Birational geometry for d-critical loci and wall-crossing in Calabi-Yau 3-folds

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1. Motivation

Donaldson-Thomas invariants

Let X be a smooth projective Calabi-Yau 3-fold, i.e. $K_X = 0$. Its Donaldson-Thomas invariant is defined by

$$\mathrm{DT}_\sigma(v) = \int_{[M_\sigma(v)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Here $v \in H^{2*}(X, \mathbb{Q})$ and σ is a Bridgeland stability condition on the derived category of coherent sheaves $D^b(X)$, and

$$M_\sigma(v)$$

is the moduli space of σ-stable objects E with $\mathrm{ch}(E) = v$.

Questions

1. What is the geometric relation of two moduli spaces $M_{\sigma^+}(v)$ and $M_{\sigma^-}(v)$? e.g. they are flip or flop in birational geometry?

2. What is the relation of $D^b(M_{\sigma^+}(v))$ and $D^b(M_{\sigma^-}(v))$?

These questions are related to wall-crossing phenoma of DT invariants.

However these questions do not make sense in usual birational geometry, because of possible bad singularities of $M_\sigma(v)$ (non-reduced, non-irreducible, etc).

In this talk, we study the above questions via d-critical birational geometry.

2. Background

Review of birational geometry

Let Y be a smooth projective variety. A minimal model program (MMP) of Y is a sequence of birational maps

$$Y = Y_1 \dashrightarrow Y_2 \dashrightarrow \cdots \dashrightarrow Y_{N-1} \dashrightarrow Y_N$$

such that each birational map $Y_i \dashrightarrow Y_{i+1}$ is a (generalized) flip

$$Y_i \xrightarrow{f_i^+} Z_i \xleftarrow{f_i^-} Y_{i+1}$$

i.e. $f_i^{\pm}$ are birational, $-K_{Y_i}$ is f_i^+-ample, and $K_{Y_{i+1}}$ is f_i^--ample.

An output Y_N is either a minimal model, i.e. K_{Y_N} is nef (i.e. $K_{Y_N} \cdot C \geq 0$ for any curve $C \subset Y_N$), or has a Mori fiber space structure $Y_N \to S$.

We can interpret MMP as a process minimizing the canonical divisor.

D/K principle

For a MMP,

$$Y = Y_1 \dashrightarrow Y_2 \dashrightarrow \cdots \dashrightarrow Y_{N-1} \dashrightarrow Y_N$$

if each Y_i is non-singular, Bondal-Orlov, Kawamata's D/K principle predicts the existence of fully-faithful functors of derived categories

$$D^b(Y_1) \hookleftarrow D^b(Y_2) \hookleftarrow \cdots \hookleftarrow D^b(Y_{N-1}) \hookleftarrow D^b(Y_N).$$

If this is true, then we can also interpret MMP as minimizing the derived category.

Bridgeland stability condition

For a variety X, a Bridgeland stability condition (mathematical framework of Douglas's Π-stability in physics) on $D^b(X)$ consists of data

$$Z \colon K(X) \to \mathbb{C}, \ \mathcal{A} \subset D^b(X)$$

where Z is a group homomorphism (central charge), $\mathcal{A}$ is the heart of a t-structure, satisfying some axioms, e.g.

$$Z(E) \subset \mathcal{H} \cup \mathbb{R}_{<0}, \ 0 \neq E \in \mathcal{A}$$

where $\mathcal{H}$ is the upper half plane. Given $\sigma = (Z, \mathcal{A})$, an object $E \in \mathcal{A}$ is σ-(semi)stable if for any non-zero subobject $F \subsetneq E$, we have

$$\arg Z(F) < (\leq) \arg Z(E), \quad \text{in } (0, \pi].$$

The space of stability conditions

Giving a pair $(Z, \mathcal{A})$ is also equivalent to giving data

$$Z \colon K(X) \to \mathbb{C},\ \mathcal{P}(\phi) \subset D^b(X),\ \phi \in \mathbb{R}$$

satisfying some axioms. Here $\mathcal{P}(\phi)$ is the subcategory of semistable objects with phase ϕ.

