On the gauge invariant path-integral measure for the overlap Weyl fermions in <u>16</u> of SO(10)

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based on :
Y.K., arXiv:1710.11618[hep-lat], arXiv:1710.11101[hep-lat]

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plan of this talk

1. Chiral lattice gauge theories ?   — A proposal for SO(10) with **16**

2. Set up — Overlap Weyl fermion / the Ginsparg-Wilson relation

3. Approaches — "*All things merge into one, and a river runs through it*"

   local cohomology problem — "*it's tough*"

   1) mirror Ginsparg-Wilson fermions — "*Mission: Impossible?*"

   2) vector-like domain-wall fermions with boundary interactions

   3) 4D TI/TSC with gapped boundary phase

   4) Saturation of the measure of right-handed modes — "*A New Hope*"

   use of the gradient flow — "*it is orthogonal to all the above ones*"

4. Discussion:

   1) more on the saturation of the measure

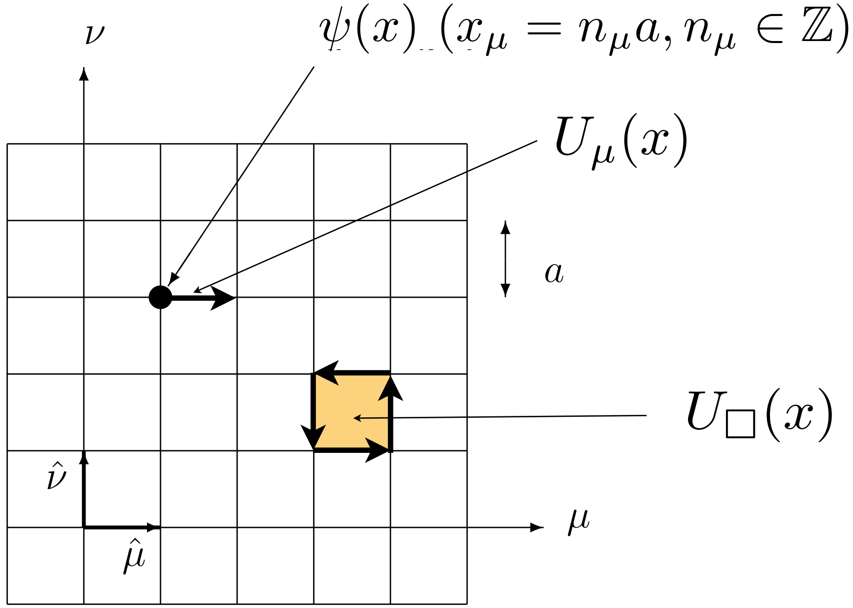
   2) check of locality   cf. 2dim. U(1) chiral gauge theories

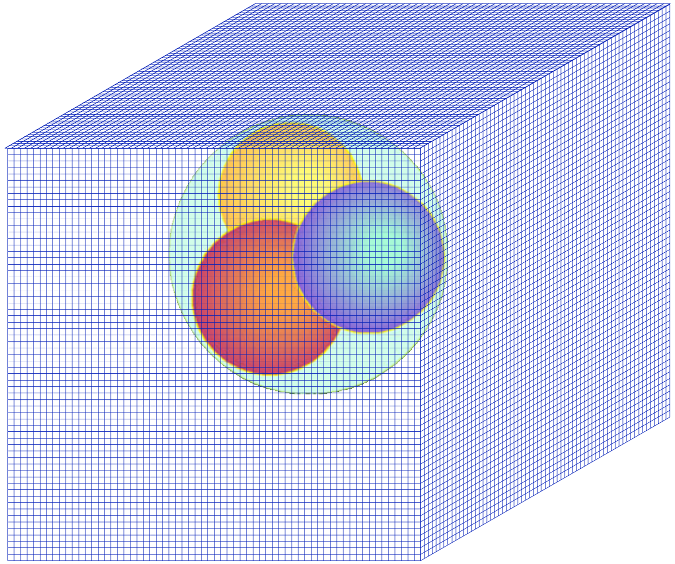
   3) sign problem ?   cf. Langevin Eq. / generalized Lefschetz thimble

1. Chiral Lattice Gauge Theory ?

— A proposal for SO(10) with <u>16</u>

Lattice Gauge Theories





gauge transformation

$$\psi(x) \longrightarrow g(x)\psi(x) \quad g(x) \in G$$

$$U_\mu(x) \to g(x)U_\mu(x)g^{-1}(x+\hat{\mu}) \quad U_\mu(x) \in G$$

$$\nabla_\mu \psi(x) = \frac{1}{a}\left(U_\mu(x)\psi(x+\hat{\mu}a) - \psi(x)\right)$$

$$[\nabla_\mu, \nabla_\nu]\psi(x) = \Big(1 - U_\Box(x)\Big)U_\mu(x)U_\nu(x+\hat{\mu}a)\psi(x+\hat{\mu}a+\hat{\nu}a)$$

$$U_\Box(x) = U_\mu(x)U_\nu(x+\hat{\mu}a)U_\mu(x+\hat{\nu}a)^{-1}U_\nu(x)^{-1}$$

Action and Path Integral measure

$$S_G = \frac{1}{g^2}\sum_{x\mu\nu} \mathrm{ReTr}\left(1 - U_\Box(x)\right)$$

$$\mathcal{D}[U_\mu(x)] = \prod_{x,\mu} dU_\mu(x)$$

compact and group-invariant

Continuum limit :

$$m_{\text{phys}} \ll \Lambda = \frac{1}{a} \quad (\nearrow \infty)$$

$$x_\mu = n_\mu a \iff p_\mu \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]$$

$$\langle \phi(x)\phi(y) \rangle \simeq \mathrm{e}^{-m_{\text{phys}}|x-y|} \quad \Big(|x-y| \gg a \Big)$$

$$\xi = \frac{1}{m_{\text{phys}}\, a} \nearrow \infty$$

second order phase transition

scaling law (universal)

$$g_0 = g_0(\Lambda/m_{\text{phys}})$$

Renormalization

RG approach : Fixed point relevant ops. renormalized traj.

Renormalization theory defined non-perturbatively

"Definition" of Quantum Field Theories

Lattice

Framework to "perform" Path Integral
(cf. Monte Carlo method)

(Naive regularization, but generic and powerful)

Chiral Lattice Gauge Theories ? Why not?

Chiral Lattice Gauge Theories ? Why not?

