S-duality, Quadratic Reciprocity, and Double Janus Configurations

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Based on works:
Ori J. Ganor, Nathan Moore, H.-Y. S., and Nesty Torres-Chicon,
arXiv: 1403.2365

and upcoming papers with:
Ori J. Ganor, H.-Y. S., and Nesty Torres-Chicon;

Law of Quadratic Reciprocity

$$x^2 \equiv a \pmod{p}? \qquad \Longrightarrow \qquad x^2 \equiv p \pmod{a}?$$

If p, a are odd primes,

$$\left(\tfrac{a}{p}\right) = \begin{cases} \left(\tfrac{p}{a}\right) & \text{if } p \equiv 1 \text{ or } a \equiv 1 \pmod{4} \\ -\left(\tfrac{p}{a}\right) & \text{if } p \equiv a \equiv 3 \pmod{4} \end{cases}$$

Conjectured by Euler and Legendre.
Proved by Gauss in 1801 in six ways!

The roadmap

Quadratic Reciprocity

$$\left(\frac{p}{q}\right) = \pm\left(\frac{q}{p}\right)$$

Calculate a certain partition function

in two different ways

How do we get String Theory to answer questions **mod** p?

Review: Janus Solution

$4D\ N=4$ SYM with gauge group $G=U(n)$

$$I = \mathrm{tr}\int\left(\frac{1}{4g^2}F\wedge^* F+\theta F\wedge F\right)+\text{scalars \& gluinos.}$$

$$\text{complex coupling :} \qquad \tau=\frac{4\pi i}{g^2}+\frac{\theta}{2\pi}$$

Janus: dilatonic deformation of AdS_5 geometry to type IIB SUGRA, and its gauge dual is

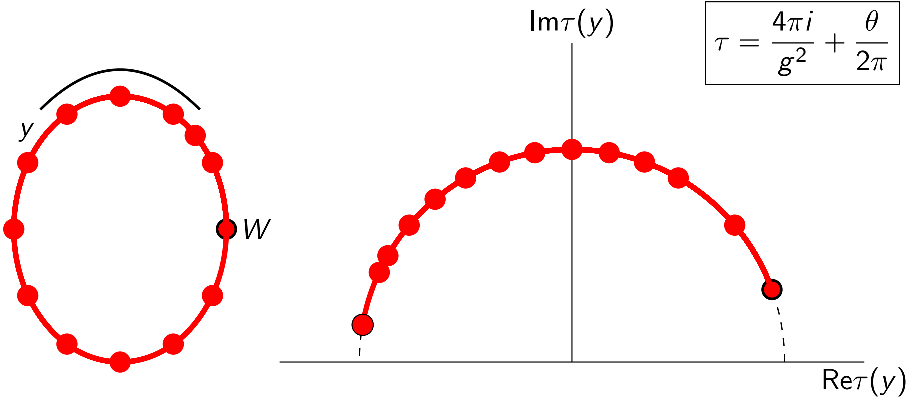
$$I = \mathrm{tr}\int\left[\frac{1}{4g(y)^2}F\wedge^* F+\theta F\wedge F\right]dy\,d^3x+\cdots\ ,\ \theta\text{ fixed}$$

[Bak, Gutperle, Hirano] : g jumps at the codim-1 interface, non-SUSY
[Clark, Karch] : one quarter of supersymmetries preserved by breaking SO
[D'Hoker, Estes, Gutperle] : smoothed the jump

Supersymmetric Janus Configurations

[Gaiotto & Witten, 2008]

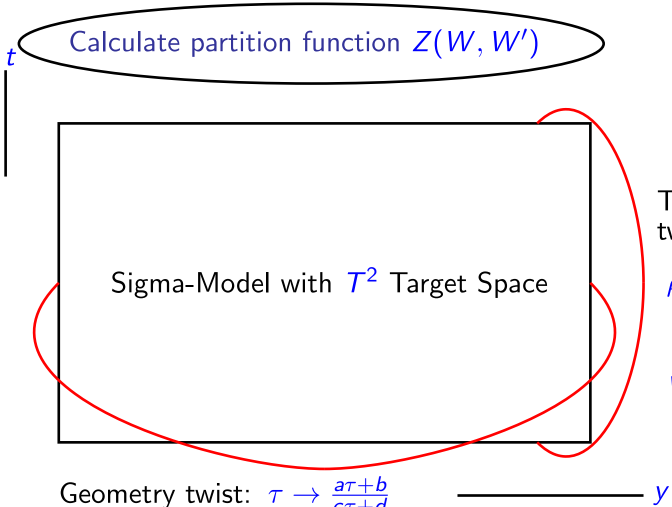
$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$$



$\tau(y)$ *arbitrary, as long as it traces a big circle in upper half plane*

$$I = \mathrm{tr} \int \left[\frac{1}{4g(y)^2} F \wedge^* F + \theta(y) F \wedge F \right] dy d^3x + O(\tau') + O(\tau'') + O(\tau'^2)$$

Quadratic Reciprocity from Double-Janus



T-duality twist:

$\rho \to \frac{a'\rho+b'}{c'\rho+d'}$

$$W' \equiv \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

Geometry twist: $\tau \to \frac{a\tau+b}{c\tau+d}$

$$W \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$Z(W, W')$

$$Z(W, W') = \mathrm{tr}_{H_W}(W') = \exp(i\phi_{\mathrm{Berry}})\,\mathrm{tr}_{H_{W'}}(W)$$

$$H_W = \text{Hilbert space with } t \text{ as time}$$

$$H_{W'} = \text{Hilbert space with } y \text{ as time}$$

Z localizes on $X = \mathrm{const.}$

Landsberg-Schaar Identity

$$Z(W, W') = \mathrm{tr}_{H_W}(W') = \exp(i\phi_{\mathrm{Berry}})\,\mathrm{tr}_{H_{W'}}(W)$$

Take

$$W = \begin{pmatrix} (p+2) & -1 \\ 1 & 0 \end{pmatrix} = T^{p+2}S$$

$$W' = \begin{pmatrix} (q+2) & -1 \\ 1 & 0 \end{pmatrix} = T^{q+2}S$$

$$\boxed{\frac{e^{\pi i/4}}{\sqrt{p}} \sum_{n=0}^{p-1} \exp\left(-\frac{\pi i n^2 q}{p}\right) = \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} \exp\left(\frac{\pi i n^2 p}{q}\right)} \qquad \begin{array}{l} p \in 2\mathbb{Z} \\ q \in 2\mathbb{Z}+1 \end{array}$$

holds for all positive even p and odd q.

$\Rightarrow$ Implies Quadratic Reciprocity.

$\star$ Can easily be derived from modular properties of $\theta(\frac{p}{q} + i\epsilon)$.

Generalization for more complicated W

For $G = U(N)$ replace $T^2 \to (T^2)^N/S_N$
$N = 2$ identity reduces to $N = 1$ with

$$W' = T^{q+2}ST^{q+2}S\,, \qquad W = T^{p+2}S\,.$$

$$-\frac{i}{q}\sum_{m,n=0}^{q-1} e^{-\frac{2\pi i}{q}(2mn-am^2-bn^2)} = \left(\frac{1+i^{qb}}{2\sqrt{ab-1}}\right)\sum_{n=0}^{2ab-3}\exp\left(-\frac{\pi iqan^2}{2(ab-1)}\right)\,,$$

$$q \in 2\mathbb{Z}_+, \qquad a, b \in \mathbb{Z}\,, \qquad ab > 1.$$

Hence, in principle, we can generalize this to triple sum and so on.