Brief Summary of Calabi-Yau geometry

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Introduction

A compact Kähler manifold (X^n, ω) is called Calabi-Yau if $c_1(X) = 0 \in H^2(X, \mathbb{Z})$. Usually we assume $\pi_1(X)$ is finite.

- In 1954, 1957, Calabi conjectured that X carries Ricci-flat metric.
- Calabi reduced his conjecture to a complex Monge-Ampère equation.

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}\right) = e^f \det\left(g_{i\bar{j}}\right)$$

  with $g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi$ being positive definite.
- Using the tools of geometric analysis, I proved the conjecture in 1976 and applied the proof to solve some problems in algebraic geometry. This is the major starting point of the development of geometric analysis.

- This provides first known non-locally-homogeneous Einstein manifolds.
- Chern number inequality $c_2 \cap [\omega]^n \geqslant 0$ and the equality holds if and only if the universal cover $\tilde{X} \cong \mathbb{C}^n$.
- The holonomy group is contained in $SU(n)$. This leads to the structure theorem of Calabi-Yau manifolds.
- There are many other applications in complex and algebraic geometry. For example, K3 is Kähler by Siu.
- There are abundant examples contributed by both physicists and mathematicians.
- There is no similar theorem in $G2$ manifolds. It is more difficult to construct examples of $G2$ manifolds.

The Ricci-flat metric on Calabi-Yau manifolds are solutions of the Einstein field equation with no matter. The theory of motions of loops inside a Calabi-Yau manifold provide a model of a conformal field theory. So Calabi-Yau manifolds became very important in the study of superstring theory. Physics theories have been inspiring a lot of mathematical studies of Calabi-Yau geometry. One of the most important topics contributed by both physicists and mathematicians is mirror symmetry. I will discuss the details in next session.

1. BCOV Theory and Gromov-Witten invariants

Mirror Symmetry

Mirror symmetry is a duality in string theory originally discovered by **Dixon, Lerche, Vafa, Warner, Witten** and many other physicists based on their study of Calabi-Yau manifolds in physics. Based on this, **Greene, Plesser** found nontrivial examples of the mirror relationship in terms of Gepner models. It is found a very rich theory relating symplectic geometry of a Calabi-Yau threefold X and complex geometry of its mirror Calabi-Yau threefold $\check{X}$. Around 1990, the detailed mathematical calculation was carried out by **Candelas, de la Ossa, Green and Parkes** to obtain a conjectural formula of the number of rational curves of arbitrary degree in a quintic Calabi-Yau threefold by relating it to period integrals of the quintic mirror. The BCOV theory, by **Bershadsky-Cecotti-Ooguri-Vafa**, is a vast generalization of this theory to all genus, all Calabi-Yau threefolds.

Nearly a quarter of century later, it continues to inspire new ideas in mathematical physics. Furthermore, many of its mathematical concepts and consequences are only beginning to be realized and proved.

Gromov-Witten invariants of Calabi-Yau threefolds

By late 1990s mathematicians had established the foundation of **Gromov-Witten** (GW) theory as a mathematical theory of A-model topological strings. In this context, the genus g free energy $F_g^X(\mathbf{t})$ on a Calabi-Yau 3-fold X collects the virtual counting $N_{g,\beta}^X$ of stable holomorphic maps from genus g curve to X, which is a function on a neighborhood around the large radius limit in the complexified Kähler moduli $\mathbf{t}$ of X.

$$F_g^X(\mathbf{t}) = \begin{cases} \dfrac{1}{6}\displaystyle\int_X \mathbf{t}^3 + \sum_{\beta>0} N_{0,\beta}^X e^{\int_\beta \mathbf{t}}, & g=0; \\[2em] -\dfrac{1}{24}\displaystyle\int_X \mathbf{t}c_2(X) + \sum_{\beta>0} N_{1,\beta}^X e^{\int_\beta \mathbf{t}}, & g=1; \\[2em] N_{g,0}^X + \displaystyle\sum_{\beta>0} N_{g,\beta}^X e^{\int_\beta \mathbf{t}}, & g\geqslant 2. \end{cases}$$

where $\mathbf{t}\in H^2(X;\mathbb{C})$ is the complexified Kähler class, and

$$N_{g,\beta}^X := \int_{[\overline{\mathcal{M}}_{g,0}(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q}$$

J. Li-Tian constructed the virtual fundamental class $[\overline{\mathcal{M}}_{g,0}(X,\beta)]^{\mathrm{vir}}$ in algebraic geometry. See also **Behrend-Fantechi**.