SU(3)xSU(2)xU(1) xU(1)$_{B-L}$

$(\underline{3}, \underline{2})_{1/6}$ $(\underline{1}, \underline{2})_{-1/2}$

$(\underline{3^*}, \underline{1})_{-2/3}$ $(3^*, 1)_{1/3}$ $(\underline{1}, \underline{1})_{1}$ $(\underline{1}, \underline{1})_{0}$

SU(5) xU(1)$_Q$

$(\underline{10})_{1}$

$(\underline{5^*})_{-3}$

$(\underline{1})_{5}$

SU(4)xSU(2)xSU(2)

$(\underline{4}, \underline{2}, \underline{1})$

$(\underline{4^*}, \underline{1}, \underline{2})$

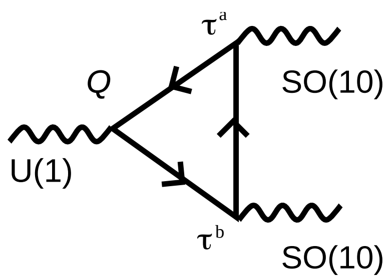
SO(10)

$(\underline{16})$

- **complex rep. of the gauge group**
- **gauge anomaly cancellation :** $Tr[\, T^a \{T^b, T^c\}\,]=0$
- **gauge inv. fermion bilinear terms** $\psi\varepsilon\psi$ **forbidden**
- **no gauge inv. regularization, but "local" counter terms in all orders of perturbation theory**
- **fermion # symmetry broken due to chiral anomaly**
- **various realizations of gauge/flavor symmetries, etc.**

Fermion # symmetry broken due to chiral anomaly [SO(10)]

chiral anomaly for 16 of SO(10)



Index theorem for 16 of SO(10)

when embedding SU(2) instanton solutions

$$\mathrm{Index}\, D = \sum_{\mathrm{SU}(2)} m\, q$$

q: topological charge of the instanton
m: integer multiple of 4

$\mathrm{SU}(5)$	$\{\underline{10}, \underline{5}^*\}$
$\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$	$\{(\underline{4},\underline{2},\underline{1}), (\underline{4}^*,\underline{1},\underline{2})\}$
$\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$	$\{(\underline{3},\underline{2})_{1/6}, (\underline{3}^*,\underline{1})_{-2/3}, (\underline{3}^*,\underline{1})_{1/3}, (\underline{1},\underline{2})_{-1/2}, (\underline{1},\underline{1})_{1}, (\underline{1},\underline{1})_{0}, \}$

zero modes => VEV of 't Hooft vertex => Fermion # symmetry breaking

Gauge field of SO(10)

$$\tilde{U}(x,\mu) = \mathrm{e}^{i\theta^{ab}(x,\mu)\tilde{\Sigma}_{ab}/2} \quad \in \mathrm{SO}(10)$$

$$U(x,\mu) = \mathrm{e}^{i\theta^{ab}(x,\mu)\Sigma_{ab}/2} \quad \in \mathrm{Spin}(10)$$

$$\{\tilde{\Sigma}_{ab}\}_{cd} = i(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})$$

$$\Sigma_{ab} = -\frac{i}{4}\left[\Gamma^a, \Gamma^b\right]$$

$$\{\Gamma^a \,|\, a = 1, 2, \cdots, 10\}$$

$$\Gamma^a\Gamma^b + \Gamma^b\Gamma^a = 2\delta^{ab}\,; \quad \Gamma^{a\dagger} = \Gamma^a$$

Dirac field in 16-dim. spinor rep. of SO(10)

$$\psi(x) = \mathrm{P}_+\psi(x), \qquad \bar{\psi}(x) = \bar{\psi}(x)\mathrm{P}_+$$

$$\mathrm{P}_+ = \frac{1+\Gamma^{11}}{2}, \qquad \Gamma^{11} = -i\Gamma^1\Gamma^2\cdots\Gamma^{10}$$

$$\Gamma^a\Gamma^b + \Gamma^b\Gamma^a = 2\delta^{ab}\,; \quad \Gamma^{a\dagger} = \Gamma^a \quad (a = 1, \cdots, 10),$$
$$\Gamma^{11} = -i\Gamma^1\Gamma^2\cdots\Gamma^{10}, \quad [\Gamma^{11}, \Gamma^a] = 0 \quad (a = 1, \cdots, 10),$$
$$\mathrm{C}\Gamma^a\mathrm{C}^{-1} = -\{\Gamma_a\}^T,\ \mathrm{C}\Gamma^{11}\mathrm{C}^{-1} = -\Gamma_{11},\ ;\ \mathrm{C}^T = \mathrm{C}^{-1} = \mathrm{C}^\dagger = -\mathrm{C}$$

Fermion # symmetry broken due to chiral anomaly [SO(10)]

zero modes => VEV of ’t Hooft vertex => Fermion # symmetry breaking

$$\boxed{\underline{16} \times \underline{16} \times \underline{16} \times \underline{16} \Rightarrow \underline{1}}$$

$\underline{16} \times \underline{16} = \underline{10} + \underline{120} + \underline{126}$

$\underline{10} \times \underline{10} = \underline{1} + \underline{45} + \underline{54}$

$(\underline{16} \times \underline{16} \Rightarrow \underline{10})$

$$\mathrm{T}^a = \mathrm{C}\Gamma^a \qquad \mathrm{T}^{aT} = \mathrm{T}^a$$

$$V_-^a(x) = \psi_-(x)^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a \psi_-(x)$$

$$\bar{V}_-^a(x) = \bar{\psi}_-(x) i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{\psi}_-(x)^{\mathrm{T}}$$

$$\boxed{T_-(x) = \frac{1}{2} V_-^a(x) V_-^a(x)} \qquad \boxed{\bar{T}_-(x) = \frac{1}{2} \bar{V}_-^a(x) \bar{V}_-^a(x)}$$

[’t Hooft vertices in the SO(10) theory]

Fermion # symmetry broken due to chiral anomaly [SU(5)]

SO(10) **SU(5) x U(1)$_Q$**

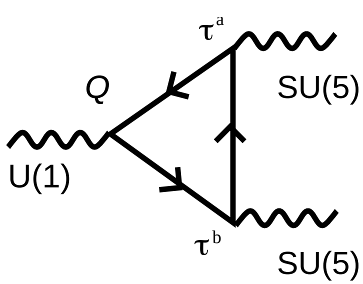
$\underline{16}$ = $\underline{10}_{1}$ + $\underline{5^*}_{-3}$ + $\underline{1}_{5}$ $\underline{1}_{5}$: spectator fermion

$(\underline{16}$ x $\underline{16}$ x $\underline{16}$ x $\underline{16})$ = $(\underline{10}$ x $\underline{10}$ x $\underline{10}$ x $\underline{5^*})_{0}$ + $(\underline{10}$ x $\underline{5^*}$ x $\underline{5^*})_{-5}$ x $\underline{1}_{5}$

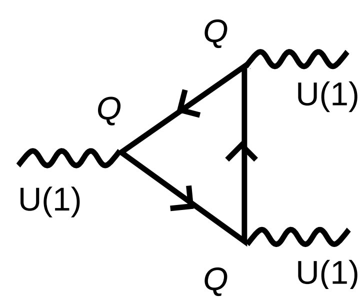
't Hooft vertex in SU(5)   U(1)$_Q$ -breaking operator

U(1)$_Q$ -anomaly matched

U(1)$_Q$ "global" symmetry



"Would-be gauge anomlay"



