Knot Categorification from Geometry, via String Theory

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Based on joint work to appear with Andrei Okounkov, and further work with Dimitri Galakhov.

This talk is about
categorification of knot invariants.

I will explain how
a six dimensional "little" string theory
leads to a unified approach to the problem.

More precisely,

we will find that it leads to

three a-priori different approaches,

two of which are based in geometry.

String theory also explains how to relate them.

Some of the approaches we will find,

have the same flavor as those in work of

Kamnitzer and Cautis,

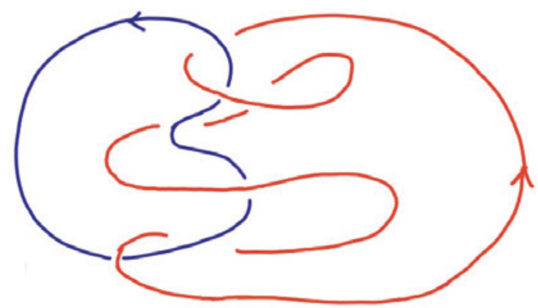
Seidel and Smith,

Webster

although it is possible that details differ.

To begin with, it is useful to recall
some well known aspects of knot invariants.

To get a quantum invariant of a link K



one starts with a Lie algebra,

$$^L\mathfrak{g}$$

and a coloring

of its strands by representation of $^L\mathfrak{g}$.

The link invariant,

in addition to the choice of a group

$$^{L}\mathfrak{g}$$

and

representations,

depends on one parameter

$$\mathfrak{q} = e^{\frac{2\pi i}{\kappa}}$$

Witten showed in his famous ’89 paper,

that the knot invariant

comes from

Chern-Simons theory with gauge group based on the Lie algebra

$$ {}^L\mathfrak{g} $$

and (effective) Chern-Simons level

$$ \kappa $$

In the same paper, he showed that underlying Chern-Simons theory is a two-dimensional conformal field theory, with

$$\widehat{{}^L\mathfrak{g}}_\kappa$$

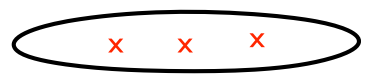
affine current algebra symmetry.

The space conformal blocks of

$$\widehat{{}^L\mathfrak{g}_\kappa}$$

on a Riemann surface $\mathcal{A}$ with punctures

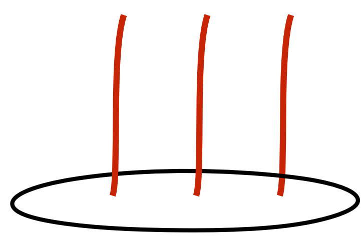
$\mathcal{A}$



is the Hilbert space of Chern-Simons theory on

$$\mathcal{A} \times \text{time}$$

$\mathcal{A}$



To eventually get invariants of knots in $\mathbb{R}^3$ or S^3

we want to take

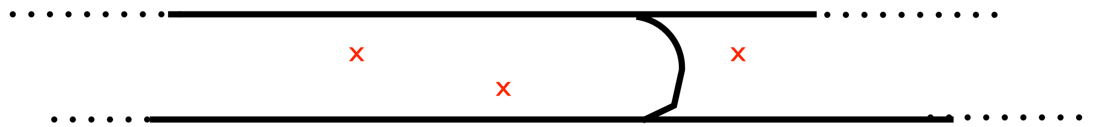
$$\mathcal{A}$$

to be a complex plane with punctures,

$\mathcal{A}$

or equivalently, a punctured infinite cylinder.

The corresponding $\widehat{{}^L\mathfrak{g}}_\kappa$ conformal blocks



are correlators of chiral vertex operators:

$$\Psi(a_1, \ldots, a_n) = \langle \mu | \; \Phi_{{}^L\rho_1}(a_1) \cdots \Phi_{{}^L\rho_n}(a_n) \; | \mu' \rangle$$

A chiral vertex operator

$$\Phi_{{}^L\rho_I}(a_I)$$

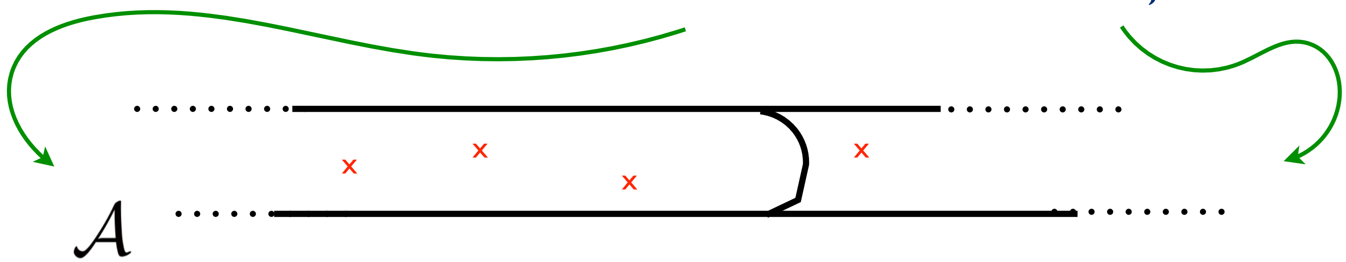
associated to a finite dimensional representation

$${}^L\rho_I$$

of ${}^L\mathfrak{g}$ adds a puncture at a finite point on $\mathcal{A}$

$$x = a_I$$

Punctures at $x = 0$ and ∞ ,



are labeled by a pair of highest weight vectors

$$|\mu\rangle \quad \text{and} \quad |\mu'\rangle$$

of Verma module representations of the algebra.

$$\Psi(a_1, \ldots, a_n) = \langle\mu|\ \Phi_{L_{\rho_1}}(a_1) \cdots \Phi_{L_{\rho_n}}(a_n)\ |\mu'\rangle$$

The space of conformal blocks corresponding to

$$\Psi(a_1,\ldots,a_n) = \langle\mu|\ \Phi_{{}^L\rho_1}(a_1)\cdots\Phi_{{}^L\rho_n}(a_n)\ |\mu'\rangle$$

is a vector space.

Its dimension is that of the weight

$$\nu = \mu - \mu'$$

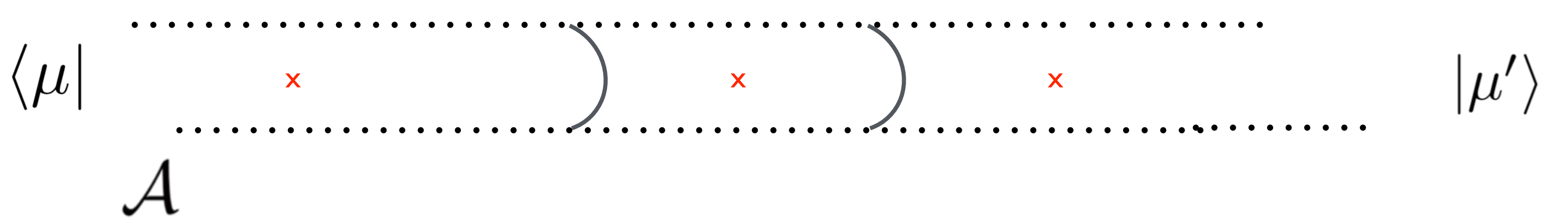
subspace of representation

$${}^L\rho = \otimes_I {}^L\rho_I$$

The finite dimensional space of conformal blocks

$$\Psi(a_1, \ldots, a_n) = \langle \mu | \, \Phi_{L_{\rho_1}}(a_1) \cdots \Phi_{L_{\rho_n}}(a_n) \, | \mu' \rangle$$

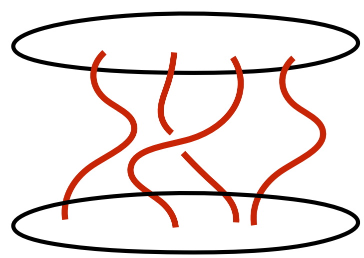
comes from choices of intermediate Verma module representations when one sews the chiral vertex operators together.



The Chern-Simons path integral on

$$\mathcal{A} \times \text{interval}$$

in the presence of a braid



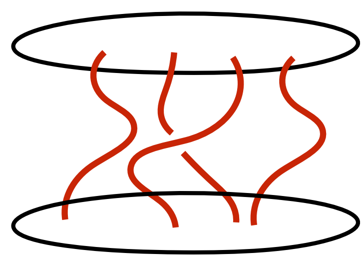
$$\mathfrak{B} = \mathfrak{B}(B)$$

gives the corresponding

quantum braid invariant.

The braid invariant one gets

$$\mathfrak{B} = \mathfrak{B}(B)$$

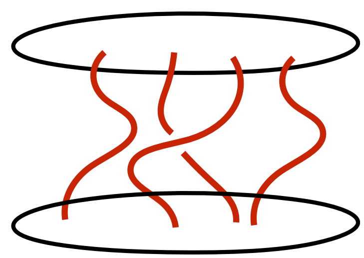


is a matrix that describes

transport of the space of conformal blocks,

along the braid B

To understand what it means to the transport the space of conformal blocks along a path corresponding to the braid



a different perspective on

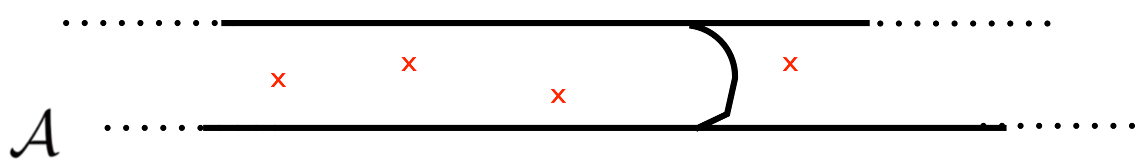
$$\widehat{{}^L\mathfrak{g}}_\kappa$$

conformal blocks is helpful.

Instead of characterizing $\widehat{{}^L\mathfrak{g}_\kappa}$ conformal blocks

$$\Psi(a_1,\ldots,a_n) = \langle\mu|\ \Phi_{{}^L\rho_1}(a_1)\cdots\Phi_{{}^L\rho_n}(a_n)\ |\mu'\rangle$$

in terms of vertex operators and sewing,



we can equivalently describe them as solutions to a differential equation.