GW invariants for quintic Calabi-Yau threefolds

Genus $g = 0$. (1990) **Candelas, de la Ossa, Green and Parkes** derived the generating functions F_0^X, for X a smooth quintic Calabi-Yau threefolds. Later, **Lian-Liu-Yau** and **Givental** independently proved the genus zero mirror formula for the quintic Calabi-Yau threefold and further for Calabi-Yau complete intersections in projective toric manifolds.

Genus $g = 1$. (1993) **Bershadsky, Cecotti, Ooguri, and Vafa** derived the generating functions F_1^X, for X a smooth quintic Calabi-Yau threefolds. This was first proved by **Zinger**, using genus-one reduced GW theory

BCOV Theory. (1993) **Bershadsky, Cecotti, Ooguri, and Vafa** the so called BCOV theory, setting the foundation to obtain higher genus generating functions F_g^X for X a smooth quintic Calabi-Yau threefolds.

Genus $g \geqslant 2$. **Yamaguchi-Yau** analyzed BCOV's holomorphic anomaly equation and argued the B-model generating function lies in a finitely generated differential ring. Based on this, **Huang, Klemm, and Quackenbush** derived an algorithm to determine F_g^X, for $g \leqslant 51$,on smooth quintic threefolds.

Two recent approaches to the BCOV genus-two mirror formula, and Yamaguchi-Yau polynomiality conjecture for F_g in all genera:

- (**Guo-Janda-Ruan**, and **Chen-Janda-Ruan**): via an extnsion of moduli of stable maps with p-fields (developed by **Chang-Li**)) that connects GW theory to LG theory
- (**Chang-Li-Li-Liu**, and **Chang-Li-Guo**): via Mixed-Spin-P fields for a master moduli space that describes the gauged linear sigma model.

B-model topological strings: BCOV theory

Bershadsky-Cecotti-Ooguri-Vafa (BCOV) theory proposes a description of higher genus B-model based on a gauge theory on Calabi-Yau threefold. The free energy of this gauge theory gives the generating function $\mathcal{F}_g^{\check{X}}(\tau,\bar{\tau})$ of the topological B-model living over the moduli of the mirror Calabi-Yau $\check{X}$.

Mirror symmetry predicts that Gromov-Witten generating function $F_g^X(\mathbf{t})$ coincides with $\mathcal{F}_g^{\check{X}}(\tau,\bar{\tau})$ of the mirror under the so-called mirror map $t \leftrightarrow \tau$ and limit $\bar{\tau} \to 0$ around the large complex structure

$$F_g^X(\mathbf{t}) = \lim_{\bar{\tau}\to\infty} \mathcal{F}_g^{\check{X}}(\tau,\bar{\tau})$$

The mirror map and $\mathcal{F}_0$ are determined by period integrals of a holomorphic 3-form of the mirror. We will come back to the period integrals later.

Holomorphic anomaly equations

The BCOV theory, among other things, produced the celebrated holomorphic anomaly equations which the B-model free energies $\mathcal{F}_g$ satisfies:

$$\overline{\partial}_i \mathcal{F}_g = \frac{1}{2}\overline{C}_{\overline{i}\overline{j}\overline{k}} e^{2K} G^{i\bar{j}} G^{k\bar{k}} \left(D_j D_k \mathcal{F}_{g-1} + \sum_{r=1}^{g-1} D_j \mathcal{F}_r D_k \mathcal{F}_{g-r} \right)$$

Yamaguchi-Yau analyzed BCOV's holomorphic anomaly equation and argued the B-model generating function lies in a finitely generated differential ring. **Huang-Klemm-Quackenbush** analyzed the polynomial structure of Yamaguchi-Yau and the gap condition at the conifold point to determine the free energies of B-model on the mirror quintic threefold up to genus 51. **Alim-Scheidegger-Yau-Zhou** showed that for the anti-canonical line bundle of a toric semi-Fano surface, the BCOV-Yamaguchi-Yau ring is essentially identical to the ring of almost-holomorphic modular forms in the sense of Kaneko-Zagier (a closely related notion of nearly holomorphic modular forms was systematically studied by Shimura), and the Yamaguchi-Yau functional equation reduces to an equation for modular forms.
Recent mathematical development of Yamaguchi-Yau functional equation:

- by **Lho-Pandharipande** for local Calabi-Yau K_{P^2}
- by **Ruan et al.** for $g = 2$ GW generating functions for quintic CY

Using the theory of Mixed-Spin-P field (**Chang-Li-Li-Liu**), **Chang-Guo-Li** has observed, via computation, that for all genera (the torus localization of) MSP reproduces BCOV Feymann diagram directly. If this can be proved, then MSP should settle Yamaguchi-Yau functional equation for all genera.