Bridgeland showed that the set of stability conditions

$$\mathrm{Stab}(X) = \{(Z, \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}})\}$$

forms a complex manifold. When X is a CY manifold, $\mathrm{Stab}(X)$ is expected to be related to the stringy Kahler moduli space of X.

Moduli spaces of (semi)stable objects

For an algebraic class v and a choice of a stability condition σ

$$v \in H^{2*}(X, \mathbb{Q}), \ \sigma \in \mathrm{Stab}(X)$$

on $D^b(X)$, let $M_\sigma(v)$ be the moduli space of σ-stable objects $E \in D^b(X)$ with $\mathrm{ch}(E) = v$.

The moduli space $M_\sigma(v)$ depends on a choice of σ. In general, we have wall-crossing phenomena: i.e. there is a wall-chamber structure on $\mathrm{Stab}(X)$ such that $M_\sigma(v)$ is constant if σ lies in a chamber but changes when σ crosses a wall.

Questions

Here is a more precise version of our main questions.

1. For two moduli spaces $M_{\sigma^+}(v)$, $M_{\sigma^-}(v)$, we can connect them by wall-crossing diagrams
$$M_{\sigma^+}(v) \dashrightarrow M_{\sigma_1}(v) \dashrightarrow M_{\sigma_2}(v) \dashrightarrow \cdots \dashrightarrow M_{\sigma^-}(v).$$
Is this a MMP?

2. Do we have a sequence of fully-faithful functors
$$D^b(M_{\sigma^+}(v)) \hookleftarrow D^b(M_{\sigma_1}(v)) \hookleftarrow D^b(M_{\sigma_2}(v)) \hookleftarrow \cdots \hookleftarrow D^b(M_{\sigma^-}(v))?$$

Issues

- When X is a K3 surface, these questions were solved by Bayer-Macri (1-st question), announced by Halpern-Leistner (2nd question). In this case, $M_\sigma(v)$ is non-singular holomorphic symplectic manifold, birational and derived equivalent under wall-crossing.
- However when X is a CY 3-fold, $M_\sigma(v)$ is singular (worse than (log) terminal singularities, (log) canonical singularities, etc). And even if they are smooth they are not birational in general.
- Therefore the questions on the relationships of $M_{\sigma^\pm}(v)$, $D^b(M_{\sigma^\pm}(v))$ do not make sense in a classical birational geometric viewpoint.

Idea of d-critical birational geometry

- Although the moduli space $M_\sigma(v)$ is usually highly singular, it has a special structure called d-critical structure, which is a classical shadow of (-1)-shifted symplectic structure in derived algebraic geometry.
- Instead of usual birational geometry, we introduce the concept of d-critical birational geometry. It involves the notion of d-critical flips, flops, etc. They are not birational maps of underlying spaces, but rather should be understood as virtual birational maps.
- We will see that several wall-crossing diagrams in CY 3-folds are described in terms of d-critical birational geometry.

3. D-critical birational geometry

(-1)-shifted symplectic structure

By [Pantev-Toen-Vaquie-Vezzosi], the moduli space $M_\sigma(v)$ on a CY 3-fold X is a truncation of a derived stack $\mathfrak{M}_\sigma(v)$ with a (-1)-shifted symplectic structure.

By [Ben-Bassat-Brav-Bussi-Joyce], the (-1)-shifted symplectic structure on $\mathfrak{M}_\sigma(v)$ is locally of Darboux form. This in particular implies that $M_\sigma(v)$ has local chart of the form

$$\begin{tikzcd} U = \{dw = 0\} \arrow[r] \arrow[d] & V \arrow[d, "w"] \\ M_\sigma(v) & \mathbb{A}^1 \end{tikzcd}$$

such that V is a smooth scheme.

d-critical structure

By [Joyce], there is a canonical sheaf $\mathcal{S}_M$ of $\mathbb{C}$-vector spaces on $M_\sigma(v)$ such that,

$$\mathcal{S}_M|_U = \mathrm{Ker}\left(\mathcal{O}_V/(dw)^2 \stackrel{d_{\mathrm{DR}}}{\to} \Omega_V/(dw)\cdot\Omega_V.\right)$$

The existence of a global (-1)-shifted symplectic form on $\mathfrak{M}_\sigma(v)$ implies that the local section $w \in \Gamma(U, \mathcal{S}_M|_U)$ glue to give a global section

$$s \in \Gamma(M_\sigma(v), \mathcal{S}_M).$$

The pair $(M_\sigma(v), s)$ is called a d-critical locus and s is called a d-critical structure of $M_\sigma(v)$.