The equation solved by conformal blocks of $\widehat{{}^L\mathfrak{g}}_\kappa$,

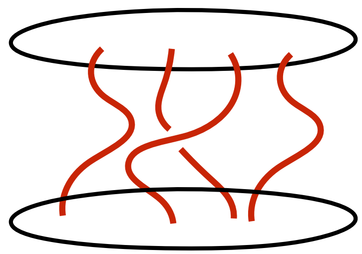
$$\Psi(a_1, \ldots, a_n) = \langle\mu|\ \Phi_{{}^L\rho_1}(a_1) \cdots \Phi_{{}^L\rho_n}(a_n)\ |\mu'\rangle$$

is the Knizhnik-Zamolodchikov equation:

$$\kappa\, a_I \frac{\partial}{\partial a_I} \Psi = \left(\sum_{J \neq I} r_{IJ}(a_I/a_J) + r_{I0} + r_{I\infty} \right) \Psi$$

The quantum invariant of the braid

$$\mathfrak{B}(B)$$



is the monodromy matrix of the Knizhnik-Zamolodchikov equation, along the path in the parameter space corresponding to braid B .

The monodromy problem of the $\widehat{{}^L\mathfrak{g}}_\kappa$ Knizhnik-Zamolodchikov equation

$$\kappa\, a_I \frac{\partial}{\partial a_I} \Psi = \sum_{J \neq I} r_{IJ}(a_I/a_J)\, \Psi$$

was solved by Drinfeld and Kohno in ’89.

They showed that its monodromy matrices are given in terms of the R-matrices of the quantum group

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

corresponding to ${}^L\mathfrak{g}$

Action by monodromies

turns the space of conformal blocks into a module for the

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

quantum group in representation,

$${}^L\rho = \otimes_I {}^L\rho_I$$

The representation ${}^L\rho$ is viewed here as a representation of $U_{\mathfrak{q}}({}^L\mathfrak{g})$ and not of ${}^L\mathfrak{g}$, but we will denote by the same letter.

The monodromy action

is irreducible only in the subspace of

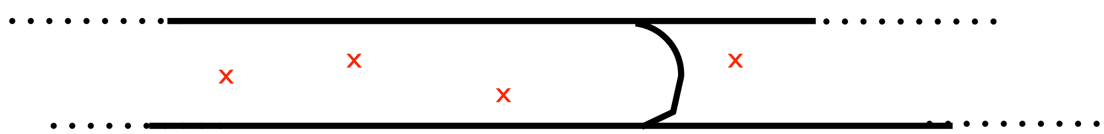
$$^L\rho = \otimes_I {}^L\rho_I$$

of fixed weight

$$\nu$$

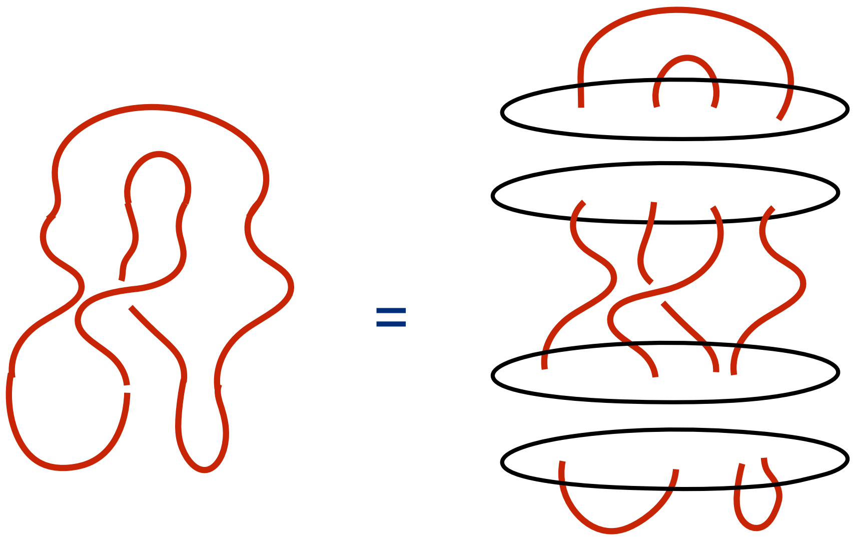
which in our setting equals to

$$\nu = \mu - \mu'$$

$\langle \mu |$  $| \mu' \rangle$

This perspective leads to
quantum invariants of not only braids
but knots and links as well.

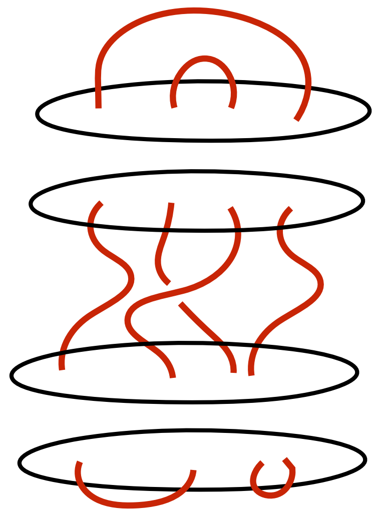
Any link K can be represented as a



a closure of some braid B

The corresponding quantum link invariant is the matrix element

$$\left(\Psi_{\mathcal{L}_{out}}\right|\,\mathfrak{B}\,\left|\Psi_{\mathcal{L}_{in}}\right)$$



$\mathfrak{B}(B)$

of the braiding matrix,

taken between a pair of conformal blocks

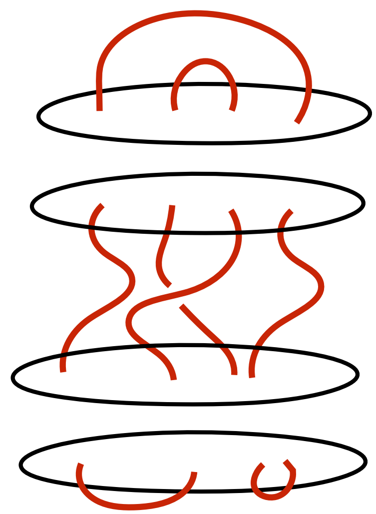
$$\Psi_{\mathcal{L}_{in}}, \qquad \Psi_{\mathcal{L}_{out}}$$

The pair of conformal blocks

$$\Psi_{\mathcal{L}_{in}}, \qquad \Psi_{\mathcal{L}_{out}}$$

that define the matrix element

$$\left(\Psi_{\mathcal{L}_{out}}\right| \mathfrak{B} \left|\Psi_{\mathcal{L}_{in}}\right)$$



$\mathfrak{B}(B)$

correspond to the top and the bottom of the picture.

The conformal blocks

$$\Psi_{\mathcal{L}_{in}}, \qquad \Psi_{\mathcal{L}_{out}}$$

are specific solutions to KZ equations

which describe pairwise fusing vertex operators

into copies of trivial representation

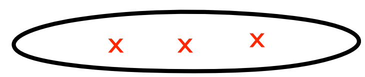
Necessarily they correspond to subspace of $\quad {}^L\rho = \otimes_I \, {}^L\rho_I$

of weight $\quad \nu = 0$

To categorify quantum knot invariants,

one would like to associate

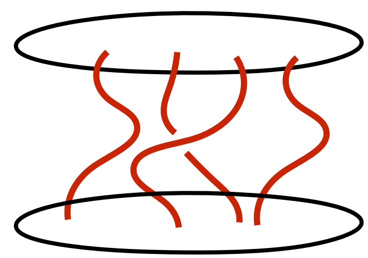
to the space conformal blocks one obtains at a fixed time slice

$\mathcal{A}$ 

a bi-graded category,

and to each conformal block an object of the category.

To braids,



one would like to associate

functors between the two categories

corresponding to the

top and the bottom.

Moreover,

we would like to do that in the way that

recovers the quantum knot invariants upon

de-categorification.

One typically proceeds by coming up with a category,
and then one has to work to prove
the de-categorification gives
the quantum knot invariants one aimed to categorify.

In the at least two of the approaches
we are about to describe,
the second step is automatic.

The starting point for us is

a geometric realization of conformal blocks,

with origin in supersymmetric quantum field theory.

This is not readily available.

We will eventually find not
one but two such interpretations.

However, to explain how they come about,
and to find a relation between them,
it is useful to ask a slightly different question first.

Namely, we will first ask for a

geometric interpretation of

q-conformal blocks of

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

the quantum affine algebra that is a q-deformation of

$$\widehat{{}^L\mathfrak{g}}$$

the affine Lie algebra.

The q-conformal blocks of

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

are q-deformations of conformal blocks of $\widehat{{}^L\mathfrak{g}}$

which I. Frenkel and Reshetikhin

discovered in the '80's.

They are defined as a correlation functions

$$\Psi(a_1, \ldots, a_n) = \langle \mu | \; \Phi_{L_{\rho_1}}(a_1) \cdots \Phi_{L_{\rho_n}}(a_n) \; | \mu' \rangle$$

of chiral vertex operators,

like in the conformal case, except all the operators are q-deformed.

$\langle \mu |$  $| \mu' \rangle$

Just like conformal blocks of

$$\widehat{{}^L\mathfrak{g}}$$

may be defined as solutions of the Knizhnik-Zamolodchikov equation,

the q-conformal blocks of

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

are solutions of the quantum Knizhnik-Zamolodchikov equation.

The quantum Knizhnik-Zamolodchikov (qKZ) equation

is a difference equation

$$\begin{aligned}\Psi(a_1,\ldots pa_I,\ldots a_n) =& R_{II-1}(pa_I/a_{I-1})\cdots R_{I1}(pa_I/a_{I-1})(\hbar^\rho)_I \\ &\times R_{In}(a_I/a_n)\ldots R_{II+1}(a_I/a_{I+1})\Psi(a_1,\ldots a_I,\ldots a_n)\end{aligned}$$

which reduces to the Knizhnik-Zamolodchikov equation

$$\kappa\, a_I \frac{\partial}{\partial a_I}\Psi = \left(\sum_{J\neq I} r_{IJ}(a_I/a_J) + r_{I0} + r_{I\infty}\right)\Psi$$

in the conformal limit.

It turns out that q-conformal blocks of

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

have a geometric realization,

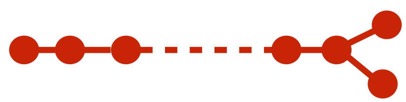
and one that originates from a supersymmetric gauge theory.