Mathematical development of BCOV theory

Must of the work on mathematical foundation of BCOV theory has been led by **Costello-S. Li.**

Costello-Li developed a mathematical framework of quantizing BCOV theory rigorously in terms of effective renormalization method. It also generalized BCOV theory to Calabi-Yau's of arbitrary dimension by incorporating with gravitational descendant.

Costello-Li further extended BCOV theory into open-closed string field theory in topological B-model by coupling with Witten's holomorphic Chern-Simons theory and revealed its connection with large N duality.

Homological mirror symmetry (HMS) for CY

The homological mirror symmetry conjecture by **Kontsevich** for Calabi-Yau mirror pairs X, $\check{X}$ states that Fukaya category of X is equivalent to the derived category of coherent sheaves on $\check{X}$. It is proved by **Seidel** for the genus-two curves and quartic surfaces and generalized by **Sheridan** for all Calabi-Yau and Fano hypersurfaces in projective spaces. A fully faithful mirror functor is achieved via family Floer theory by **Abouzaid** using Lagrangian fibers in the SYZ picture.

2. Donaldson-Thomas Invariants

DT invariants of Calabi-Yau threefolds

Let X be a non-singular projective Calabi-Yau threefold. Donaldson-Thomas (DT) invariants are virtual counts of ideal sheaves of curves in X with holomorphic Euler characteristic n and curve class $\beta \in H_2(X;\mathbb{Z})$

$$DT^X_{n,\beta} := \int_{[\mathrm{Hilb}^{n,\beta}(X)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Donaldon-Thomas constructed the virtual fundamental class $[\mathrm{Hilb}^{n,\beta}(X)]^{\mathrm{vir}}$ and **K. Behrend** proved that $DT^X_{n,\beta}$ is a weighted Euler characteristic.

GW/DT Correspondence and the Topological Vertex

The GW/DT correspondence, conjectured by **Maulik-Nekrasov-Okounkov-Pandharipande** (MNOP), relates the GW theory and DT theory of a threefold. For nonsingular toric Calabi-Yau threefolds, this is equivalent to the **Aganagic-Klemm-Mariño-Vafa** algorithm of the Topological Vertex derived from the large N duality (by the work of **Diaconescu-Florea, Li-Liu-Liu-Zhou, MNOP, and Okounkov-Reshetikhin-Vafa**).

MNOP conjecture for dimension zero DT invariants has been proved by

- MNOP: toric threefolds
- **Behrend-Fantechi**: Calabi-Yau threefolds
- **Levine-Pandharipande**: projective threefolds
- **J. Li**: compact complex threefolds

The vertex of GW/DT correspondence for nonsingular toric threefolds

- 1-leg vertex (framed unknot): **Liu-Liu-Zhou, Okounkov-Pandhariande**,
- 2-leg vertex (framed Hopf link): **Liu-Liu-Zhou**,
- The full 3-leg case: **Maulik-Oblomkov-Okounkov-Pandharipande**

DT/SW correspondence

The DT/SW correspondence was first conjectured using Gauge theory reduction approach by **Gukov-Liu-Sheshmani-Yau** (GLSY). It relates the DT theory of sheaves with 2 dimensional support in a non-compact local surface threefold $X : L \to S, L \in Pic(S)$ over a smooth projective surface S, to the Seiberg-Witten (SW) invariants of the surface S and invariants of nested Hilbert scheme on S.

- Gholampour-Sheshmani-Yau (GSY) proved GLSY conjecture and showed modular property of invariants of Nested Hilbert scheme of points.
- GSY proved that Nested Hilbert scheme invariants can specialize to Poincaré invariants of Dürr-Kabanov-Okonek (DKO) and the stable pair invariants of Kool-Thomas (KT).
- GSY related the Vafa-Witten (VW) invariants to SW invariants of surface $+$ correction terms governed by invariants of nested Hilbert schemes.