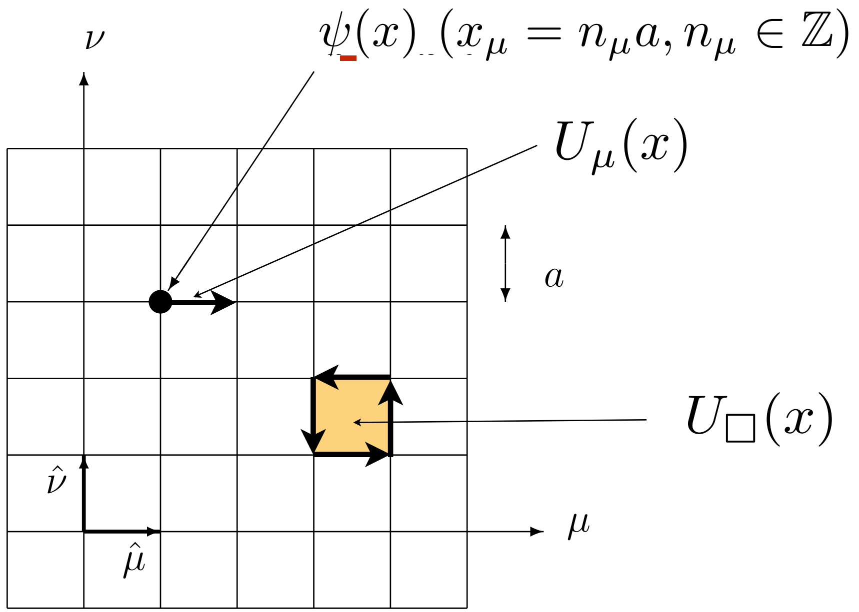
(-5)^3

(1)x Tr$_{\underline{10}}$[$T^a T^b$] + (-3)x Tr$_{\underline{5*}}$[$T^a T^b$] = 0

B_{-5} =$(\underline{10}$ x $\underline{5^*}$ x $\underline{5^*})_{-5}$

$\underline{1}_{5}$: spectator fermion

Chiral Lattice Gauge Theories ? Why not?



Weyl fermions ?!

Action and Path Integral measure

$$S_G = \frac{1}{g^2}\sum_{x\mu\nu} \mathrm{ReTr}\,(1 - U_\Box(x))$$

$$S_W = a^4 \sum_x \bar\psi(x)\, D\, \psi(x)$$

$$\mathcal{D}[U_\mu(x)] = \prod_{x,\mu} dU_\mu(x)$$

$$\mathcal{D}[\psi]\mathcal{D}[\bar\psi] = \prod_x d\psi(x) d\bar\psi(x)$$

gauge transformation

$$\psi(x) \longrightarrow g(x)\psi(x) \quad g(x) \in G$$

$$U_\mu(x) \to g(x)U_\mu(x)g^{-1}(x+\hat\mu) \quad U_\mu(x) \in G$$

$$\nabla_\mu \psi(x) = \frac{1}{a}\left(U_\mu(x)\psi(x+\hat\mu a) - \psi(x)\right)$$

$$[\nabla_\mu, \nabla_\nu]\psi(x) = \Big(1 - U_\Box(x)\Big)U_\mu(x)U_\nu(x+\hat\mu a)\psi(x+\hat\mu a+\hat\nu a)$$

$$U_\Box(x) = U_\mu(x)U_\nu(x+\hat\mu a)U_\mu(x+\hat\nu a)^{-1}U_\nu(x)^{-1}$$

exact gauge-invariance ?!

species doubling ?!

Overlap Dirac operator : gauge-covariant solution to GW rel.

$$\boxed{D = \frac{1}{2a}\left(1 + X\frac{1}{\sqrt{X^\dagger X}}\right)}, \quad X = aD_{\rm w} - m_0, \quad X^\dagger = \gamma_5 X \gamma_5$$

$$D_{\rm w} = \sum_{\mu=1}^{4}\left\{\gamma_\mu \frac{1}{2}\left(\nabla_\mu - \nabla_\mu^\dagger\right) + \frac{a}{2}\nabla_\mu \nabla_\mu^\dagger\right\}$$

$$\boxed{\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D}$$

Neuberger (1998)

Ginsparg-Wilson relation: "chiral limit of lattice fermion action"

$$S = a^4 \sum_x \bar\psi(x)\, D\, \psi(x)$$

$$= a^4 \sum_x \left\{\bar\psi(x) P_+ D \hat{P}_- \psi(x) + \bar\psi(x) P_- D \hat{P}_+ \psi(x)\right\}$$

Overalp Weyl fermions

chiral symmetry is preserved exactly!

Luscher (1999)

$$\delta S = 0 \qquad \boxed{\delta_\alpha \psi(x) = i\alpha\, \gamma_5 (1 - 2aD)\psi(x), \quad \delta_\alpha \bar\psi(x) = i\alpha\, \bar\psi(x)\gamma_5}$$

$$\hat\gamma_5 = \gamma_5(1 - 2aD) \qquad {\hat\gamma_5}^2 = \mathbb{I} \qquad \hat{P}_\pm = \left(\frac{1 \pm \hat\gamma_5}{2}\right), \quad P_\pm = \left(\frac{1 \pm \gamma_5}{2}\right)$$

Overlap Weyl field in 16-dim. spinor rep. of SO(10)

$$\psi(x) = \mathrm{P}_+\psi(x), \qquad \bar\psi(x) = \bar\psi(x)\mathrm{P}_+ \qquad \mathrm{P}_+ = \frac{1+\Gamma^{11}}{2}, \qquad \Gamma^{11} = -i\Gamma^1\Gamma^2\cdots\Gamma^{10}$$

$$D = \frac{1}{2}\Big(1 + X/\sqrt{X^\dagger X}\Big), \qquad X = \gamma_\mu\frac{1}{2}(\nabla_\mu - \nabla_\mu^\dagger) + \frac{1}{2}\nabla_\mu\nabla_\mu^\dagger - m_0 \qquad 0 < m_0 < 2$$

$$\nabla_\mu\psi(x) = U(x,\mu)\psi(x+\hat\mu) - \psi(x)$$

$$\hat\gamma_5 \equiv \gamma_5(1-2D), \qquad (\hat\gamma_5)^2 = \mathbb{I} \qquad \hat P_\pm = \left(\frac{1\pm\hat\gamma_5}{2}\right), \quad P_\pm = \left(\frac{1\pm\gamma_5}{2}\right)$$

$$\psi_-(x) = \hat P_-\psi(x), \qquad \bar\psi_-(x) = \bar\psi(x)P_+$$

$$S_{\mathrm{W}}[\psi_-,\bar\psi_-] = \sum_{x\in\Lambda}\bar\psi_-(x)D\psi_-(x)$$

A gauge invariant path-integral measure for the overlap Weyl fermions in 16 of SO(10)

YK (2017)

$$\mathcal{D}[U] \equiv \prod_{x\in\Lambda}\prod_{\mu=0}^{3} dU(x,\mu)$$

products of 't Hooft vertex (right-handed)

$$\mathcal{D}[\psi_-]\mathcal{D}[\bar\psi_-] \equiv \mathcal{D}[\psi]\mathcal{D}[\bar\psi] \prod_{x\in\Lambda} F(T_+(x)) \prod_{x\in\Lambda} F(\bar T_+(x))$$

Gauge-invariance!

$$F(w) \equiv 4!\,(z/2)^{-4} I_4(z)\Big|_{(z/2)^2=w} = 4! \sum_{k=0}^{\infty} \frac{w^k}{k!(k+4)!}$$

$$Z \equiv \int \mathcal{D}[U]\, \mathrm{e}^{-S_G[U]+\Gamma_W[U]}$$

$$\boxed{\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D}$$

$$\hat P_\pm = \left(\frac{1\pm\hat\gamma_5}{2}\right), \quad P_\pm = \left(\frac{1\pm\gamma_5}{2}\right)$$

$$\mathrm{e}^{\Gamma_W[U]} \equiv \int \mathcal{D}[\psi_-]\mathcal{D}[\bar\psi_-]\, \mathrm{e}^{-S_W[\psi_-,\bar\psi_-]}$$

$$= \int \mathcal{D}[\psi]\mathcal{D}[\bar\psi] \prod_{x\in\Lambda} F(T_+(x)) \prod_{x\in\Lambda} F(\bar T_+(x))\, \mathrm{e}^{-S_W[\psi_-,\bar\psi_-]}$$

the GW rel. => Chiral zero modes of *D* => Fermion # symm. breaking !