D-critical birational transformation

Let $(M^{\pm}, s^{\pm})$ be two d-critical loci, e.g. $M^{\pm} = M_{\sigma^{\pm}}(v)$, and consider a diagram of schemes

$$M^+ \stackrel{\pi^+}{\to} A \stackrel{\pi^-}{\leftarrow} M^-.$$

The above diagram is called d-critical flip (flop) if for any $p \in A$, there exists an open neighborhood $p \in U \subset A$, a commutative diagram

$$\begin{tikzcd}
(\pi^{\pm})^{-1}(U) \arrow[r] \arrow[d] & Y^{\pm} \arrow[r, "\mathrm{id}"] \arrow[d, "f^{\pm}"] & Y^{\pm} \arrow[d, "w^{\pm}"] \\
U \arrow[r] & Z \arrow[r, "w"] & \mathbb{C}
\end{tikzcd}$$

where $Y^{\pm}$ are smooth, the left horizontal arrows are closed immersions, satisfying the following conditions:

D-critical birational transformation

- We have isomorphisms as d-critical loci

$$(\pi^{\pm})^{-1}(U) \stackrel{\cong}{\to} \{dw^{\pm} = 0\} \subset Y^{\pm}.$$

- The diagram

$$Y^{+} \stackrel{f^{+}}{\to} Z \stackrel{f^{-}}{\leftarrow} Y^{-}$$

  is a flip (flop) of smooth varieties, i.e. $f^{\pm}$ are birational, $-K_{Y^{+}}$ is f^{+}-ample and $K_{Y^{-}}$ is f^{-}-ample (resp. $K_{Y^{\pm}}$ are $f^{\pm}$-trivial).

The notion of d-critical divisorial contraction, d-critical Mori fiber space are also defined in a similar way.

Example: d-critical flip

Let us consider the following commutative diagram

$$\begin{tikzcd}
\widehat{\mathbb{C}^2} \arrow[r, "f"] \arrow[d, "w^+"'] & \mathbb{C}^2 \arrow[d, "g"'] & \mathbb{C}^2 \arrow[l, "\mathrm{id}"'] \arrow[d, "w^-"] \\
\mathbb{C} \arrow[r, "\mathrm{id}"] & \mathbb{C} & \mathbb{C} \arrow[l, "\mathrm{id}"']
\end{tikzcd}$$

where f is the blow-up at the origin and $g(u,v) = uv$. Then we have the diagram

$$M^+ = \{dw^+ = 0\} \to \operatorname{Spec} \mathbb{C} \leftarrow M^- = \{dw^- = 0\}.$$

The above diagram is a d-critical flip. We have $M^- = \operatorname{Spec} \mathbb{C}$, $(M^+)^{\mathrm{red}} = \mathbb{P}^1$ with non-reduced structures at $\{0\}$ and $\{\infty\}$.

Example: symmetric products of curves

Let C be a smooth projective curve with genus g, and let $S^k(C)$ be the k-th symmetric product of C. Let $\mathrm{Pic}^k(C)$ be the moduli space of line bundles on C with degree k. For each $n > 0$, we have the classical diagram of Abel-Jacobi maps

$$S^{n+g-1}(C) \stackrel{\pi^+}{\to} \mathrm{Pic}^{n+g-1}(C) \stackrel{\pi^-}{\leftarrow} S^{-n+g-1}(C).$$

Here the morphisms $\pi^{\pm}$ are given by

$$\pi^+(Z \subset C) = \mathcal{O}_C(Z), \ \pi^-(Z' \subset C) = \omega_C(-Z').$$

At a point in $\mathrm{Im}\,\pi^-$, the above diagram is a d-critical flip, though they are not birational.