3. Periods and Mirror Map

Geometric set-up

Consider a family $\pi : \mathcal{Y} \to B$ of d-dimensional compact complex manifolds, with $Y_b := \pi^{-1}(b)$. The $H^d(Y_b)$ form a vector bundle $E \to B$ with a flat connection ∇. Fix a global section $\omega \in \Gamma(B, E)$, it generates a period sheaf $\mathbf{\Pi}(\omega)$ given by integrals over geometric cycles

$$\int_\gamma \omega.$$

The study of period map has a long history by Euler, Gauss, Riemann, and there were works of Picard, Fuchs, Leray, Griffiths, Dwork and also Gelfand-Kapranov-Zelevinsky. Period integrals is a very important component of computations in mirror symmetry as pioneered by Candelas, de la Ossa, Green and Parkes.

Large Complex Structure Limits

- **Hosono-Lian-Yau** '96 showed for hypersurfaces Y in a toric variety X, a Large Complex Structure Limit (LCSL) Y_∞ is given by the toric divisor $D = \cup D_i$ of X. They also computed the periods of Y near Y_∞ by first finding complete solutions to a GKZ system.
- By [**Lian-Todorov-Yau** '00], a LCSL Y_∞ is characterized by the property that, there is a global meromorphic top form, such that its restriction to Y_∞ has the form $(z_1 ... z_n)^{-1} dz_1 \wedge ... \wedge dz_n$ in a neighborhood.
- Mirror Symmetry Conjecture:

$$\text{periods near } \; Y_\infty \Longrightarrow \text{counting curves on mirror } \; Y^*$$

.

General Problem: Construct an explicit *complete* linear PDE system τ for the sheaf $\mathbf{\Pi}(\omega)$. That is a system τ such that

$$\boxed{sol(\tau) = \mathbf{\Pi}(\omega).}$$

Main goals & applications:

- To compute explicitly periods $\int_\gamma \omega$ as power series and determine local, and even global monodromy of periods
- To count curves in algebraic varieties by mirror symmetry

Tautological systems

Lian-Yau in 2010 wrote global residue formula for period integrals. Period integrals for the family $\mathcal{Y}$ are annihilated by the Fourier transform of the defining ideal of the tautological embedding of X, and first order symmetry operators induced by the ambient space symmetry. The resulting system of differential equations τ is called a *tautological system*. τ specializes to a **GKZ system** if X is a toric variety and $G = T$ is the dense torus.

The tautological system τ is complete for $X = \mathbb{P}^n$ by **Bloch-Huang-Lian-Srinivas-Yau** 2013. Moreover, for any homogeneous G-variety, τ is complete iff $PH^n(X) = 0$ (**Huang-Lian-Zhu** 2014) . The construction also describes solutions to the differential system τ in topological terms. This result is applied to the large complex structure limit for homogeneous G-variety of a semisimple group G for the purpose of mirror symmetry.

A recent application: Hasse-Witt matrices and periods

- Let X be a Fano toric variety or G/P of dimension n defined over $\mathbb{Z}$, and $\pi : \mathcal{Y} \to B$ the universal family of Calabi-Yau (or general type) hypersurfaces in X. Given any prime p, after reduction mod p, we have Frobenius endomorphism $F \colon \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{Y}}$ by raising functions to p-th power. This gives a p-semilinear morphism

$$R^{n-1}\pi_*(\mathcal{O}_{\mathcal{Y}}) \to R^{n-1}\pi_*(\mathcal{O}_{\mathcal{Y}}).$$

  Under certain basis by adjunction formula, this gives rise to the Hasse-Witt matrices HW_p.
- [Huang-Lian-Yau-Yu'18] An appropriate limit of a rescalling of HW_p as $p \to \infty$ recovers the unique holomorphic period of the family over the complex numbers, at the rank 1 point (LCSL candidate) s_∞. Conversely, HW_p is equal to an appropriate truncation of Taylor series expansion of the complex period at s_∞ mod p.
- In the case of X being a toric variety, this explains part of the work by Candelas, de la Ossa, Rodriguez-Villegas in 2000, regarding Calabi-Yau varieties over finite fields.

4. SYZ Conjecture and Special Lagrangian Geometry

T-duality and SYZ

- T-duality relates
  - type IIA string theory on the circle S_A of radius R

    T-dual $\updownarrow$
  - type IIB string theory on the circle S_B of radius $1/R$.

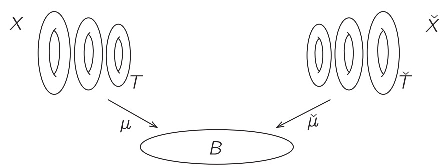
Inspired by the role of D-branes under T-duality on Calabi-Yau

Conjecture (**Stominger-Y.-Zaslow, 1996)**

Let X and $\check{X}$ be a mirror pair of CY manifolds near the large structure limits.