Let ${}^L\mathfrak{g}$ be a simply laced Lie algebra so in particular

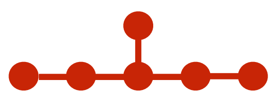
$$ {}^L\mathfrak{g} = \mathfrak{g} $$

and of the following types:

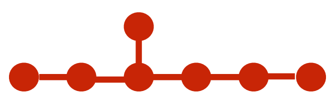
$\mathfrak{g} = A_n$



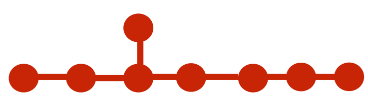
$\mathfrak{g} = D_n$



$\mathfrak{g} = E_6$



$\mathfrak{g} = E_7$



$\mathfrak{g} = E_8$

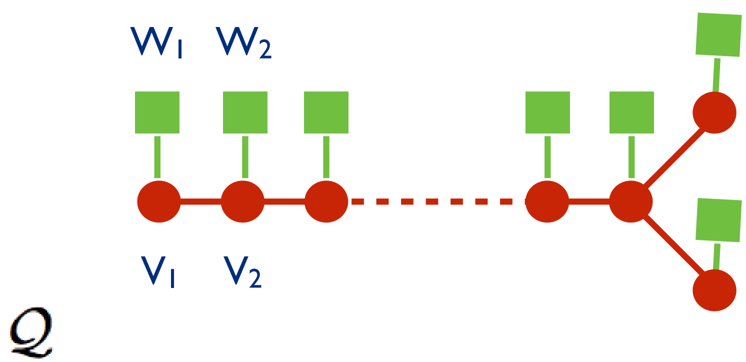
The non-simply laced cases can be treated,

but I will not have time to describe this in this talk.

The gauge theory we need

is a

three dimensional quiver gauge theory

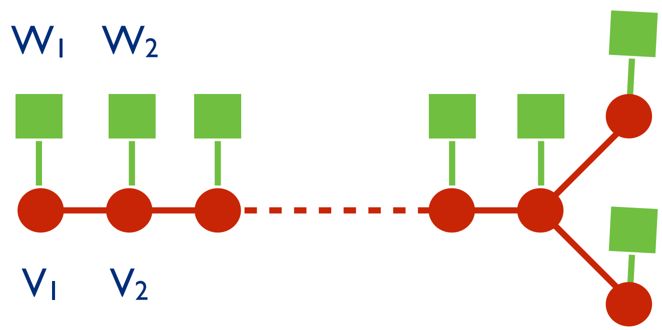


with N=4 supersymmetry.

The quiver diagram $\mathcal{Q}$

defining the theory, is a collection of nodes and arrows between them,

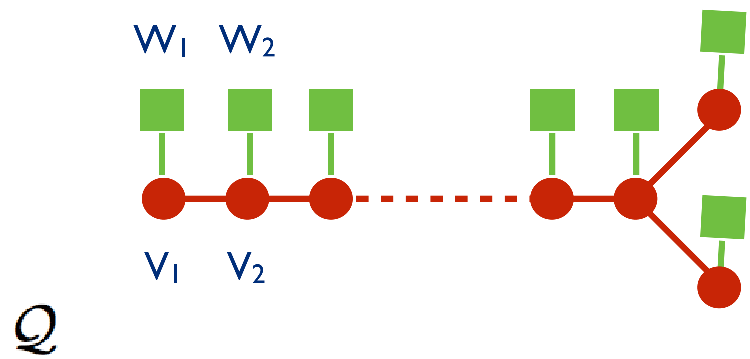
based on the Dynkin diagram of $\mathfrak{g}$



$\mathcal{Q}$

$\mathfrak{g} = D_n$

The quiver encodes



the gauge symmetry and global symmetry groups

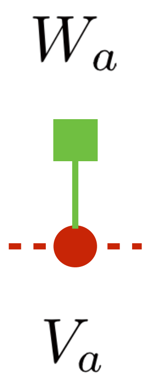
$$G_{\mathcal{Q}} = \prod_a GL(V_a) \qquad G_W = \prod_a GL(W_a)$$

and representations of matter fields charged under it

$$\mathrm{Rep}\,\mathcal{Q} = \oplus_{a\to b}\,\mathrm{Hom}(V_a, V_b) \oplus_a \mathrm{Hom}(V_a, W_a)$$

The ranks of the vector spaces

$$\dim V_a = d_a \,, \qquad \dim W_a = m_a$$



are determined by a pair of weights

$$\lambda, \quad \nu$$

associated to the representation $\quad {}^L\rho = \otimes_I \, {}^L\rho_I$

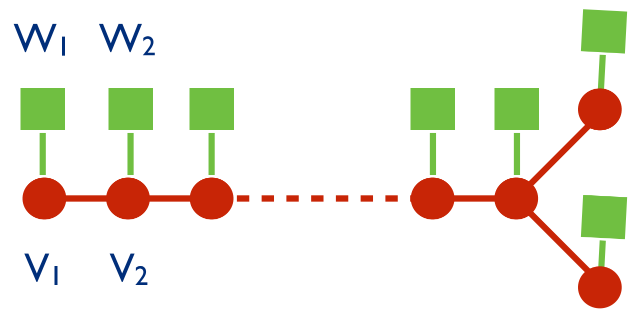
which q-conformal blocks transform in:

$$\text{highest weight } \lambda = \sum_I \lambda_I \; = \sum_a m_a \, {}^L w_a$$

$$\text{weight } \; \nu = \mu - \mu' = \sum_a m_a \, {}^L w_a - \sum_a d_a \, {}^L e_a$$

The solutions to qKZ equation

turn out to be supersymmetric partition functions



$\mathcal{Q}$

of the three dimensional quiver gauge theory on

$$\mathrm{C} \times S^1$$

where $\mathrm{C} = \mathbb{C}$

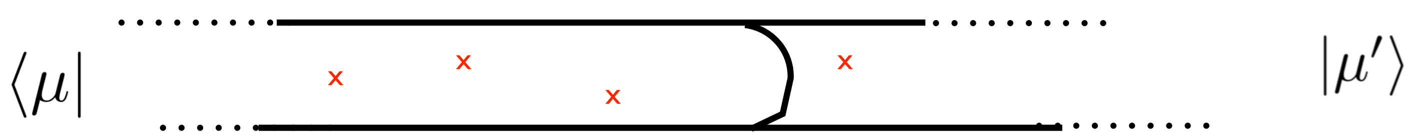
To define the supersymmetric partition function we need, one wants to use the

$$U(1)_H \times U(1)_V \in SU(2)_H \times SU(2)_V$$

R-symmetry of the 3d N=4 theory.

All the ingredients in the q-conformal block

$$\Psi(a_1, \ldots, a_n) = \langle \mu | \; \Phi_{L_{\rho_1}}(a_1) \cdots \Phi_{L_{\rho_n}}(a_n) \; | \mu' \rangle$$

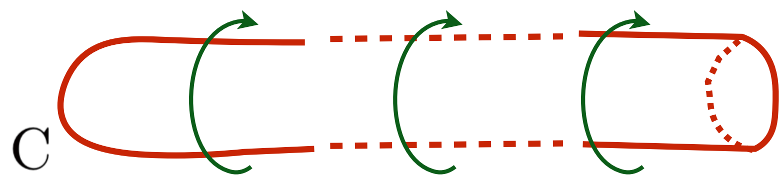


have a gauge theory interpretation.

The step

$$p = \hbar^{-\kappa}$$

of the qKZ equation is the parameter by which $\mathbb{C}$ rotates,



as we go around the S^1 in $\mathbb{C} \times S^1$

To preserve supersymmetry,

this is accompanied by a holonomy around the S^1,

of the $U(1)_V$ R- symmetry.

The parameter $\hbar$ in

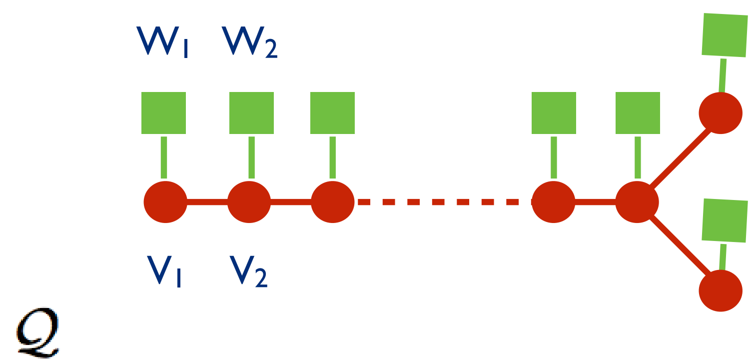
$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

is the holonomy around the S^1 in $\mathrm{C} \times S^1$

of the off-diagonal

$$U(1) \in U(1)_H \times U(1)_V$$

The holonomies associated to masses of fundamental hypermultiplets



are the positions of vertex operators,

$$\Psi(a_1,\ldots,a_n) = \langle\mu|\ \Phi_{{}^L\rho_1}(a_1)\cdots\Phi_{{}^L\rho_n}(a_n)\ |\mu'\rangle$$

The highest weight vector of Verma module $\langle\mu|$ in

$$\Psi(a_1,\ldots,a_n) = \langle\mu|\ \Phi_{^L\rho_1}(a_1)\cdots\Phi_{^L\rho_n}(a_n)\ |\mu'\rangle$$

is related to the holonomy $z \in (\mathbb{C}^\times)^{\mathrm{rk}(^{\mathrm{L}}\mathfrak{g})}$ of global symmetry associated with Fayet-Iliopoulos parameters:

$$z = \hbar^{\mu}$$

The partition function

of the quiver gauge theory on

$$\mathrm{C} \times S^1$$

has a precise mathematical formulation,

in terms of

quantum K-theory

of the Nakajima quiver variety X.

The quiver variety X ,

is the Higgs branch of the gauge theory:

$$X = T^*\mathrm{Rep}\,\mathcal{Q}////G_{\mathcal{Q}}$$

$$\mathrm{Rep}\,\mathcal{Q} = \oplus_{a\to b}\,\mathrm{Hom}(V_a, V_b) \oplus_a \mathrm{Hom}(V_a, W_a)$$

$$G_{\mathcal{Q}} = \prod_a GL(V_a)$$

Quantum K-theory

of Nakajima quiver varieties,

generalizing their Gromov-Witten theory

was developed a few years ago

by Okounkov (with Maulik and Smirnov).

The theory is a variant of the theory

Givental put forward earlier, but uses crucially the fact

that these are holomorphic-symplectic varieties.

The supersymmetric partition function of the gauge theory on

$$\mathrm{C} \times S^1$$

for $\mathrm{C} = \mathbb{C}$ is the K-theoretic "vertex function"

$$\mathsf{Vertex}^K(X)$$

This is the generating function of equivariant, K-theoretic counts of "quasi-maps" (i.e. vortices)

$$\mathrm{C} \dashrightarrow X$$

of all degrees.

One works equivariantly with respect to:

$$\mathsf{T} = \mathsf{A} \times \mathbb{C}^{\times}_{\hbar}$$

A is the maximal torus of rotations of X that preserve the symplectic form, and $\mathbb{C}^{\times}_{\hbar}$ scales it,

and with respect to

$$\mathbb{C}^{\times}_{p}$$

which rotates the domain curve.