4. Analytic neighborhood theorem

Moduli spaces of polystable objects

Let X be a smooth projective CY 3-fold, and $\mathcal{M}_\sigma(v)$ be the moduli stack of σ-semistable objects E with $\mathrm{ch}(E) = v$. If it is an algebraic stack of finite type, then there is a good moduli space (Alper-H. Leistner-Heinloth)

$$\mathcal{M}_\sigma(v) \to \overline{M}_\sigma(v)$$

where $\overline{M}_\sigma(v)$ is a proper algebraic space of finite type, which parametrizes σ-polystable objects E with $\mathrm{ch}(E) = v$, i.e. direct sum of the form

$$E = \bigoplus_{i=1}^{k} E_i^{\oplus m_i}, \ E_i \text{ is } \sigma\text{-stable}, \ \phi(E_i) = \phi(E_j), \ E_i \neq E_j.$$

We have an open immersion $M_\sigma(v) \subset \overline{M}_\sigma(v)$, which is an isomorphism if v is primitive and σ is general.

Analytic neighborhood theorem

Suppose that v is primitive, $\sigma \in \mathrm{Stab}(X)$ lies in a wall and $\sigma^{\pm}$ lie on its adjacent chambers. Then the open immersions

$$\mathcal{M}_{\sigma^+}(v) \subset \mathcal{M}_{\sigma}(v) \supset \mathcal{M}_{\sigma^-}(v)$$

induce the diagram of proper algebraic spaces

$$M_{\sigma^+}(v) \stackrel{\pi^+}{\to} \overline{M}_{\sigma}(v) \stackrel{\pi^-}{\leftarrow} M_{\sigma^-}(v).$$

Theorem (T)

For any $p \in \overline{M}_{\sigma}(v)$ corresponding to a polystable object $\oplus_{i=1}^{k} E_i^{\oplus m_i}$, there is an analytic open neighborhood $p \in \mathcal{U} \subset \overline{M}_{\sigma}(v)$, and isomorphisms as d-critical loci

$$(\pi^{\pm})^{-1}(\mathcal{U}) \stackrel{\cong}{\to} \mathrm{Rep}_{\theta^{\pm}}^{(Q,W)}(\vec{m}).$$

Analytic neighborhood theorem

- Q is a quiver with vertex $\{1, 2, \dots, k\}$, and the number of edges from i to j is the dimension of $\mathrm{Ext}^1(E_i, E_j)$.
- W is a convergent super-potential of Q, i.e. W is a formal super-potential which has a convergence radius.
- $\theta^{\pm}$ are stability conditions on Q-representations, given by the central charge

$$K(\mathrm{Rep}Q) \to \mathbb{C}, \ S_i \mapsto Z^{\pm}(E_i).$$

  Here S_i is the simple Q-representation corresponding to the vertex i, $Z^{\pm}$ are the central charges for $\sigma^{\pm}$.
- $\mathrm{Rep}_{\theta^{\pm}}^{(Q,W)}(\vec{m})$ is the moduli space of $\theta^{\pm}$-stable Q-representations satisfying the relation ∂W with dimension vector $\vec{m} = (m_i)_{1 \le i \le k}$ which is a complex analytic space.

Analytic neighborhood theorem

We have

$$\mathrm{Rep}^{(Q,W)}_{\theta^{\pm}}(\vec{m}) = \{d(\mathrm{tr}W) = 0\} \subset \mathrm{Rep}^{Q}_{\theta^{\pm}}(\vec{m}) \stackrel{\mathrm{tr}W}{\to} \mathbb{C}.$$

Here $\mathrm{Rep}^{Q}_{\theta^{\pm}}(\vec{m})$ is the moduli space of $\theta^{\pm}$-stable Q-representations without the relation ∂W. As $\mathrm{Rep}^{Q}_{\theta^{\pm}}(\vec{m})$ is smooth and birational, we have the following corollary:

Corollary

The diagram

$$M_{\sigma^{+}}(v) \stackrel{\pi^{+}}{\to} \overline{M}_{\sigma}(v) \stackrel{\pi^{-}}{\leftarrow} M_{\sigma^{-}}(v).$$

is a d-critical flip (flop) at a point $p = \oplus_{i=1}^{k} E_i^{\oplus m_i}$ *if*

$$\mathrm{Rep}^{Q}_{\theta^{+}}(\vec{m}) \dashrightarrow \mathrm{Rep}^{Q}_{\theta^{-}}(\vec{m})$$

is a flip (flop) of smooth varieties.