1. *X and $\check{X}$ admit dual special Lagrangian torus fibrations $\mu : X \to B$ and $\check{\mu} : \check{X} \to B$ over the same base B.*
2. *There exists a fiberwise Fourier-Mukai transform which maps Lagrangian submanifolds of X to coherent sheaves on $\check{X}$.*

e.g. D3: SLag torus fibers $\longleftrightarrow$ D0: skyscraper sheaves.



mirror symmetry
is T-duality!

FM transform of special Lagrangian equation

- **Leung-Yau-Zaslow** worked out fiberwise Fourier-Mukai transform of supersymmetric cycles when quantum correction is absent (semi-flat).

Theorem (Leung-Y.-Zaslow)

Let X and $\check{X}$ be a mirror pair of CY manifolds (in the sense of SYZ). Then the fiberwise Fourier-Mukai transform of a special Lagrangian section in X produces a holomorphic line bundle on $\check{X}$ which satisfies the deformed Hermitian Yang-Mills equation (dHYM in short).

- More specifically,

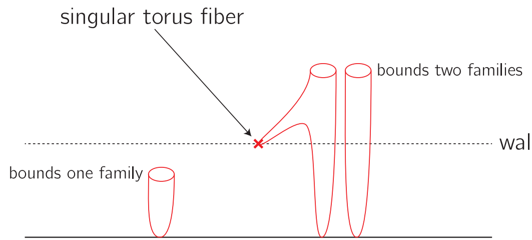
$$\begin{array}{l} (i)\ \omega|_C = 0 \\ (ii)\ \operatorname{Im}\Omega|_C = \tan\theta\operatorname{Re}\Omega|_C \end{array} \xrightarrow{FM} \begin{array}{l} (i)\ F_A^{0,2} = 0 \\ (ii)\ \operatorname{Im}(\omega + F_A)^k = \tan\theta\operatorname{Re}(\omega + F_A)^k \end{array}$$

Construction of SYZ fibrations

- **W.D. Ruan** asserted a Lagrangian torus fibration on the quintic CY constructed by gradient flow and toric degenerations. **Goldstein** and **Gross** constructed Lagrangian fibrations for toric Calabi-Yau n-folds, generalizing the fibrations of **Harvey-Lawson** on $\mathbb{C}^3$. They showed that these fibrations are special with respect to certain holomorphic volume forms.
- **Castano-Bernard** and **Matessi** constructed Lagrangian fibrations on local models around singular fibers in dimension 2 and 3, and applied symplectic methods to glue them to give a global Lagrangian fibration.
- Near a singular fiber, the genuine CY metric should be approximated by gluing local CY models around singular fibers with the *semi-flat metric*.
- The semi-flat metric with singularities were constructed by **Greene-Shapere-Vafa-Yau**. The local CY model around each singular fiber is constructed by **Ooguri-Vafa**. For K3 surface, **Gross-Wilson** attempts to glue together the models based on the original proof of Calabi conjecture, but failed to contain enough information of instanton corrections.

SYZ construction with quantum corrections

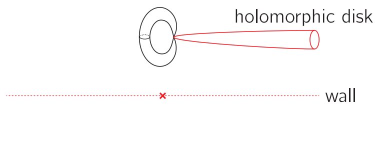
- **Auroux** proposed that the counting of holomorphic disks (that bounds smooth torus fibers) drastically changes when one goes across a certain wall in the SYZ base, displaying a *wall-crossing phenomenon*.

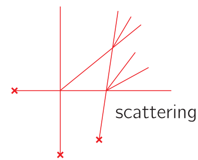


- **Chan-Lau-Leung** used this technique to construct the SYZ mirrors of toric CY manifolds. A typical example is $K_{\mathbb{P}^n}$.
  Abouzaid-Auroux-Katzarkov constructed toric CY manifolds as SYZ mirrors of blow-ups of toric varieties.
- Recently **Lau-Zheng** constructed the SYZ mirrors of hypertoric varieties. A typical example is $T^*\mathbb{P}^n$. They serve as local models for holomorphic symplectic manifolds.

Quantum corrections and scattering

- Quantum correction come from holomorphic disks emanating from singular fibers, forming the *wall structure* on the SYZ base.