VEV of 't Hooft vertex (left-handed)

The saturation of lattice fermion measures due to 't Hooft vertices

$$\text{lattice units } a = 1$$

$$\Lambda = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \,|\, 0 \leq x_\mu < L \,(\mu = 0,1,2,3)\}$$

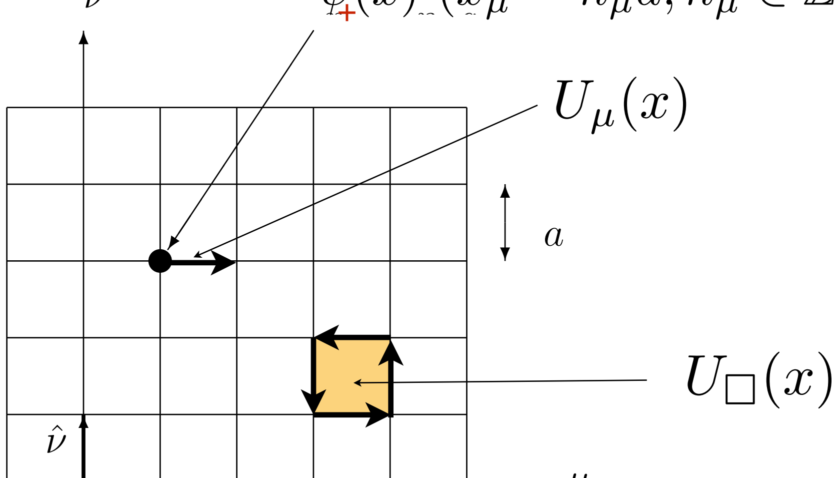
$$\int \prod_{x\in\Lambda} \prod_{\alpha=1}^{2} \prod_{s=1}^{16} d\psi_{\alpha s}(x) \prod_{x\in\Lambda} \frac{4!}{8!12!} \left\{ \frac{1}{2}\, \psi(x)^{\mathrm{T}} P_+ i\gamma_5 C_D \mathrm{T}^a \psi(x) \quad \psi(x)^{\mathrm{T}} P_+ i\gamma_5 C_D \mathrm{T}^a \psi(x) \right\}^8 = 1$$

$$\int \prod_{x\in\Lambda} \prod_{\alpha=3}^{4} \prod_{s=1}^{16} d\bar\psi_{\alpha s}(x) \prod_{x\in\Lambda} \frac{4!}{8!12!} \left\{ \frac{1}{2}\, \bar\psi(x) P_- i\gamma_5 C_D \mathrm{T}^a \bar\psi(x)^{\mathrm{T}} \, \bar\psi(x) P_- i\gamma_5 C_D \mathrm{T}^a \bar\psi(x)^{\mathrm{T}} \right\}^8 = 1$$

[*Eichten-Preskill(1986)*]

32-components at a site !

$\psi_+(x)$ $(x_\mu = n_\mu a, n_\mu \in \mathbb{Z})$ *in* **16**



(16 x 16 x 16 x 16)
= (16 x 16 x 16) x 16

= (10 x 10 x 10 x 5*) $_0$
+ (10 x 5* x 5*) $_{-5}$ **x 1** $_5$

B_{-5} **=(10 x 5* x 5*)** $_{-5}$

1 $_5$ **(spectator fermion)**

The saturation of lattice fermion measures due to 't Hooft vertices

lattice units $a = 1$

$$\Lambda = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \,|\, 0 \leq x_\mu < L\,(\mu = 0, 1, 2, 3)\}$$

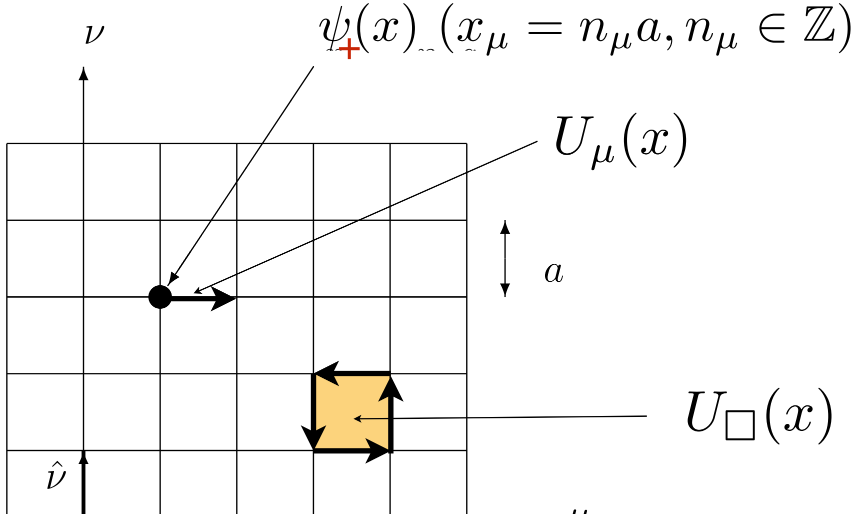
$$\int \prod_{x\in\Lambda} \prod_{\alpha=1}^{2} \prod_{s=1}^{16} d\psi_{\alpha s}(x) \prod_{x\in\Lambda} \frac{4!}{8!12!} \left\{ \frac{1}{2}\, \psi(x)^{\mathrm{T}} P_{+} i\gamma_5 C_D \mathrm{T}^a \psi(x) \quad \psi(x)^{\mathrm{T}} P_{+} i\gamma_5 C_D \mathrm{T}^a \psi(x) \right\}^8 = 1$$

$$\int \prod_{x\in\Lambda} \prod_{\alpha=3}^{4} \prod_{s=1}^{16} d\bar\psi_{\alpha s}(x) \prod_{x\in\Lambda} \frac{4!}{8!12!} \left\{ \frac{1}{2}\, \bar\psi(x) P_{-} i\gamma_5 C_D \mathrm{T}^a \bar\psi(x)^{\mathrm{T}}\, \bar\psi(x) P_{-} i\gamma_5 C_D \mathrm{T}^a \bar\psi(x)^{\mathrm{T}} \right\}^8 = 1$$

[*Eichten-Preskill(1986)*]

32-components at a site !