A key result of the theory,

due to Okounkov,

is that the K-theoretic vertex function of X

$$\mathsf{Vertex}^K(X)$$

solves the quantum Knizhnik-Zamolodchikov equation

corresponding to

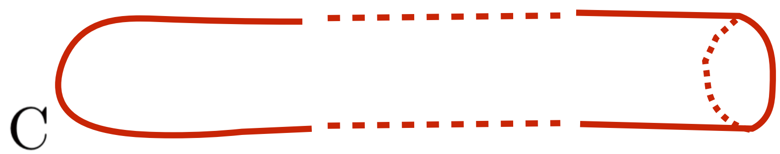
$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

Which solution of the qKZ equation

$$\mathsf{Vertex}^{K}(X)$$

computes depends on the choice of data

at infinity of



This choice means vertex functions should be thought of as valued in

$$\mathsf{Vertex}^{K}(X) \in Ell_{\mathrm{T}}(X)$$

While

$$\Psi = \mathsf{Vertex}^K(X)$$

solve the quantum Knizhnik-Zamolodchikov equation,

$$\Psi(a_1, \ldots pa_I, \ldots a_n) = R_{II-1}(pa_I/a_{I-1}) \cdots R_{I1}(pa_I/a_{I-1})(\hbar^\rho)_I$$
$$\times R_{In}(a_I/a_n) \ldots R_{II+1}(a_I/a_{I+1})\Psi(a_1, \ldots a_I, \ldots a_n)$$

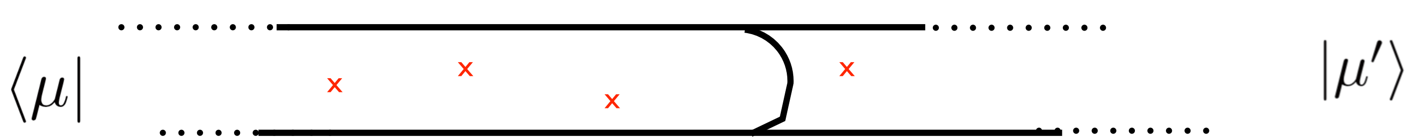
they are not the q-conformal blocks of $U_\hbar(\widehat{{}^L\mathfrak{g}})$

q-conformal blocks are the solutions of the qKZ equation

which are holomorphic in a chamber such as

$$\mathfrak{C}: \qquad |a_5| > |a_2| > |a_7| > \ldots .$$

corresponding to choice of ordering of vertex operators in



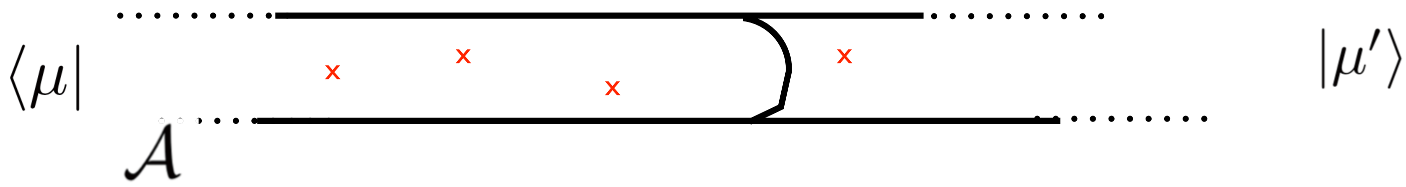
This is a choice of mass parameters of the gauge theory.

Instead,

$$\Psi = \mathsf{Vertex}^{K}(X)$$

are holomorphic in a chamber of Kahler moduli of X

$$z = \hbar^{\mu}$$



and Fayet-Iliopoulus parameters of the gauge theory.

So, this does not give an answer to the
question we are after,
namely to find a geometric interpretation of conformal blocks,
even after q-deformation.

It turns out that

$$X = T^* \mathrm{Rep}\, \mathcal{Q} /\!\!/\!\!/\!\!/ G_{\mathcal{Q}}$$

is not the only geometry that underlies solutions to the qKZ equation corresponding to our problem.

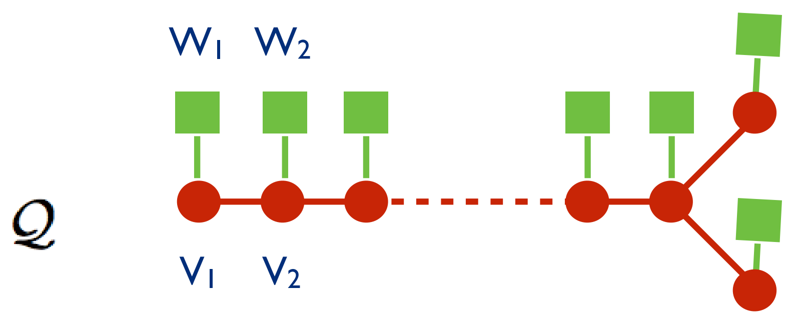
There is a second one,
which turns out to the relevant one.

There are two natural holomorphic symplectic

varieties one can associate to

the 3d quiver gauge theory

with quiver



and N=4 supersymmetry.

One such variety is
the Nakajima quiver variety

$$X = T^* \mathrm{Rep}\, \mathcal{Q} /\!/\!/\!/ G_{\mathcal{Q}}$$

This is the
Higgs branch of vacua
of the 3d gauge theory.

The other is the Coulomb branch,

which we will denote by

$$X^{\vee}$$

The Coulomb branch

$$X^\vee$$

of our gauge theory is a certain intersection of slices in the (thick) affine Grassmanian of G

$$\mathrm{Gr}_G = G((z))/G[z]$$

Here, G is the adjoint form of a Lie group with Lie algebra $\mathfrak{g}$

Braverman, Finkelberg, Nakajima
Bullimore, Dimofte, Gaiotto

The Coulomb branch

$$X^\vee = \mathrm{Gr}^{\overline{\lambda}}{}_\nu$$

is an intersection of a pair of orbits

$$\mathrm{Gr}^{\overline{\lambda}}{}_\nu = \overline{\mathrm{Gr}}^\lambda \cap \mathrm{Gr}_\nu$$

in the affine Grassmanian of complex dimension $\dim(\mathrm{Gr}^{\overline{\lambda}}{}_\nu) = 2\,\mathrm{rk}\,G_{\mathcal{Q}}$

$$\mathrm{Gr}^\lambda = G[z]z^{-\lambda} \qquad \mathrm{Gr}_\nu = G_1[[z^{-1}]]z^{-\mathrm{w}_0\nu}$$

where $\lambda,\ \nu$ are the pair of weights defining the quiver $\mathcal{Q}$

with $\lambda \geq \nu \geq 0$

The two holomorphic symplectic varieties

$$X \quad , \qquad X^{\vee}$$

in general live in different dimensions,

and between them,

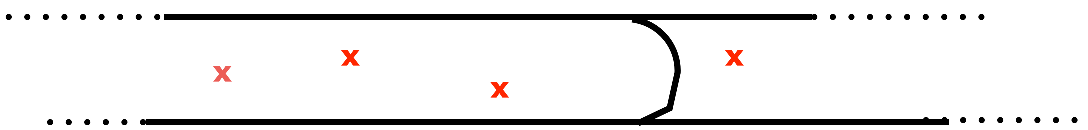
the roles of Kahler (Fayet-Illiopolus) and equivariant (mass) moduli,

get exchanged:

$$\Lambda = A^{\vee}, \qquad A = \Lambda^{\vee}$$

The masses of fundamental hypermultiplets

are the positions of vertex operators on



$\mathcal{A}$

They are equivariant moduli of

$$X$$

and the Kahler moduli of

$$X^{\vee}$$

Another way to think about

$$X^\vee$$

is as the moduli space of

G -monopoles,

on

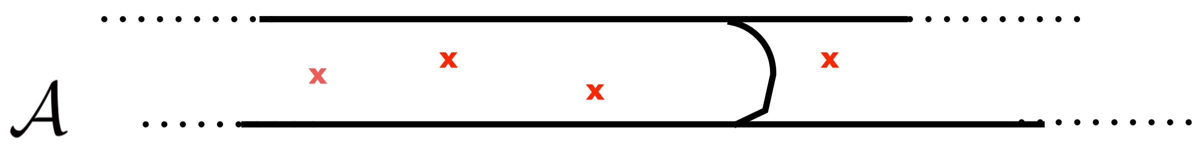
$$\mathbb{R} \times \mathbb{C}_\hbar$$

where λ is the charge of singular monopoles,

and ν the total monopole charge.

Hanany, Witten

In this language, the positions of vertex operators on



are the (complexified) positions of singular monopoles

on $\mathbb{R}$ in $\mathbb{R} \times \mathbb{C}_\hbar$

(monopoles sit at the origin of $\mathbb{C}_\hbar$ to preserve the $\mathbb{C}_\hbar^\times$ symmetry)

Three dimensional mirror symmetry

leads to relations between certain computations on

X and $X^\vee$

This is referred to as

"symplectic duality",

in some of the literature that explores it.

Physically, one expects to be able to compute the supersymmetric partition function of the gauge theory on

$$\mathrm{C} \times S^1$$

by starting with the sigma model to either

X or $X^\vee$

with suitable boundary conditions at infinity.

We prove that, whenever it is defined,

the K-theoretic vertex function of $X^\vee$

$$\mathsf{Vertex}^K(X^\vee)$$

solves the same qKZ equation

as

$$\mathsf{Vertex}^K(X)$$

the K-theoretic vertex function of X

From perspective of

$$X^\vee$$

the qKZ equation

$$\Psi(a_1,\ldots pa_I,\ldots a_n) = R_{II-1}(pa_I/a_{I-1})\cdots R_{I1}(pa_I/a_{I-1})(\hbar^\rho)_I$$
$$\times R_{In}(a_I/a_n)\ldots R_{II+1}(a_I/a_{I+1})\Psi(a_1,\ldots a_I,\ldots a_n)$$

is the quantum difference equation since the a -variables are the Kahler variables of $X^\vee$.

The “quantum difference equation"

is the K-theory analogue of

of the quantum differential equation of Gromov-Witten theory.

Here “quantum” refers to the

quantum cohomology cup product on

$$H^*(X^\vee)$$

used to define it.

While

$\mathsf{Vertex}^K(X)$ and $\mathsf{Vertex}^K(X^\vee)$

solve the same qKZ equation,

they provide two different basis of its solutions.

While

$\text{Vertex}^K(X)$

leads to solutions of qKZ which are analytic in

z -variables, but not in a -variables,

$\text{Vertex}^K(X^\vee)$

does the opposite.