5. Wall-crossing of stable pair moduli spaces

Stable pair moduli spaces

Let X be a CY 3-fold. By definition, a Pandharipande-Thomas stable pair is data

$$(F, s), \ s \colon \mathcal{O}_X \to F$$

where F is a pure one dimensional coherent sheaf on X and s is surjective in dimension one. For $n \in \mathbb{Z}$ and $\beta \in H_2(X, \mathbb{Z})$, the moduli space of stable pairs

$$P_n(X, \beta) = \{(F, s) : [F] = \beta, \chi(F) = n\}$$

is a projective scheme. It is isomorphic to the moduli space of two term complexes $(\mathcal{O}_X \to F)$ in the derived category.

The case of irreducible curve class

Suppose that β is an irreducible curve class, i.e. it is not written as $\beta_1 + \beta_2$ for effective classes β_i. In this case, we have the diagram

$$P_n(X, \beta) \stackrel{\pi^+}{\to} M_n(X, \beta) \stackrel{\pi^-}{\leftarrow} P_{-n}(X, \beta).$$

Here $M_n(X, \beta)$ is the moduli space of one dimensional Gieseker stable sheaves F on X with $([F], \chi(F)) = (\beta, n)$, and the maps $\pi^{\pm}$ are defined by

$$\pi^+(F, s) = F, \ \pi^-(F', s') = \mathcal{E}xt^2_X(F', \mathcal{O}_X).$$

The above diagram is isomorphic to the wall-crossing diagram

$$M_{\sigma^+}(v) \to M_{\sigma}(v) \leftarrow M_{\sigma^-}(v)$$

for weak stability conditions, and $v = (1, 0, -\beta, -n)$.

SOD for stable pair moduli spaces

Theorem (T)

For an irreducible curve class β and $n > 0$, we have the following:

1. The diagram

$$P_n(X,\beta) \stackrel{\pi^+}{\to} M_n(X,\beta) \stackrel{\pi^-}{\leftarrow} P_{-n}(X,\beta).$$

is a d-critical flip.

2. Suppose that $M_n(X,\beta)$ is non-singular. Then $P_{\pm n}(X,\beta)$ are also non-singular, and we have the following semiorthogonal decomposition

$$D^b(P_n(X,\beta)) =$$

$$\langle \overbrace{D^b(M_n(X,\beta)), \dots, D^b(M_n(X,\beta))}^{n}, D^b(P_{-n}(X,\beta)) \rangle.$$

Remarks

- The SOD of $D^b(P_n(X,\beta))$ gives a categorification of the wall-crossing formula of PT invariants

$$P_{n,\beta} - P_{-n,\beta} = (-1)^{n-1} n N_{n,\beta}$$

  where $P_{n,\beta}$, $N_{n,\beta}$ are the integrations of the zero dimensional virtual classes on $P_n(X,\beta)$, $M_n(X,\beta)$. The above formula is used to show the rationality of the generating series of PT invariants.

- The existence of fully-faithful functor

$$D^b(P_{-n}(X,\beta)) \hookrightarrow D^b(P_n(X,\beta))$$

  indicates d-critical version of D/K principle.

In a general case

When β is not necessary irreducible, there exist sequences of wall-crossing diagrams in $D^b(X)$

$$P_n(X,\beta) = M_1 \dashrightarrow M_2 \dashrightarrow \cdots \dashrightarrow M_N,$$
$$P_{-n}(X,\beta) = M_{N+N'} \dashrightarrow M_{N+N'-1} \dashrightarrow \cdots \dashrightarrow M_N$$

where each dotted arrows are d-critical flips. The above diagrams were used to show the formula (T)

$$\sum_{n,\beta} P_{n,\beta} q^n t^\beta = \prod_{n>0,\beta>0} \exp((-1)^{n-1} n N_{n,\beta} q^n t^\beta) \left(\sum_{n,\beta} L_{n,\beta} q^n t^\beta \right)$$

which implies the rationality of PT generating series. Here $L_{n,\beta}$ is the integration of the virtual classes on M_N.