$\psi(x)\ (x_\mu = n_\mu a, n_\mu \in \mathbb{Z})$ *in* **16**



for the naive lattice fermion
with species doublers

OK also for
the Overlap Weyl fermion /
the Ginsparg-Wilson relation

Symmetry structure

- **gauge symmetry**

  manifest because

$$\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] \equiv \prod_{x\in\Lambda}\prod_{\alpha=1}^{4}\prod_{s=1}^{16} d\psi_{\alpha s}(x) \prod_{x\in\Lambda}\prod_{\alpha=1}^{4}\prod_{s=1}^{16} d\bar{\psi}_{\alpha s}(x)$$

- **fermion number symmetry of Left-handed fields**

  anomalous due to

$$\delta_\alpha \mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-] = -i\sum_{x\in\Gamma}\alpha(x)\mathrm{tr}\{\hat{P}_- - P_+\}(x,x)\times\mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-]$$

VEV of

$$T_-(x) = \frac{1}{2}V_-^a(x)V_-^a(x) \qquad V_-^a(x) = \psi_-(x)^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a \psi_-(x)$$

$$\bar{T}_-(x) = \frac{1}{2}\bar{V}_-^a(x)\bar{V}_-^a(x) \qquad \bar{V}_-^a(x) = \bar{\psi}_-(x) i\gamma_5 C_D \mathrm{T}^{a\dagger}\bar{\psi}_-(x)^{\mathrm{T}}$$

- **fermion number symmetry of Right-handed fields**

  Z_4 x Z_4 due to the 't Hooft vertices and the absence of the kinetic term

$$\prod_{x\in\Lambda} F(T_+(x)) \prod_{x\in\Lambda} F(\bar{T}_+(x))$$

- **CP invarinace**

  maintained thanks to the choice

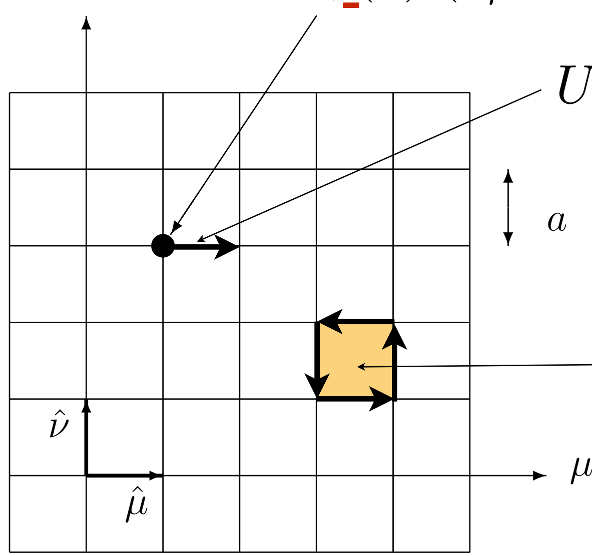
$$F(w) \equiv 4!\,(z/2)^{-4} I_4(z)\Big|_{(z/2)^2=w} = 4!\sum_{k=0}^{\infty}\frac{w^k}{k!(k+4)!}$$

$$F(w)\Big|_{w=(1/2)u^a u^a} = (\pi^5/12)^{-1}\int\prod_{a=1}^{10} de^a \delta(\sqrt{e^b e^b}-1)\,\mathrm{e}^{e^c u^c}$$

SO(10) Chiral Lattice Gauge Theories
with Overalp Weyl fermions/the Ginsparg-Wilson relation

$\psi(x)\ (x_\mu = n_\mu a, n_\mu \in \mathbb{Z})$ Overlap Weyl fermions !

$U_\mu(x)$



$U_\Box(x)$

gauge transformation

$$\psi(x) \longrightarrow g(x)\psi(x) \quad g(x) \in G$$

$$U_\mu(x) \to g(x)U_\mu(x)g^{-1}(x+\hat{\mu}) \quad U_\mu(x) \in G$$

$$\nabla_\mu \psi(x) = \frac{1}{a}\left(U_\mu(x)\psi(x+\hat{\mu}a) - \psi(x)\right)$$

$$[\nabla_\mu, \nabla_\nu]\psi(x) = \Big(1 - U_\Box(x)\Big)U_\mu(x)U_\nu(x+\hat{\mu}a)\psi(x+\hat{\mu}a+\hat{\nu}a)$$

$$U_\Box(x) = U_\mu(x)U_\nu(x+\hat{\mu}a)U_\mu(x+\hat{\nu}a)^{-1}U_\nu(x)^{-1}$$

Action and Path Integral measure

$$S_G = \frac{1}{g^2}\sum_{x\mu\nu} \mathrm{ReTr}\left(1 - U_\Box(x)\right)$$

$$S_W = a^4 \sum_x \bar{\psi}(x)\, D\, \psi(x)$$

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

$$\psi_-(x) = \hat{P}_-\psi(x) \quad \bar{\psi}_-(x) = \bar{\psi}(x)P_+$$

$$\mathcal{D}[U_\mu(x)] = \prod_{x,\mu} dU_\mu(x)$$

$$\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] = \prod_x d\psi(x)d\bar{\psi}(x)$$

$$\prod_{x\in\Lambda} F(T_+(x)) \prod_{x\in\Lambda} F(\bar{T}_+(x))$$

Gauge-invariance!

Fermion # symm. breaking !

cf. Eichten-Preskill model

[*Eichten-Preskill(1986)*]

$$S_{\rm EP} = \sum_{x\in\Lambda}\{\bar\psi(x)\gamma_\mu P_-([\nabla_\mu - \nabla_\mu^\dagger]/2)\psi(x)$$

$$\boxed{\begin{aligned}&-\sum_{x\in\Lambda}\{\frac{\lambda}{24}\left[\psi_-^{\rm T}(x)i\gamma_5 C_D {\rm T}^a\psi_-(x)\right]^2 + \frac{\lambda}{24}\left[\bar\psi_-(x)i\gamma_5 C_D {\rm T}^{a\dagger}\bar\psi_-(x)^T\right]^2\}\\ &-\sum_{x\in\Lambda}\{\frac{r}{48}\Delta\left[\psi^{\rm T}(x)i\gamma_5 C_D {\rm T}^a P_-\psi(x)\right]^2 + \frac{r}{48}\Delta\left[\bar\psi(x)P_+ i\gamma_5 C_D {\rm T}^{a\dagger}\bar\psi(x)^T\right]^2\}\end{aligned}}$$

Explicit Breaking of chiral symmetry by 't Hooft vertices

generalized Wilson-term

$$\begin{aligned}&\Delta\{A(x)B(x)C(x)D(x)\}\\ &\quad\equiv +\frac{1}{2}\sum_\mu\Big\{\big(\nabla_\mu\nabla_\mu^\dagger A(x)\big)B(x)C(x)D(x) + A(x)\big(\nabla_\mu\nabla_\mu^\dagger B(x)\big)C(x)D(x)\\ &\qquad + A(x)B(x)\big(\nabla_\mu\nabla_\mu^\dagger C(x)\big)D(x) + A(x)B(x)C(x)\big(\nabla_\mu\nabla_\mu^\dagger D(x)\big)\Big\}.\end{aligned}$$

{ **(<u>16</u>)**- + **(<u>16</u>)**+ } x 8 —> **(<u>16</u>)**- + heavy { **(<u>16</u>)**- x 7 + **(<u>16</u>)**+ x 8} ?

Attempts/Approaches to "exactly gauge-invariant formulation"

1. To solve the local cohomology problem [*Luscher (2000)*]

   - to determine the measure term $\Sigma_i(v_i, \delta_\eta v_i)$, based on the topological properties of the measure term (gauge anomaly) in 4+2 dim.
     cf. Wess-Zumino's Descent relation through 4+2, 4+1, 4 dim.