Kahler for X
and
equivariant for $X^\vee$

Kahler for $X^\vee$
and
equivariant for X

Now we can return to

our main interest,

which is obtaining a geometric realization of

$$\widehat{L_{\mathfrak{g}}}$$

conformal blocks.

The conformal limit is the limit which takes

$$U_\hbar(\widehat{{}^L\mathfrak{g}}) \longrightarrow \widehat{{}^L\mathfrak{g}}$$

and the qKZ equation to the corresponding KZ equation.

It is amounts to

$$\begin{aligned} \hbar &\to 1 \\ p = \hbar^{-\kappa} &\to 1 \qquad\qquad \kappa, a,\ \mu \text{ fixed} \\ z = \hbar^{\mu} &\to 1 \end{aligned}$$

This corresponds to keeping the data of the conformal block fixed.

The conformal limit treats

X and $X^\vee$

very differently,

since it treats the

z - and the a -variables,

differently:

Kahler for X

Kahler for $X^\vee$

$$z \to 1, \qquad a \text{ fixed}$$

The conformal limit,

is not a geometric limit from perspective of the Higgs branch

$$X = T^*\mathrm{Rep}\,\mathcal{Q}/\!\!/\!\!/\!\!/G_{\mathcal{Q}}$$

The limit results in a badly singular space,

since the z -variables

which go to $z \to 1$ in the limit are its Kahler variables.

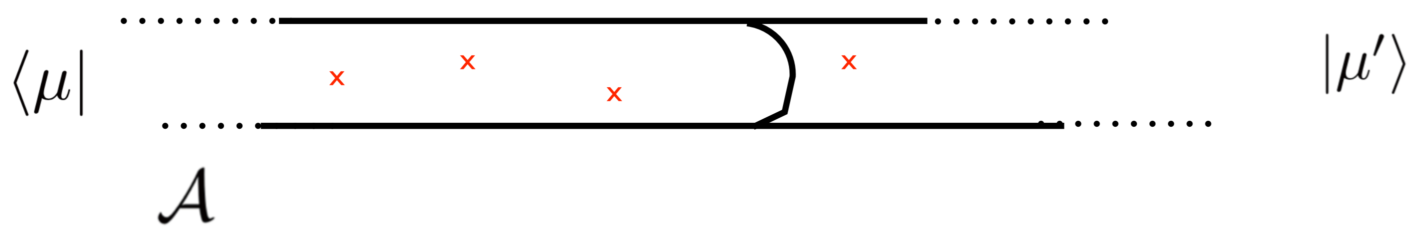
By contrast, from perspective of the Coulomb branch

$$X^{\vee} = \mathrm{Gr}^{\overline{\lambda}}{}_{\nu}$$

the limit is perfectly geometric.

Its Kahler variables are the a -variables,

the positions of vertex operators,



which are kept fixed.

From perspective of $X^\vee$,

the conformal limit,

$$U_\hbar(\widehat{{}^L\mathfrak{g}}) \longrightarrow \widehat{{}^L\mathfrak{g}}$$

is the cohomological limit

taking:

$$\text{quantum K-theory of } X^\vee \to \text{quantum cohomology of } X^\vee$$

The Knizhnik-Zamolodchikov equation

we get in the conformal limit

$$\kappa\, a_I \frac{\partial}{\partial a_I} \Psi = \sum_{J \neq I} r_{IJ}(a_I/a_J)\, \Psi$$

becomes the quantum differential equation

of $X^\vee$

It follows that conformal blocks of

$$\widehat{{}^{L}\mathfrak{g}}$$

have a geometric interpretation as

cohomological vertex functions

$$\Psi = \mathsf{Vertex}(X^\vee)$$

computed by equivariant Gromov-Witten theory of

$$X^\vee$$

The cohomological vertex function counts holomorphic maps

$$\mathrm{C} \dashrightarrow X^\vee$$

equivariantly with respect to

$$\mathrm{T}^\vee = \Lambda \times \mathbb{C}^\times_{\mathfrak{q}}$$

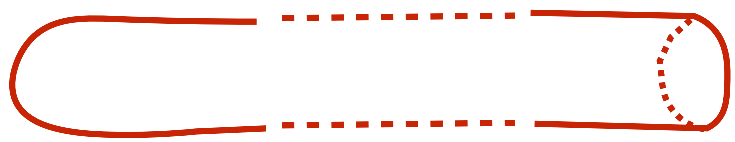
where one scales the holomorphic symplectic form of $X^\vee$ by

$$\mathfrak{q} = e^{\frac{2\pi i}{\kappa}}$$

The domain curve

$$\mathbb{C}$$

is best thought of as an infinite cigar with an S^1 boundary at infinity.



The boundary data is a choice of a K-theory class

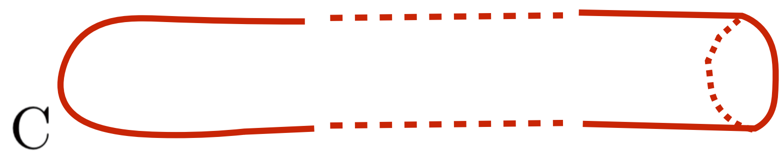
$$[\mathcal{F}] \in K_{\mathrm{T}^\vee}(X^\vee)$$

as in the work of Iritani.

The K-theory class

$$[\mathcal{F}] \in K_{\mathrm{T}^\vee}(X^\vee)$$

inserted at infinity determines which solution of



the quantum differential equation, and thus which conformal block,

$$\mathsf{Vertex}(X^\vee)[\mathcal{F}]$$

computes.

Underlying

the Gromov-Witten theory of

$$X^\vee$$

is a two-dimensional supersymmetric sigma model with

$X^\vee$ as a target space.

The geometric interpretation of

conformal blocks of

$$\widehat{{}^L\mathfrak{g}}$$

in terms of the supersymmetric sigma model to

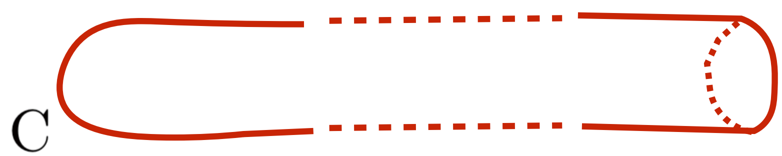
$$X^\vee$$

has far more information than the conformal blocks themselves.

The physical meaning of

$$\mathsf{Vertex}(X^\vee)[\mathcal{F}]$$

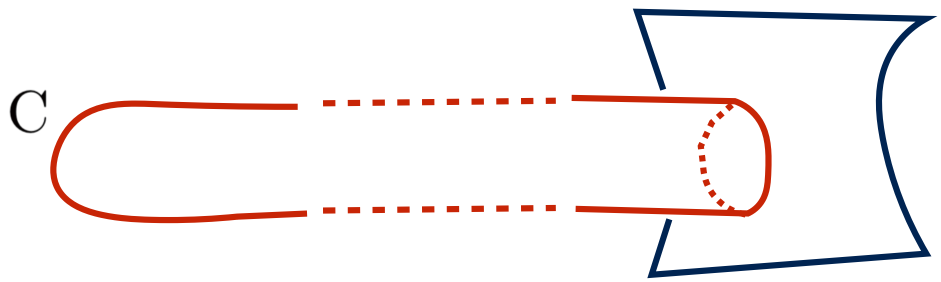
is the partition function of the supersymmetric sigma model with target $X^\vee$ on C



To get

$$\mathsf{Vertex}(X^\vee)[\mathcal{F}]$$

one has A-type twist in the interior of C

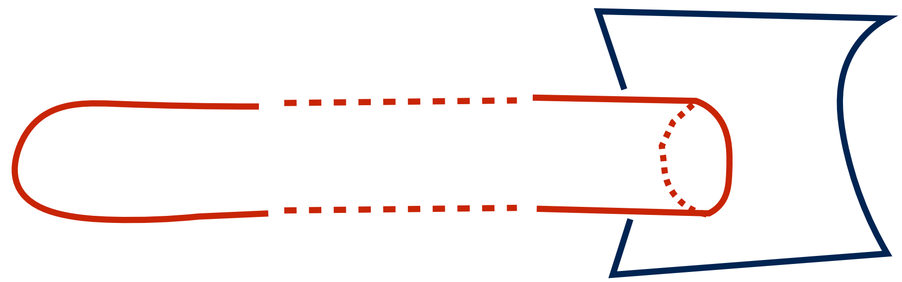


and at infinity, one places a B-type boundary condition, corresponding to the choice of

$$[\mathcal{F}] \in K_{\mathrm{T}^\vee}(X^\vee)$$

The B-type boundary condition

C



is a B-type brane on $X^\vee$. The brane we need to get

$$\mathsf{Vertex}(X^\vee)[\mathcal{F}]$$

is an object

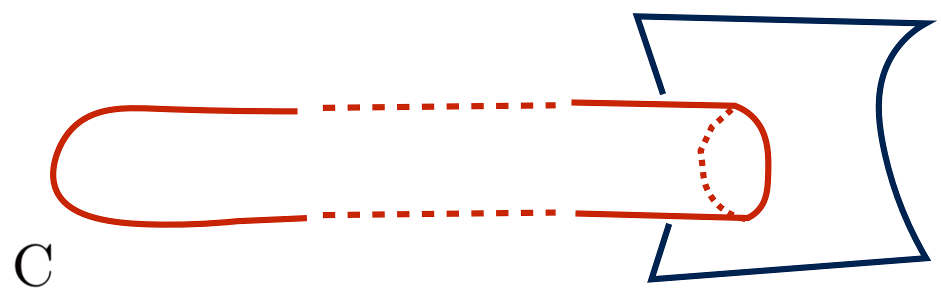
$$\mathcal{F} \in D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$

of the derived category of $\mathrm{T}^\vee$ -equivariant coherent sheaves on $X^\vee$
whose K-theory class is $[\mathcal{F}] \in K_{\mathrm{T}^\vee}(X^\vee)$

The choice of a B-type brane

$$\mathcal{F} \in D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$

at infinity of C determines which

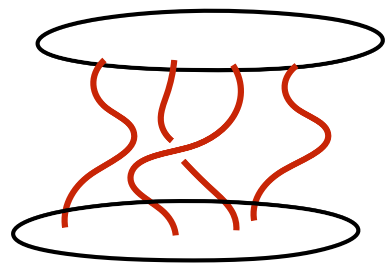


conformal block of $\widehat{{}^L\mathfrak{g}}$

$$\mathsf{Vertex}(X^\vee)[\mathcal{F}]$$

computes.

From perspective of $X^\vee$,



B

the action of braiding

$$\mathfrak{B} = \mathfrak{B}(B)$$

on the space of conformal blocks

is the monodromy of the quantum differential equation,

along the path B in its Kahler moduli.