- **Kontsevich-Soibelman** provided a universal *wall-crossing formula* and asserted that the scattering of holomorphic disks obey the formula.
- **Gross-Siebert** developed a reconstruction program of mirrors which uses the wall crossing formula to capture quantum corrections combinatorially.
- **Chan-Cho-Lau-Tseng** computed all the quantum corrections for toric CY via the mirror map. **Chan-Lau-Leung-Tseng** found a relation with Seidel representations for semi-Fano toric manifolds.

5. Non-Kähler Calabi-Yau Manifolds

Non-Kähler Calabi-Yau Manifolds

It is interesting to also consider manifolds that are not Kähler but whose canonical bundle is still trivial. These non-Kähler, almost Hermitian manifolds with $c_1 = 0$ are referred to as non-Kähler Calabi-Yau manifolds. Interest in them within the context of complex threefolds dates back to the mid-1980s.

In 1987, **Reid** made a proposal (commonly called Reid's fantasy) that all Calabi-Yau threefolds, that can be deformed to a Moishezon manifold, fit into a single universal moduli space where Calabi-Yaus of different homotopy types are connected to one another through conifold transitions. These conifold transition made use of the work of **Clemens** and **Friedman**: first contract disjoint rational curves which results in a Calabi-Yau threefold with double-point singularities and then deform the singular space into a smooth complex manifold.

The complex manifold that results from a Clemens-Friedman conifold however need not be Kähler. For instance, we may contract enough rational curves such that the second Betti number becomes zero. The smoothed-out complex manifold then can only be non-Kähler. In fact, it would be diffeomorphic to a k-connected sum of $S^3 \times S^3$ with $k \geqslant 2$.

Non-Kähler Calabi-Yau manifolds and the Strominger system

For non-Kähler Calabi-Yaus, we would like to impose some additional geometric conditions which replaces the Kähler condition. One such set of conditions is that from a system of supersymmetry equations from heterotic strings known as the **Strominger** system. This system requires the consideration of also a stable bundle over the complex manifold. The hermitian metric is required to satisfy the balanced condition plus an anomaly equation that relates the hermitian metric with the Hermitian-Yang-Mills metric on the stable bundle.

Many solutions of the Strominger systems on complex threefolds are now known. **Fu-Yau** (2006) gave the first solution on a compact, non-Kähler threefold – a torus bundle over a $K3$ surface. Recently, **Fei-Huang-Picard** (2017) found Strominger system solutions on compact, non-Kähler threefolds of infinitely many topological types and Hodge numbers.

Non-Kähler symplectic Calabi-Yaus and mirror symmetry

For symplectic manifolds, we can use a compatible almost complex structure on it to define a first Chern class. A symplectic manifold is called symplectic Calabi-Yau if its first Chern class is trivial. Symplectic Calabi-Yau manifolds are generally non-Kähler manifolds. Consideration of such non-Kähler symplectic Calabi-Yau spaces is also natural as mirror duals of non-Kähler complex Calabi-Yau spaces.

Indeed, a symplectic mirror of the Clemens-Friedman's conifold transition for complex threefolds was proposed by **Smith-Thomas-Yau** in 2002. In the symplectic version, disjoint Lagrangian three-spheres of a Kähler Calabi-Yau would be collapsed and replaced by symplectic two-spheres. The resulting real six-manifold would be still symplectic but its third Betti number may be zero. Hence, such a manifold would be in general be non-Kähler, but nevertheless, a sympletic Calabi-Yau.

Smith-Thomas-Yau used such symplectic conifold transitions to construct many real six-dimensional symplectic Calabi-Yaus. They showed that if such a conifold transition collapsed all disjoint three-spheres, then the resulting space is a manifold that is diffeomorphic to connected sums of $\mathbb{CP}^3$. This mirrors the complex case, where after the collapsed of all disjoint rational curves gives a connected sums of $S^3 \times S^3$.

Symplectic Calabi-Yaus can also be considered from the perspective of SYZ mirror symmetry of non-Kähler Calabi-Yaus. In the semi-flat limit, **Lau-Tseng-Yau** (2015) related symplectic, non-Kähler, $SU(3)$ supersymmetry conditions of type IIA strings to those of complex, non-Kähler, $SU(3)$ supersymmetry conditions of type IIB strings via SYZ and Fourier-Mukai transform. In real dimensions eight and higher, they also presented a mirror pair system of equations - a symplectic system and a complex system for non-Kähler Calabi-Yaus - that are dual to each other under SYZ mirror symmetry.