6. Categorification of Kawai-Yoshioka formula

Stable pairs on K3 surfaces

Let S be a smooth projective K3 surface such that

$$\mathrm{Pic}(S) = \mathbb{Z}[\mathcal{O}_S(H)]$$

for an ample divisor H on S with $H^2 = 2g - 2$. Let

$$\pi \colon \mathcal{C} \to |H| = \mathbb{P}^g$$

be the universal curve and

$$\mathcal{C}^{[n+g-1]} \to \mathbb{P}^g$$

the π-relative Hilbert schemes of $(n + g - 1)$-points. It is known that $\mathcal{C}^{[n+g-1]}$ is isomorphic to the stable pair moduli space $P_n(S, [H])$ on S.

Moduli spaces on K3 surfaces

For each $k \geq 0$, let M_k be the moduli space of H-Gieseker stable sheaves E on S such that

$$v(E) := \mathrm{ch}(E)\sqrt{\mathrm{td}_S} = (k, [H], k+n) \in H^{2*}(S, \mathbb{Z}).$$

The moduli space M_k is an irreducible holomorphic symplectic manifold.

We also denote by $\mathcal{P}_k$ be the moduli space of coherent systems

$$\mathcal{P}_k = \{(E, s) : [E] \in M_k,\ 0 \neq s \colon \mathcal{O}_S \to E\}.$$

Kawai-Yoshioka proved that $\mathcal{P}_k$ is a smooth projective variety.

Diagrams of moduli spaces on K3 surfaces

We have morphisms

$$\mathcal{P}_k \stackrel{\pi_k^+}{\to} M_k \stackrel{\pi_k^-}{\leftarrow} P_{k+1}$$

where $\pi_k^{\pm}$ are defined by

$$\pi_k^+(E, s) = E,\ \pi_k^-(E', s') = \mathrm{Cok}(s').$$

For $N \gg 0$, we have the sequence of diagrams

$$\mathcal{C}^{[n+g-1]} = \mathcal{P}_0 \dashrightarrow \mathcal{P}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{P}_{N-1} \dashrightarrow \mathcal{P}_N = \emptyset.$$

The above diagram multiplied by an elliptic curve C is identified with wall-crossing diagram in $D^b(X)$, where X is a CY 3-fold $X = S \times C$.

SOD of stable pairs on K3 surfaces

Theorem (T)

- The sequence

$$\mathcal{C}^{[n+g-1]} = \mathcal{P}_0 \dashrightarrow \mathcal{P}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{P}_{N-1} \dashrightarrow \mathcal{P}_N = \emptyset.$$

  is a sequence of d-critical flips.

- We have the SOD

$$D^b(\mathcal{P}_k) = \langle \overbrace{D^b(M_k), D^b(M_k), \ldots, D^b(M_k)}^{n+2k}, D^b(\mathcal{P}_{k+1}) \rangle.$$

Relation to Kawai-Yoshioka formula

Corollary

The moduli space of stable pairs $P_n(S, [H]) = \mathcal{C}^{[n+g-1]}$ *admits a SOD*

$$D^b(\mathcal{C}^{[n+g-1]}) = \langle \mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_N \rangle$$

where each $\mathcal{A}_k$ *admits a SOD* $\mathcal{A}_K = \langle \overbrace{D^b(M_k), D^b(M_k), \dots, D^b(M_k)}^{n+2k}\rangle$.

The SOD of $\mathcal{C}^{[n+g-1]}$ categorifies Kawai-Yoshioka's formula

$$e(\mathcal{C}^{[n+g-1]}) = \sum_{k=0}^{N} (n+2k) e(M_k).$$

Together with $e(M_k) = e(\mathrm{Hilb}^{g-k(k+n)}(S))$ and Gottsche formula, the above formula led to the Katz-Klemm-Vafa formula for stable pairs with irreducible curve classes.

Thank you very much