2. To decouple the mirror degrees of freedom out of Overlap Dirac fields

   1) Mirror Ginsparg-Wilson fermions [*Poppitz et al (2006)*]

      - by Multi-fermion interactions or Yukawa-interactions

   2) Mirror modes of Domain-wall fermion [*Creutz et al (1997)*]

      - by boundary interactions

   3) 4D TI / TSC with Gapped Boundary Phases

      [*Wen(2013), You-BenTov-Xu(2014), You-Xu (2015)*]

      - by boundary/bulk interactions

   cf. Eichten-Preskill model [*Eichten-Preskill(1986)*]

3. To Saturate the r.h.p. of Dirac-measure by 't Hooft vertices [*YK(2017): SO(10)*]

2. Set Up

— Overlap Dirac operator/the Ginsparg-Wilson relation

Overlap Dirac operator : gauge-covariant solution to GW rel.

$$D = \frac{1}{2a}\left(1 + X\frac{1}{\sqrt{X^\dagger X}}\right), \quad X = aD_{\mathrm{w}} - m_0, \quad X^\dagger = \gamma_5 X \gamma_5$$

$$D_{\mathrm{w}} = \sum_{\mu=1}^{4}\left\{\gamma_\mu \frac{1}{2}\left(\nabla_\mu - \nabla_\mu^\dagger\right) + \frac{a}{2}\nabla_\mu \nabla_\mu^\dagger\right\}$$

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

Neuberger (1998)

Ginsparg-Wilson relation: "chiral limit of lattice fermion action"

$$\begin{aligned} S &= a^4 \sum_x \bar{\psi}(x)\, D\, \psi(x) \\ &= a^4 \sum_x \left\{\bar{\psi}(x) P_+ D \hat{P}_- \psi(x) + \bar{\psi}(x) P_- D \hat{P}_+ \psi(x)\right\} \end{aligned}$$

Luscher (1999)

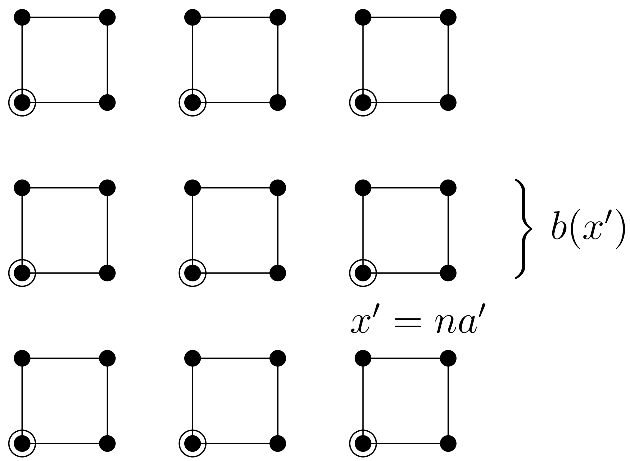
chiral symmetry is preserved exactly!

$$\delta S = 0 \qquad \delta_\alpha \psi(x) = i\alpha\, \gamma_5 (1 - 2aD)\psi(x), \quad \delta_\alpha \bar{\psi}(x) = i\alpha\, \bar{\psi}(x)\gamma_5$$

$$\hat{\gamma}_5 = \gamma_5(1 - 2aD) \qquad {\hat{\gamma}_5}^{\,2} = \mathbb{I} \qquad \hat{P}_\pm = \left(\frac{1 \pm \hat{\gamma}_5}{2}\right), \quad P_\pm = \left(\frac{1 \pm \gamma_5}{2}\right)$$

Block-spin transformation

Ginsparg-Wilson(1982)



$$\psi'(x') \;\Leftarrow\; \frac{Z}{2^4} \sum_{x \in b(x')} \psi(x) = \Psi(x';\psi)$$

$$e^{-S'[\psi',\bar{\psi}']} = \int \prod_x d\psi(x) d\bar{\psi}(x) \; e^{-S_W[\psi,\bar{\psi}]} \times$$

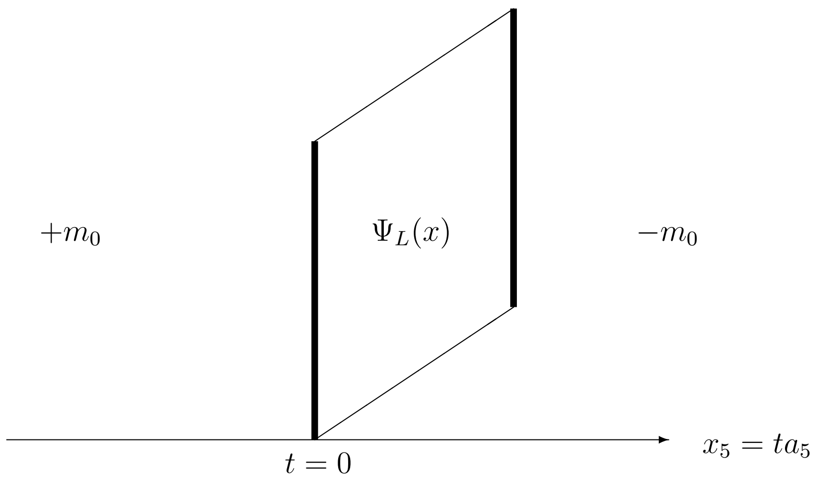
$$\exp\left\{ -\alpha \sum_{x'} \left(\bar{\psi}'(x') - \bar{\Psi}(x';\bar{\psi})\right) \left(\psi'(x') - \Psi(x';\psi)\right) \right\}$$

IR fixed point :

$$S^* = a^4 \sum_x \bar{\psi}(x) D^* \psi(x)$$ **(local, low-energy effective action)**

$$\gamma_5 D^{*-1} + D^{*-1}\gamma_5 = \frac{2}{\alpha}\gamma_5 \, a \, \delta_{xy}$$ **(GW rel.)**

chiral fermion bound to Domain-wall *Kaplan(1992)*



5 dim. fermion coupled to Domain-wall

$$\{i\gamma_\mu D_\mu + i\gamma_5\partial_5 - m_0\epsilon(x_5)\}\psi(x,x_5) = 0$$

$$V(x,x_5) = \gamma_5 m_0 \delta(x_5) \quad \Longrightarrow \quad \psi_0(x,x_5) \simeq \psi_-(x)\ \mathrm{e}^{-m_0|x_5|}$$