6. Calabi-Yau Cones

Calabi-Yau Cones

A Kähler manifold (X^{2n}, J, g) is called a Kähler cone if

- There is a function $r : X \to \mathbb{R}_{>0}$ so that

$$g = dr^2 + r^2 \bar{g}$$

  for some metric $\bar{g}$ on the *Link* of the cone, $\{r = 1\}$.
- The vector fields $r\partial_r$, and $\xi := J(r\partial_r)$ are real holomorphic.
- In particular, $r\partial_r - \sqrt{-1}\xi$ generates a $(\mathbb{C}^*)^k$ action on X for some $k \geqslant 1$.

(X^{2n}, J, g) is a Kähler cone if and only if the link (S, g) is a Sasakian manifold with Reeb vector field ξ.

Calabi-Yau Cones

Theorem

If (X^{2n}, J, g) is a Kähler cone manifold, then (X^{2n}, J) can be embedded as an affine variety in $\mathbb{C}^N$ for some large N, so that ξ is induced by a holomorphic vector field in $\mathfrak{u}(N)$.

Motivated by the AdS/CFT correspondence, it is natural to ask:

Question

When does (X^{2n}, J, g) admit a Ricci-flat Kähler cone metric?

As in the compact case, there are restrictions on which X we can consider. X is *admissible* if :

- X is $\mathbb{Q}$-Gorenstein (ie. $K_X^{\otimes \ell} = \mathcal{O}_X$ for some $\ell > 0$)
- X has at worst log-terminal singularities.

Calabi-Yau Cones

In fact, there are many examples of such affine varieties, defined by weighted homogeneous polynomials.

Example

For each p, q *the affine variety* $Z_{p,q} = \{xy + z^p + w^q = 0\} \subset \mathbb{C}^4$ *is* $\mathbb{Q}$*-Gorenstein, log terminal, with a Reeb vector field* $\xi_{p,q}$ *induced by the action*

$$\lambda.(x, y, z, w) = (\lambda^{pq}x, \lambda^{pq}x, \lambda^{2q}z, \lambda^{2p}w)$$

Example

If Y is a projective variety with $-K_Y$ ample, then $X = \overline{K_Y \backslash \{0\}}$ is admissible, with $\mathbb{C}^*$ action by scaling the fibers. X admits a Ricci-flat cone metric with respect to the scaling Reeb field if and only if Y admits a K"ahler-Einstein metric with positive Ricci curvature.

Kähler-Einstein metrics with positive Ricci curvature are obstructed in general.

Conjecture (Yau-Tian-Donaldson)

A Kähler manifold Y admits a Kähler-Einstein metric with positive Ricci curvature if and only if $(Y, -K_Y)$ is K-stable.

The notion of K-stability is purely *algebraic*.

Theorem (Chen-Donaldson-Sun)

The Yau-Tian-Donaldson conjecture is true.

For conical Ricci-flat metrics:

- **Martelli-Sparks-Yau** found obstructions to existence of Ricci-flat cone metrics, and explained how these obstructions could be interpreted in terms of the AdS/CFT dual field theory.
- **Futaki-Ono-Wang** proved that Ricci flat Kähler cone metrics exist on all admissible toric varieties, for appropriately chosen Reeb field.
- **Collins-Székelyhidi** found a notion of K-stability for Ricci-flat cone metrics on an admissible affine variety X.
- If $X = \{f_1 = \cdots = f_n = 0\} \subset \mathbb{C}^N$, K-stability is defined in terms of subtle algebraic properties of the ring

$$\frac{\mathbb{C}[x_1, \ldots, x_N]}{(f_1, \ldots, f_n)}$$

graded by the Reeb vector field.

Theorem (Collins-Székelyhidi)

The affine variety X, with Reeb vector field ξ, admits a Ricci-flat Kähler cone metric if and only if (X, ξ) is K-stable.

Collins-Székelyhidi show that the affine variety $Z_{p,q} = \{xy + z^p + w^q = 0\} \subset \mathbb{C}^4$ with Reeb vector field a multiple of $\xi_{p,q}$ is K-stable if and only if $2p > q$ and $2q > p$.

Using the AdS/CFT correspondence, **Collins-Xie-Yau** proposed an interpretation of K-stability in field theory terms. The interpretation is in terms of the chiral ring and a generalized central charge maximization.