Monodromy of the quantum differential equation

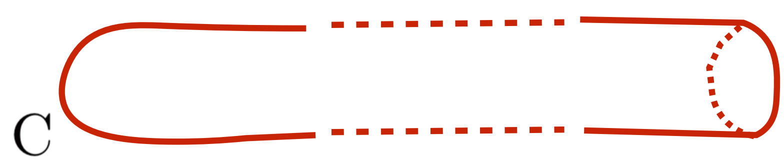
acts on

$$\Psi_{\mathcal{F}} = \mathsf{Vertex}(X^\vee)[\mathcal{F}]$$

via its action on K-theory classes

$$[\mathcal{F}] \in K_{\mathrm{T}^\vee}(X^\vee)$$

inserted at the boundary at infinity of



A theorem of Bezrukavnikov and Okounkov says that,

the action of braiding matrix on

$$K_{\mathrm{T}^\vee}(X^\vee)$$

via the monodromy of the quantum differential equation

lifts to

a derived auto-equivalence functor of

$$D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$

for any smooth holomorphic symplectic variety $X^\vee$.

This implies that, for a smooth

$$X^{\vee} = \mathrm{Gr}^{\overline{\lambda}}{}_{\nu} \ ,$$

derived auto-equivalence functors of

$$D^b Coh_{\mathrm{T}^{\vee}}(X^{\vee})$$

categorify the

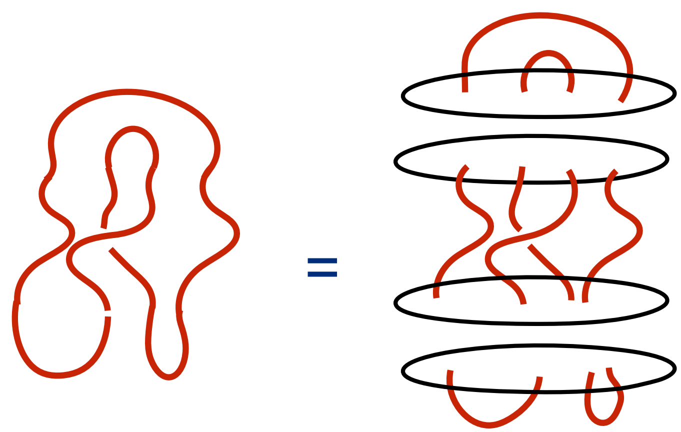
action of the $U_{\mathfrak{q}}({}^L\mathfrak{g})$ R-matrices

on conformal blocks of $\widehat{{}^L\mathfrak{g}}$

This also implies that from

$$D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$

we get categorification of quantum invariants of links



since they can be expressed as matrix elements of the braiding matrix

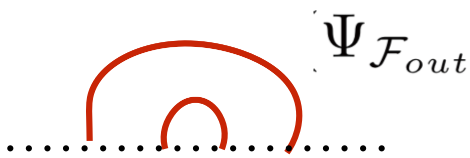
$$\left(\Psi_{\mathcal{F}_{out}} \middle| \mathfrak{B}\, \Psi_{\mathcal{F}_{in}}\right)$$

between pairs of conformal blocks.

Denote by

$$\mathcal{F}_{in}, \mathcal{F}_{out} \in D^b Coh_{\mathrm{T}}(X)$$

the branes that give rise to conformal blocks



and by

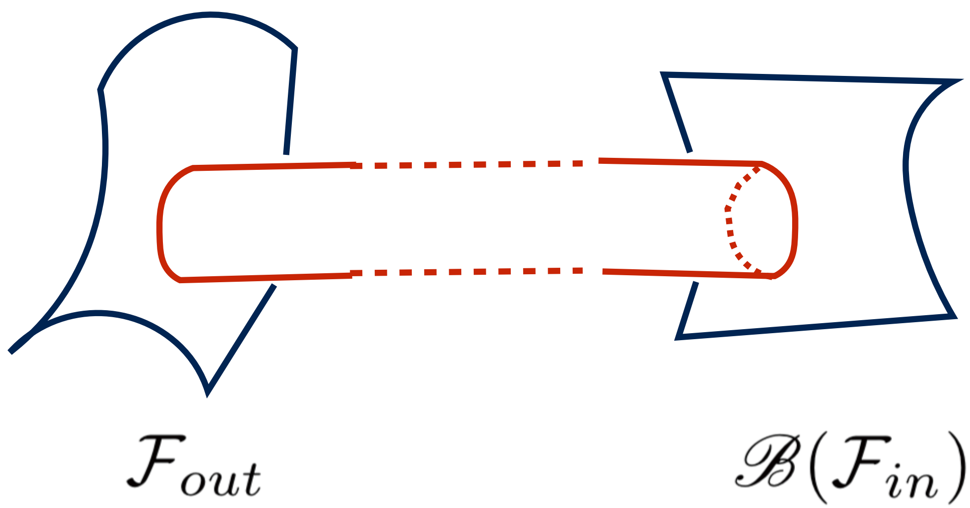
$$\mathscr{B}(\mathcal{F}_{in}) \in D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$

the image of $\mathcal{F}_{in}$ under the braiding functor.

The matrix element

$$\left(\Psi_{\mathcal{F}_{out}} \,\middle|\, \mathfrak{B}\, \Psi_{\mathcal{F}_{in}}\right)$$

is the partition function of the B-twisted sigma model to $X^\vee$ on



with the pair of B-branes at the boundary.

The corresponding categorified link invariant

is the graded Hom between the branes

$$H^{*,*}(K) = \mathrm{Ext}^*_{\mathrm{T}^\vee}(\mathcal{F}_{out}, \mathscr{B}(\mathcal{F}_{in}))$$

computed in

$$D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$

In addition to the homological grade, there is a second the grade,

coming from the $\mathbb{C}^\times_{\mathfrak{q}} \in \mathrm{T}^\vee$ -action,

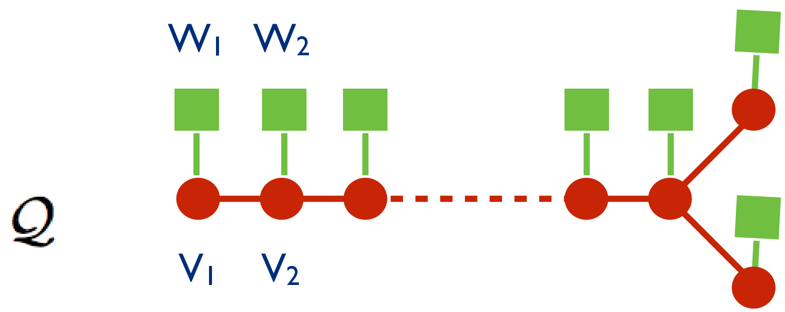
that scales the holomorphic symplectic form on $X^\vee$,

with weight

$$\mathfrak{q} = e^{\frac{2\pi i}{\kappa}}$$

The three dimensional gauge theory

we started with



leads to a

second description

of the categorified knot invariants.

It leads to a description in terms of a

two-dimensional equivariant mirror of

$$X^\vee = \mathrm{Gr}^{\overline{\lambda}}{}_{\nu}$$

The mirror description of $X^\vee$ is also

a new result.

The mirror

is a Landau-Ginzburg theory with target Y ,

and potential

W .

The Landau-Ginzburg potential

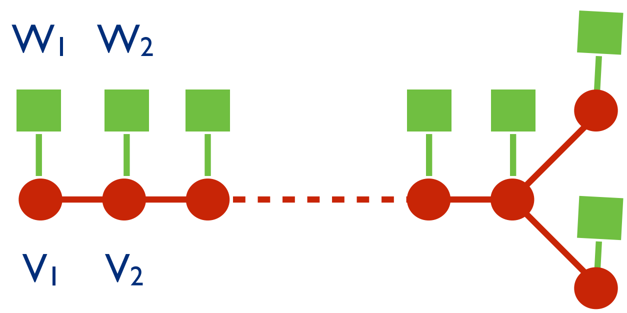
$$W$$

and the target

$$Y = \mathcal{A}^{\,\mathrm{rk},*}/\mathrm{Weyl}$$

can be derived from the 3d gauge theory.

$\mathcal{Q}$



The potential is a limit of the three dimensional effective

superpotential, given as a sum of contributions

associated to its nodes and its arrows.

One instructive, if roundabout, way to

discover the mirror description,

is as follows.

Recall that, in the conformal limit,

$$\mathsf{Vertex}^K(X)$$

has no geometric interpretation in terms of X

It does have a conformal limit.

To find it one wants to make use the integral formulation of

$$\text{Vertex}^{K}(X)$$

which one can derive by thinking of maps to

$$X = T^{*}\mathrm{Rep}\,\mathcal{Q}////G_{\mathcal{Q}}$$

in geometric invariant theory terms.

Aganagic, Frenkel, Okounkov

The integrals,

come from studying quasi-maps to the pre-quotient and,

projecting to gauge invariant configurations,

leads to integration over the maximal torus of

$$G_{\mathcal{Q}} = \prod_a GL(V_a)$$

The conformal limit of $\mathrm{Vertex}^K(X)$ has the form:

$$\Psi_{\mathcal{L}} = \int_{\mathcal{L}} \Omega \; e^{W/\kappa}$$

It gives integral solutions to the Knizhnik-Zamolodchikov equation corresponding to

$$\Psi(a_1, \ldots, a_n) = \langle \mu | \; \Phi_{L_{\rho_1}}(a_1) \cdots \Phi_{L_{\rho_n}}(a_n) \; | \mu' \rangle$$

The function W that enters

$$\Psi_{\mathcal{L}} = \int_{\mathcal{L}} \Omega \; e^{W/\kappa}$$

is the Landau-Ginzburg potential,

and Ω is a top holomorphic form on Y

The potential is a sum over three types of terms

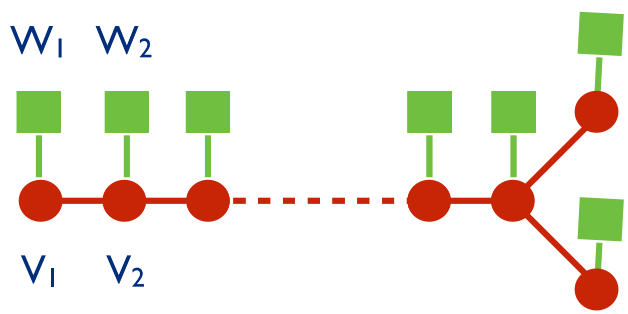
$$W = W_1 + W_2 + W_3$$

one of which come from the nodes

$$W_1 = \sum_a \sum_\alpha \ln(x_{\alpha,a})^{(^L e_a, \mu)}$$

and two from the arrows.

$$W_2 = \sum_{a,\alpha} \sum_I \ln(x_{\alpha,a} - a_I)^{(^L e_a, \lambda_I)} \qquad W_3 = -\sum_{a,b} \sum_{\alpha<\beta} \ln(x_{\alpha,a} - x_{\beta,b})^{(^L e_a, ^L e_b)}$$



The integration in

$$\Psi_{\mathcal{L}} = \int_{\mathcal{L}} \Omega \; e^{W/\kappa}$$

is over a Lagrangian cycle $\mathcal{L}$ in

$$Y = \mathcal{A}^{\mathrm{rk},*}/\mathrm{Weyl}$$

the target space of the Landau-Ginzburg model.