chiral mode bound to Domain-wall

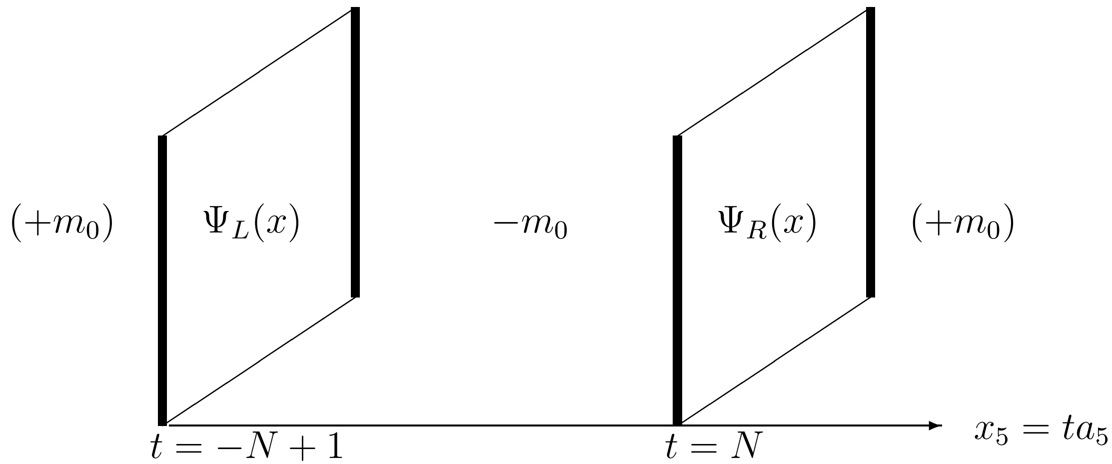
partition function of the chiral mode

$$Z = \langle v+|v-\rangle$$ **"vacuum overlap"** *Narayanan-Neuberger(1993)*

$$\hat{H}_\pm|v\pm\rangle = \epsilon_0|v\pm\rangle \qquad \hat{H}_\pm = \int dx^4 \bar{\psi}(x)\gamma_5(-i\gamma_\mu D_\mu \pm m_0)\psi(x)$$

Vector-like setup

Shamir(1993)



local, low-energy effecttive action -> overlap D.

Neuberger(1998)
Neuberger-Y.K.(1998)
Noguchi-Y.K.(1999)

$$\det(D_{5\mathrm{d}} - m_0/a)^{[dir.]} = \det D_{\mathrm{eff}} \cdot \det(D_{5\mathrm{d}} - m_0/a)^{[ap]}$$

$$\lim_{N\to\infty} D_{\mathrm{eff}} = \frac{1}{2a}\left(1+\gamma_5 \frac{H_{\mathrm{w}}}{\sqrt{H_{\mathrm{w}}^2}}\right), \quad H_{\mathrm{w}} = \gamma_5(D_{4\mathrm{d}} - m_0/a)$$

$$\langle q(x)\bar{q}(x)\rangle = D_{\mathrm{eff}}^{-1} - a \qquad q(x) = \Psi_R(x) + \Psi_L(x)$$

Weyl fermions in the framework of overlap D/GW rel.

chiral operator $(a=1)$

$$\hat{\gamma}_5 \equiv \gamma_5 \left(1 - 2D\right)$$

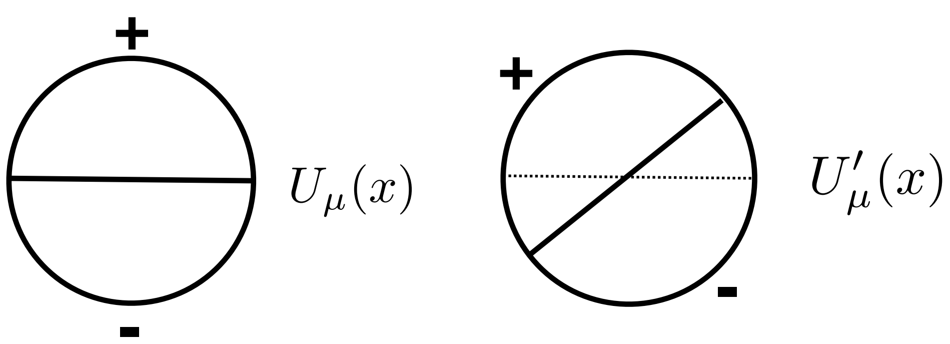
$$\gamma_5 D + D\hat{\gamma}_5 = 0, \quad \{\hat{\gamma}_5\}^2 = 1$$

chiral projection

$$\hat{\gamma}_5 \psi_\pm(x) = \pm\,\psi_\pm(x), \qquad \bar{\psi}_\pm(x)\gamma_5 = \mp\,\bar{\psi}_\pm(x)$$

$$\psi_-(x) = \sum_i v_i(x) c_i \qquad \{v_i(x) \mid \hat{\gamma}_5 v_i(x) = -v_i(x) \ (i = 1, \cdots, N_-)\}$$

$$\bar{\psi}_-(x) = \sum_i \bar{c}_i \bar{v}_i(x) \qquad \{\bar{v}_i(x) \mid \bar{v}_i(x)\gamma_5 = +\bar{v}_i(x) \ (i = 1, \cdots, \bar{N}_-)\}$$



The space of
the chiral fermion
depends on the gauge fields

Path Integral quantization

$$\psi_-(x) = \sum_i v_i(x) c_i \qquad \bar{\psi}_-(x) = \sum_i \bar{c}_i \bar{v}_i(x)$$

$$Z = \int \mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-]\, \mathrm{e}^{-a^4 \sum_x \bar{\psi}_- D \psi_-(x)}$$

$$= \int \prod_i dc_i \prod_j d\bar{c}_j\, \mathrm{e}^{-\sum_{ij} \bar{c}_j M_{ji} c_i} \qquad M_{ji} = a^4 \sum_x \bar{v}_j D v_i(x)$$

Path Integral measure depends on the gauge field !

- change of the chiral basis by a unitary transformation

$$\tilde{v}_i(x) = v_j(x) \left(\tilde{Q}^{-1}\right)_{ji} \qquad \tilde{c}_i = \tilde{Q}_{ij} c_j$$

$$\mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-] \Longrightarrow \mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-]\, \det \tilde{Q}\, [U_\mu(x)]$$

* in sharpe contrast to the case of Dirac fermions in QCD-like theories

cf. in Lattice QCD $\quad \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] = \prod_x d\psi(x) d\bar{\psi}(x)$

gauge anomaly:

variation of effective action & gauge anomaly

$$\Gamma_{\text{eff}} = \ln\det(\bar{v}_k D v_j) \qquad \delta_\eta U(x,\mu) = i\eta_\mu(x)U(x,\mu)$$

$$\delta_\eta \Gamma_{\text{eff}} = \mathrm{Tr}\left\{(\delta_\eta D)\hat{P}_- D^{-1} P_+\right\} + \sum_i (v_i, \delta_\eta v_i)$$

$$= i\mathrm{Tr}\omega\gamma_5\,(1-D) - i\sum_i (v_i, \delta_\omega v_i) \qquad \eta_\mu(x) = -i\nabla_\mu \omega(x)$$

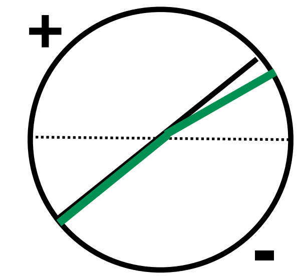
gauge anomaly!

$$\mathrm{tr}\{T^a\gamma_5(1-aD)(x,x)\} \xrightarrow{a\to 0} \frac{-1}{128\pi^2} d^{abc}\epsilon_{\mu\nu\rho\sigma}F^b_{\mu\nu}(x)F^c_{\rho\sigma}(x) + O(a)$$

$$\sum_R d^{abc} = \sum_R 2i\mathrm{tr}\{T^a[T^bT^c + T^cT^b]\}$$

Zero modes, 't Hooft vertex, Fermion number non-conservation

chiral fermion basis : $\hat{\gamma}_5 = -\dfrac{H_{\rm w}}{\sqrt{H_{\rm w}^2}}$



$U'_\mu(x)$

$$\{v_i(x) \mid \hat{\gamma}_5 v_i(x) = -v_i(x) \ (i = 1, \cdots, N_-)\}$$
$$\{\bar{v}_i(x) \mid \bar{v}_i(x)\gamma_5 = +\bar{v}_i(x) \ (i = 1, \cdots, \bar{N}_-)\}$$

$$N_+ + N_- = N = 4N_c L^4 \qquad \bar{N}_+ = \bar{N}_- = N/2$$
$$N_+ - N_- = 2Q$$

