connected by a Stokes jump at some phase of Ψ, this implies that $n_{\mathbf{a},\mathbf{a}} = n_{\mathbf{b},\mathbf{b}}$ for all $\mathbf{a}$ and $\mathbf{b}$.
- This fixes the integration cycle (up to an overall coefficient) to be the diagonal sum over all irreducible flat connections,

$$\mathcal{C} = \sum_{\mathbf{a}\in \mathrm{irred}} \mathcal{C}_{\mathbf{a}} \times \mathcal{C}_{\mathbf{a}} . \quad (*)$$

- It turns out that Stokes phenomena cancel for this combination of Lefschetz thimbles for $b^2 > 0$. Thus, on this line the integration cycle is still $(*)$. Note that the CS level v on this line is imaginary.

Integration Cycle for CS-III

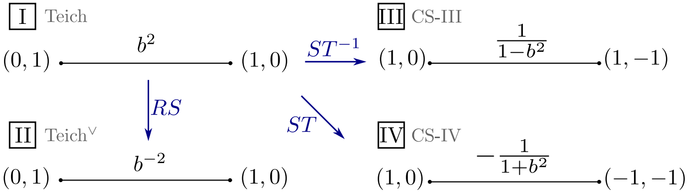
$$Z_{\mathrm{CS-III}}[M_3] = \sum_{\mathbf{a}\in \mathrm{irred}} \int_{\mathcal{C}_{\mathbf{a}}\times \mathcal{C}_{\mathbf{a}}} D\mathcal{A}D\widetilde{\mathcal{A}} \exp\left(-\frac{i}{4\pi}\Psi \mathrm{CS}(\mathcal{A}) + \frac{i}{4\pi}(\Psi-1)\mathrm{CS}(\widetilde{\mathcal{A}})\right) .$$

- At $b^2 = 1$ or $\Psi = \infty$, the partition function reduces to a one-loop exact formula
$$Z_{\mathrm{CS-III}}[M_3] = \sum_{\mathbf{a}\in \mathrm{irred}} \tau_{\mathbf{a}}^2 \exp\left(-i\mathrm{CS}(\mathbf{a})\right) , \qquad (*)$$
where $\tau_{\mathbf{a}}$ is the Ray-Singer torsion for the flat connection $\mathbf{a}$.

- For example, for $M_3 \simeq S^3 \setminus 4_1$ we have [S. Garoufalidis, R. Kashaev, arXiv:1411.6062]:
$$\begin{aligned} Z[S^3\setminus 4_1]|_{b^2=1} &= 3^{-1/2}e^{-3\pi i/4} \exp\left(-i\left(\frac{\pi}{12}-\frac{iV}{2\pi}\right)\right) \\ &+ 3^{-1/2}e^{\pi i/4} \exp\left(-i\left(\frac{\pi}{12}+\frac{iV}{2\pi}\right)\right) . \end{aligned}$$

- In fact, on the 3d-3d dual side, eq.$(*)$ is nothing but the formula of [C. Closset, H. Kim, B. Willett, arXiv:1701.03171] for the 3d $\mathcal{N}=2$ round S^3 partition function as a sum over Bethe vacua. That formula has an extension to all orders in the expansion in $b-1$, which on the Chern-Simons side corresponds to the full perturbative expansion in CS-III. [D. Gang, V. Mikhaylov, M. Yamazaki, "to appear".]

Integration Cycle for CS-IV



- Finally, for CS-IV, which has $k=1$ and $v=-i\frac{b^2-1}{b^2+1}$, it was predicted by Cordova and Jafferis that the integration cycle is the usual one, $\sum \mathcal{C}_{\mathbf{a}} \times \mathcal{C}_{\overline{\mathbf{a}}}$.

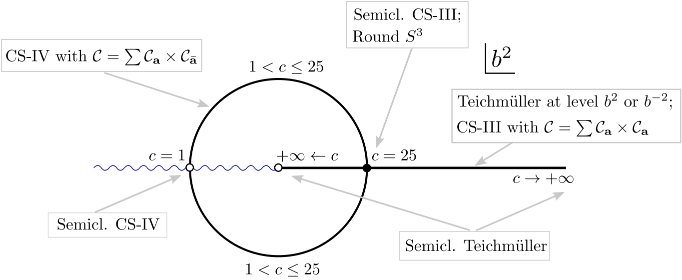
- (In fact, one can guess this from the fact that in configuration IV, the SYM boundary conditions at $y=0$ and $y=1$ are NS5 and $\overline{\text{NS5}}$-type, so $\mathcal{A}=A+i\phi$ and $\widetilde{\mathcal{A}}=A-i\phi$, up to a *positive* rescaling of ϕ.)

- The usual integration cycle makes sense for $v\in\mathbb{R}$, or $|b^2|=1$. The partition function has an essential singularity at $b^2=-1$, but this is OK, since our supergravity background is not well-defined for $b^2\in\mathbb{R}_{\leq 0}$.

- In agreement with this, for $M_3\simeq S^3\setminus 4_1$ numerical experimentation gives

$$\begin{aligned} Z[S^3\setminus 4_1] &= 3^{-1/2}e^{-i\pi/4}\exp\left(i\left((-2\epsilon^{-1}+i)\frac{V}{2\pi}+\frac{\pi}{12}\right)\right)(1+O(\epsilon)) \\ &+ 3^{-1/2}e^{-i\pi/4}\exp\left(i\left((2\epsilon^{-1}-i)\frac{V}{2\pi}+\frac{\pi}{12}\right)\right)(1+O(\epsilon))\,. \end{aligned}$$

Conclusions and Some Open Questions



- Understand the dualities at the level of mirror symmetry in the Hitchin's sigma-model. Can we interpret states in CS-III as an actual quantization of $\mathcal{M}_H$ with a real symplectic form? Teichmüller TQFT is unitary; how to see this unitarity in CS-III?

- At $c = 1$ and $c = 25$, Liouville conformal blocks are states in a B-model with a pair of space-filling (B, B, B) branes. What can we learn from this?

- Teichmüller TQFT on an interval with oper and conjugate-oper boundary conditions is Liouville. What are the corresponding 2d theories in the other two duality frames?

- We focused on S^3_b, that is, $|k| = 1$, but other values of k are also interesting, – in particular, $k = 0$ and $k = 2$.

Thank you!