We are re-discovering

from

geometry and supersymmetric gauge theory,

the integral representations of conformal blocks of

$$\widehat{{}^L\mathfrak{g}}$$

They are very well known,

and go back to work of Feigin and E.Frenkel in the '80's

and Schechtman and Varchenko.

The fact that the Knizhnik-Zamolodchikov equation which the Landau-Ginzburg integral solves

$$\Psi_{\mathcal{L}} = \int_{\mathcal{L}} \Omega \; e^{W/\kappa}$$

is also the quantum differential equation of $X^{\vee}$ ....

…..gives a Givental type proof of 2d mirror symmetry

at genus zero,

relating

equivariant A-model on $X^\vee$

to

B-model on Y with superpotential W .

The Landau-Ginzburg origin of

conformal blocks

automatically

leads to categorification of the corresponding

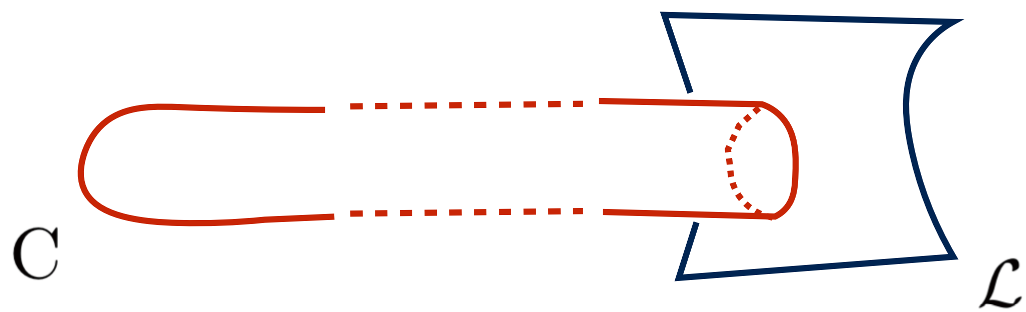
braid and link invariants.

From the Landau-Ginzburg perspective

the conformal block

$$\Psi_{\mathcal{L}} = \int_{\mathcal{L}} \Omega \; e^{W/\kappa}$$

is the partition function of the B-twisted theory on C ,

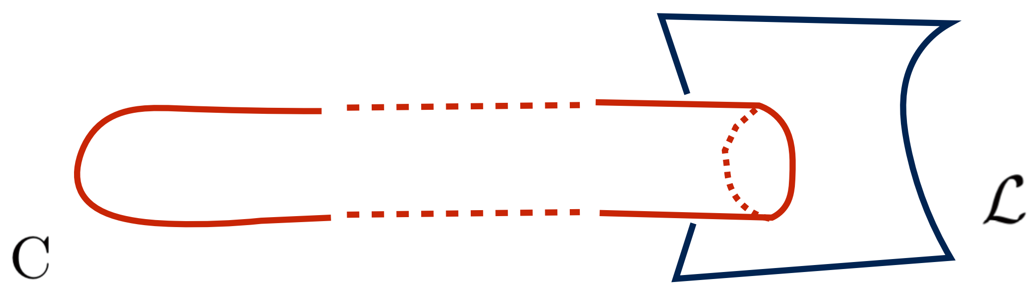


with A-type boundary condition at infinity, corresponding to the Lagrangian $\mathcal{L}$ in Y .

Thus, corresponding to a solution to the

Knizhnik-Zamolodchikov equation

is an A-brane at the boundary of $\mathcal{C}$ at infinity,



The brane is an object of

$$\mathcal{FS}(Y, W)$$

the Fukaya-Seidel category of A-branes on Y with potential W

Equivariant mirror symmetry

implies that the

Fukaya-Seidel category of A-branes on Y with potential W

$$\mathcal{FS}(Y, W)$$

should be equivalent to

$$D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$

the category of equivariant B-branes on $X^\vee$

We get a 2d mirror description of categorified knot invariants

based on

$$\mathcal{FS}(Y, W)$$

the Fukaya-Seidel category of A-branes on Y,

the target of the Landau-Ginzburg model,

with potential W.

This description is

developed in a work with Dimitrii Galakhov.

The categorified link invariant arizes as the Floer cohomology group

$$H^{*,*}(K) = HF^{*,*}(\mathcal{L}_{out}, \mathscr{B}\mathcal{L}_{in})$$

where the second grade is

is the winding number,

associated to the

non-single valued potential.

So far we have put forward two
approaches
to the problem of categorifying knot invariants.

It turns out that there is

a third approach to categorification

which is related to the other two,

though less tractable.

It is important to understand the connection,

in particular because what will emerge

is a unified picture of

the knot categorification problem,

and its solutions.

This will also demystify an aspect of the story so far

which seems strange:

The question what

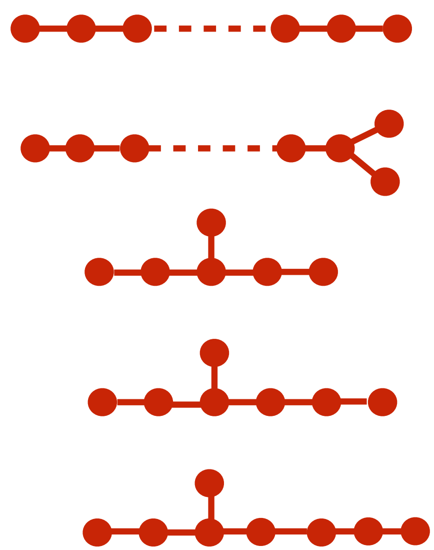
three dimensional supersymmetric gauge theories

have to do with knot invariants?

The explanation comes from the six dimensional (2,0)

little string theory

labeled by a simply laced Lie algebra $\mathfrak{g}$.



$\mathfrak{g} = A_n$

$\mathfrak{g} = D_n$

$\mathfrak{g} = E_6$

$\mathfrak{g} = E_7$

$\mathfrak{g} = E_8$

The 6d little string theory,

is a six dimensional string theory

obtained by taking a limit of IIB string theory on

$$Y \ ,$$

the ADE surface singularity of type $\mathfrak{g}$.

In the limit, one keeps only the degrees of freedom supported at the singularity of Y and decouples the 10d bulk.

The q-conformal blocks of the

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

and partition functions of the 3d gauge theories they compute

turn out to be best understood

as the supersymmetric partition functions of

the $\mathfrak{g}$ -type little string theory,

with defects that lead to knots.

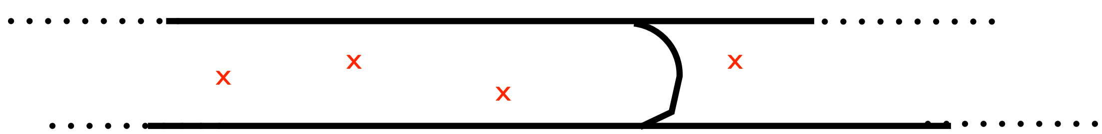
One wants to study the six dimensional (2,0) little string theory on

$$M_6 = \mathcal{A} \times \mathrm{C} \times \mathbb{C}_\hbar$$

where

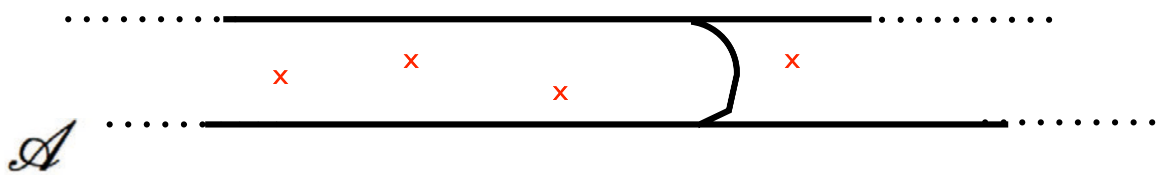
$$\mathscr{A}$$

is the Riemann surface where the conformal blocks live:



and C is the domain curve of the 2d theories we had so far.

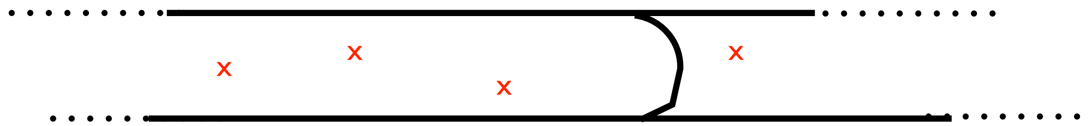
The vertex operators on the Riemann surface



come from a collection of defects in the little string theory,

which are inherited from D-branes of the ten dimensional string.