7. Stability in Calabi-Yau Geometry

Stability conditions

- Motivated by Π-stability of D-branes studied by Douglas, in 2007, Bridgeland introduced the notion of stability conditions on a triangulated category $\mathcal{D}$.
- Given a Calabi–Yau manifold X, we can associate two triangulated categories to it:
- (A): The derived Fukaya category $\mathrm{Fuk}(X)$, which only depends on the symplectic structure on X.
- (B): The derived category of coherent sheaves $\mathrm{D^b Coh}(X)$, which only depends on the complex structure on X.
- Bridgeland stability conditions are supposed to recover the “other half” of the geometric structure from the category: Stability conditions on $\mathrm{Fuk}(X)$ recover the complex structure, and stability conditions on $\mathrm{D^b Coh}(X)$ recover the symplectic structure.

Deformed Hermitian–Yang–Mills equation

- To study Bridgeland stability conditions on $\mathrm{D^bCoh}(X)$, we consider the *mirror* of special Lagrangian equation, which is the deformed Hermitian–Yang–Mills equation.
- Consider a holomorphic line bundle L over X (supposedly dual to a Lagrangian section in the mirror).
- **Q)** Does there exist a metric h on L such that

  $$\mathrm{Im} e^{-i\hat{\theta}}(\omega - F)^n = 0,$$

  where $F = -\partial\bar{\partial}\log h$, and $e^{i\hat{\theta}} \in S^1$ is a topological constant determined by $\omega, c_1(L)$?
- Note $e^{i\hat{\theta}} \in S^1$ is determined by

  $$\int_X (\omega + \sqrt{-1}c_1(L))^n \in \mathbb{R}_{>0} e^{i\hat{\theta}}$$

- **Jacob-Yau** and **Collins-Jacob-Yau** studied the existence and the uniqueness of the deformed Hermitian-Yang-Mills equation.

Lagrangian phase θ of line bundles

- Consider the relative endomorphism K of $T^{1,0}(X)$ given by

$$K := \sqrt{-1} g^{j\bar{k}} F_{\bar{k}l} \frac{\partial}{\partial z^j} \otimes dz^l.$$

- In terms of normal coordinates with $g_{\bar{k}j} = \delta_{kj}$ and $F_{\bar{k}j} = \sqrt{-1}\lambda_j \delta_{kj}$ ($\lambda_j =$ eigenvalue of K), define

$$\Theta_\omega(h) = \textstyle\sum_j \arctan(\lambda_j).$$

- The dHYM equation is $\theta(h) = \Theta$ for a constant $\Theta \in (-n\frac{\pi}{2}, n\frac{\pi}{2})$.
- If there exists a solution of dHYM, then the constant Θ is unique, and lifts $\hat{\theta}$ to $\mathbb{R}$: $e^{i\hat{\theta}} = e^{i\Theta}$.
- **Jacob–Y.**: For given a line bundle L on X, suppose that (L, h) has $\mathrm{osc}_X \Theta_\omega(h) < \pi$, then there exists a *unique* lift $\Theta \in \mathbb{R}$ of $\hat{\theta} \in [0, 2\pi)$.

Analysis on the phase of dHYM equation

Theorem (Collins–Jacob–Y.)

Suppose that the lifted angle Θ *satisfies the critical phase condition*

$$\Theta > (n-2)\frac{\pi}{2}.$$

Then there exists a solution to the deformed Hermitian-Yang-Mills equation if and only if there exists a metric $\underline{h}$ *on* $L \to X$ *so that*

- $\Theta_\omega(\underline{h}) > (n-2)\frac{\pi}{2}$, *and*
- *For every* $1 \leqslant k \leqslant n$

$$\sum_{j \neq k} \arctan(\lambda_j(\underline{h})) > \Theta - \frac{\pi}{2}$$

where $\lambda_j(\underline{h})$ *are the eigenvalues of* $\sqrt{-1}g^{j\bar{k}}F(\underline{h})_{\bar{k}l}$.

Furthermore, the solution is unique up to multiplication by a positive constant.

Obstructions to the existence of solutions of dHYM

From the conditions in the theorem, we find obstructions to the existence of a solution:

Theorem (Collins-Jacob-Y.)

If there exists a solution to the dHYM equation, then for every proper, irreducible analytic subvariety $V \subset X$ *with* $\dim V = p$ *we have*

$$Im\left(\frac{Z_V(L)}{Z_X(L)}\right) > 0$$

where $Z_V = \int_V e^{-\sqrt{-1}\omega} ch(L)$*, and* $Z_X = \int_X e^{-\sqrt{-1}\omega} ch(L)$*.*

Note: (1) This obstruction appears to be related to Bridgeland stability.
(2) In dimension 2, these inequalities are equivalent to existence of a solution to the dHYM equation (Collins-Jacob-Y.).

Thanks!