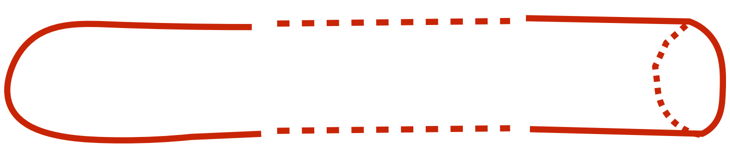
The D-branes needed are



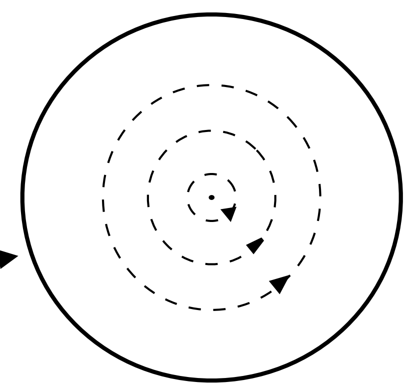
$\mathscr{A}$

two dimensional defects of the six dimensional theory on

$$M_6 = \mathcal{A} \times \mathrm{C} \times \mathbb{C}_\hbar$$

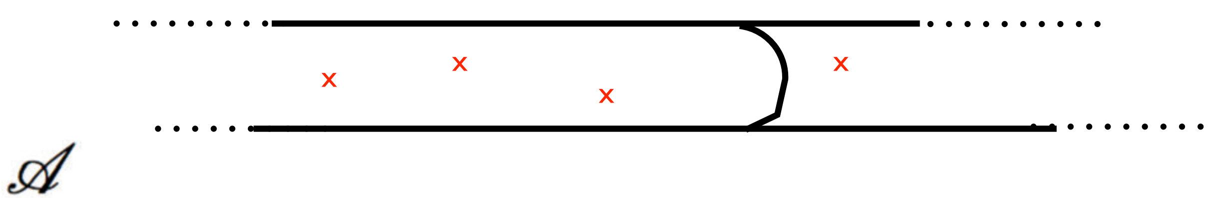


C

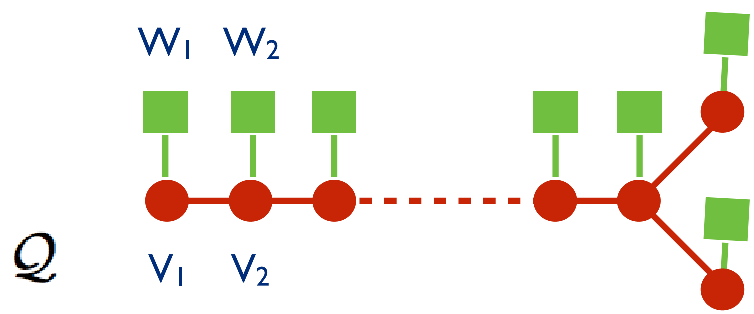


supported on C and the origin of $\mathbb{C}_\hbar$

The choice of which conformal blocks we want to study translates into choices of defects



The theory on the defects is the quiver gauge theory



that arose earlier in the talk.

This is a consequence of the familiar description of D-branes on ADE singularities due to Douglas and Moore in '96.

The theory on the defects supported on C,

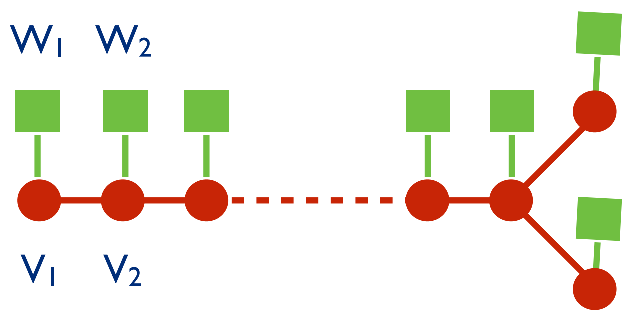
is a three dimensional quiver gauge theory on

$$\mathrm{C} \times S^1$$

rather than a two dimensional theory on C,

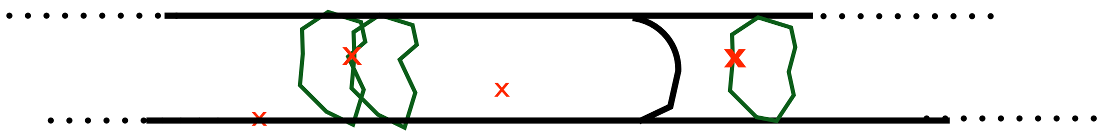
due to a stringy effect.

$\mathcal{Q}$



In a string theory,

one has to include the winding modes of strings around $\mathscr{A}$



$\mathscr{A}$

These turn the theory on the defects supported on C ,

to a three dimensional quiver gauge theory on

$$\mathrm{C} \times S^1$$

where the S^1 is the dual of the circle in $\mathscr{A}$

In general,

the partition functions of theories on defects

capture only the local physics of the defect.

In this case,

they capture all of the 6d physics.

The reason for this is

akin to localization:

Partition function of the 6d string theory we need

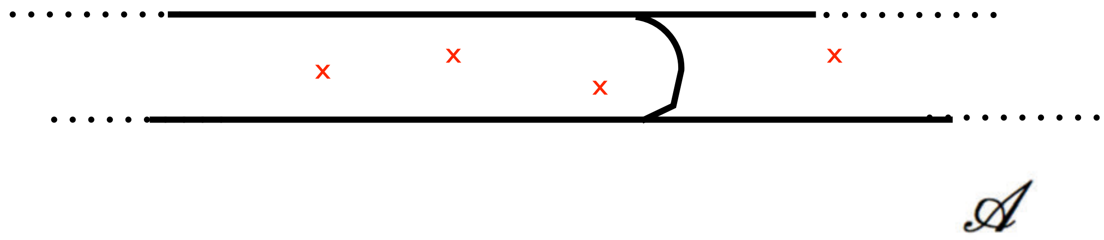
is trivial in the absence of defects,

$\mathscr{A}$

due to cancellations between bosons and fermions in the bulk.

All the non-trivial contributions come from the theories on the defects.

With the defects added,



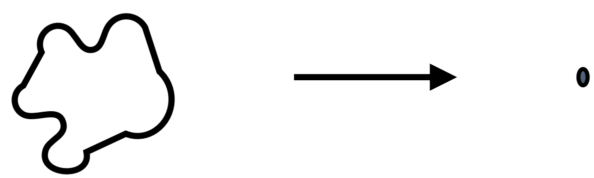
the partition function of the 6d string theory equals to the partition function of the theory supported on the defects.

Aganagic, Haouzi

The conformal limit of the algebras

$$U_\hbar(\widehat{{}^L\mathfrak{g}}) \longrightarrow \widehat{{}^L\mathfrak{g}}$$

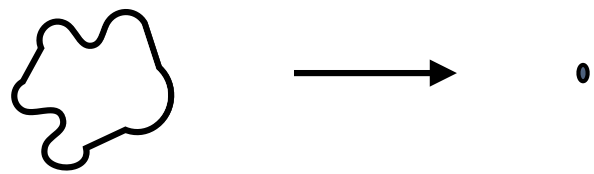
coincides with the point particle limit of little string theory



in which it becomes the six dimensional conformal field theory

of type $\mathfrak{g}$ (with (2,0) supersymmetry)

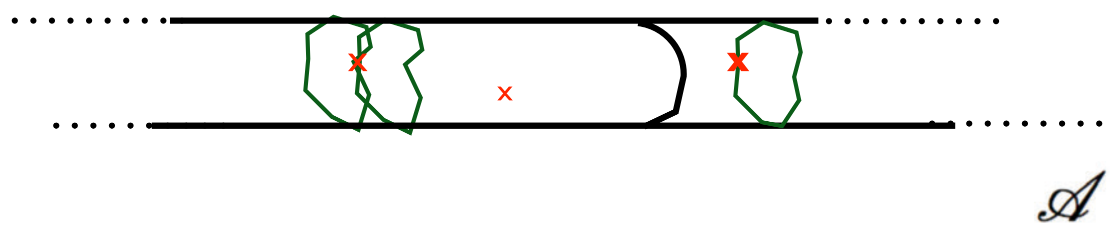
In the point particle limit,



the winding modes that made the theory

on the defects three dimensional instead of two,

become infinitely heavy.



As a result, in the conformal limit,

the theory on the defects

becomes a two dimensional theory on

$$\mathbb{C}$$

The two dimensional theory on the defects
of the six dimensional (2,0) theory was not known previously.

It is not a gauge theory,
but it has two other descriptions,
I described earlier in the talk.

One description

is based on the supersymmetric sigma model

describing maps

$$\mathrm{C} \dashrightarrow X^{\vee}$$

The other is in terms of the mirror

Landau-Ginzburg model on C with potential W

The two descriptions so far
described the categorification problem
starting with the theory on the
defects.

There is a third description,

due to Witten.

It is obtained from

the perspective of the 6d theory in the bulk.

Compactified on a small circle,

the six dimensional $\mathfrak{g}$ -type (2,0) conformal theory

with no classical description,

becomes a $\mathfrak{g}$ -type gauge theory

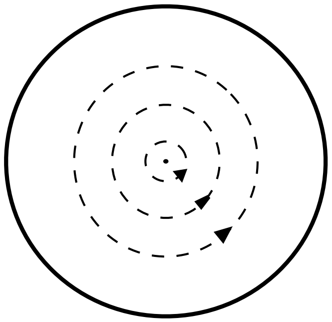
in one dimension less.

The circle the theory is compactified on corresponds to

rotations of $\mathbb{C}_\hbar$

in

$$M_6 = \mathcal{A} \times \mathrm{C} \times \mathbb{C}_\hbar$$



resulting in a five dimensional gauge theory

with gauge group

$$G$$

which is the adjoint form of a Lie group with lie algebra $\mathfrak{g}$.

on a five manifold with a boundary

$$\widetilde{M}_5 = \widetilde{M}_3 \times \mathrm{C} \qquad \widetilde{M}_3 = \mathcal{C} \times \mathbb{R}_{\geq 0}$$

The two dimensional defects become monopoles supported at the boundary of the five manifold,

$$\widetilde{M}_5 = \widetilde{M}_3 \times \mathrm{C} \qquad \widetilde{M}_3 = \mathcal{A} \times \mathbb{R}_{\geq 0} \quad ,$$

along C.

The collection of monopoles

is the same one that underlies the description of the

Coulomb-branch

as the moduli space of G - monopoles.

Witten shows that the five dimensional theory on

$$\widetilde{M}_5 = \widetilde{M}_3 \times \mathrm{C}$$

can be viewed as a gauged

Landau-Ginzburg model on C,

with potential

$$\mathcal{W}_{\mathrm{CS}} = \int_{\widetilde{M}_3} \mathrm{Tr}(A \wedge dA + A \wedge A \wedge A)$$

on an infinite dimensional target space $\mathcal{Y}_{\mathrm{CS}}$

corresponding to $\mathfrak{g}_{\mathbb{C}}$ connections on $\widetilde{M}_3 = \mathcal{A} \times \mathbb{R}_{\geq 0}$

with suitable boundary conditions (depending on the monopoles).

To obtain knot homology groups in this approach,
one would end up counting solutions to
certain five dimensional equations.

The equations arise in
constructing the Floor cohomology groups
of the five dimensional Landau-Ginzburg theory.

All three different approaches
describe the same physics
starting in six dimensions
so they are expected to be equivalent.

The approach based on

$$D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$

should be equivalent to that
of Kamnitzer and Cautis in type A.

The 2d mirror approach based on

$$\mathcal{FS}(Y, W)$$

should be related,
by Calabi-Yau/Landau-Ginzburg correspondence
to the Fukaya categories approach by Abouzaid, Smith and Seidel.

The idea that the 5d Landau-Ginzburg theory could have a 2d Landau-Ginzburg counterpart was suggested in the works of Gaiotto, Witten and Moore.