(for fund. rep. of SU(Nc))

$$\therefore \ N_- = N/2 - Q, \quad \bar{N}_- = N/2$$

$$M_{ji} = a^4 \sum_x \bar{v}_j D v_i(x) \qquad (N_- \times \bar{N}_- \ \ \text{rectangular matrix})$$

$$Z = \int \prod_i dc_i \prod_j d\bar{c}_j \, \mathrm{e}^{-\sum_{ij} \bar{c}_j M_{ji} c_i} = 0$$

Non-zero VEV of fermion number non-conserving operators

$$\psi_-(x) = \sum_i v_i(x) c_i$$

$$\langle \prod_\alpha \psi_\alpha(x) \rangle \neq 0$$

't Hooft vertex well-defined !

Chiral Lattice Gauge Theories ? Why not?

3. **Approaches**

— ***“All things merge into one, and a river runs through it”***

Given the setup of overalp Weyl fermions/the GW relation, what kind of attempts/approaches are possible to “exactly and/or manifestly gauge invariant formulation” ?

Attempts/Approaches to "exactly gauge-invariant formulation"

1. To solve the local cohomology problem [*Luscher (2000)*]

- to determine the measure term $\Sigma_i(v_i, \delta_\eta v_i)$, based on the topological properties of the measure term (gauge anomaly) in 4+2 dim.
  cf. Wess-Zumino's Descent relation through 4+2, 4+1, 4 dim.

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   1) Mirror Ginsparg-Wilson fermions [*Poppitz et al (2006)*]

      - by Multi-fermion interactions or Yukawa-interactions

   2) Mirror modes of Domain-wall fermion [*Creutz et al (1997)*]

      - by boundary interactions cf. [*Kaplan-Grabowska(2016)*]

   3) 4D TI / TSC with Gapped Boundary Phases

      [*Wen(2013), You-BenTov-Xu(2014), You-Xu (2015)*]

      - by boundary/bulk interactions

   ☆ cf. Eichten-Preskill model [*Eichten-Preskill(1986)*]

3. To Saturate the r.h.p. of Dirac-measure by 't Hooft vertices [*YK(2017): SO(10)*]

Local cohomology problem for the Path Integral measure of Overlap Weyl fermions

$$\psi_-(x) = \sum_i v_i(x) c_i \qquad \bar\psi_-(x) = \sum_i \bar c_i \bar v_i(x)$$

$$Z = \int \mathcal{D}[\psi_-]\mathcal{D}[\bar\psi_-]\, \mathrm{e}^{-a^4 \sum_x \bar\psi_- D\psi_-(x)}$$

$$= \int \prod_i dc_i \prod_j d\bar c_j\, \mathrm{e}^{-\sum_{ij} \bar c_j M_{ji} c_i} \qquad M_{ji} = a^4 \sum_x \bar v_j D v_i(x)$$

Path Integral measure depends on the gauge field !

the gauge-field dependence must be fixed ... *Luscher(98)*

1. **locality** [admissibility condition, topology of lattice gauge fields]
2. **gauge invariance** [gauge anomaly cancellations]
3. **integrability** [topology of the space of gauge fields]

* in sharpe contrast to the case of Dirac fermions in QCD-like theories

Construction of SU(2)xU(1) Electroweak theory (I)

infinite volume case

local counter term!

$$\eta_\mu(x) = \eta_\mu^{(2)}(x) \oplus \eta_\mu^{(1)}(x) \qquad U_t(x,\mu)^{(1)} = \mathrm{e}^{itA_\mu(x)} \quad t \in [0,1]$$

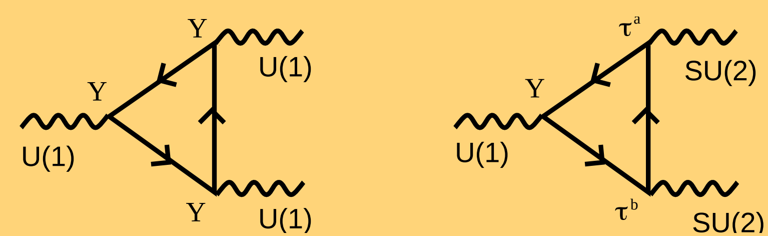
$$\begin{aligned}\mathfrak{L}_\eta &= i \sum_i (v_i, \delta_\eta v_i) = \sum_x \eta_\mu(x) j_\mu(x) \\ &= i \int_0^1 dt \, \mathrm{Tr}\left\{\hat{P}_-[\partial_t \hat{P}_-, \delta_\eta \hat{P}_-]\right\} + \delta_\eta \int_0^1 dt \sum_{x\in\mathbb{Z}^4} \{A_\mu(x)\, k_\mu(x)\}\end{aligned}$$

cf. U(1) case
Luscher(98)
Neuberger(01)

gauge anomaly cancellation

analysis of U(1) with SU(2) fixed

$$q(x) = \mathrm{tr}\,\{\gamma_5(1-aD)(x,x)\}|_{U^{(1)},U^{(2)}} \qquad x \in \mathbb{Z}^4$$



$$\sum_L Y^3 - \sum_R Y^3 = 0 \qquad \sum_{\mathrm{doublet(L)}} Y = 0, \qquad \sum_{\mathrm{singlet(R)}} Y = 0$$

$$\begin{aligned}&\sum_\alpha Y_\alpha q(x)|_{U^{(1)} \to \{U^{(1)}\}^{Y_\alpha}} \\ &= \sum_\alpha Y_\alpha q(x)|_{U^{(2)}} + \sum_\alpha Y_\alpha^3 \frac{1}{32\pi^2}\, \epsilon_{\mu\nu\lambda\rho} F_{\mu\nu}(x) F_{\lambda\rho}(x+\hat{\mu}+\hat{\nu}) + \partial_\mu^* k_\mu(x) \\ &= \partial_\mu^* k_\mu(x)\end{aligned}$$

cf. Luscher(98)
(in 4dim.)

cf. Nakayama-YK(00)
(in 6dim.)

Chiral lattice gauge theories with exact gauge invariance

- Anomaly-free U(1) chiral gauge theories
  - A complete construction on the finite-volume lattice
    M. Lüscher, Nucl. Phys. B**538**, 515 , B**549**, 295 (1999)
- SU(2)$\times$U(1) gauge theory of EW interaction
  - Local cohomology problem in 4+2 dim. is solved for $L = \infty$
    $\Rightarrow$ Exact cancellations of the gauge anomalies
    U(1)^3, SU(2)$^2\times$U(1)-mixed type
    Y. Nakayama and Y.K., Nucl. Phys. B**597**, 519 (2001)**, ...**
  - Measure term is constructed for $L < \infty$, $m_{\mu\nu} = 0$
  - Global integrability ?
- Non-abelian chiral gauge theories (SU(N), SO(10) etc.)
  - Non-perturbative construction is not obtained yet
  - In all orders in the weak coupling expansion
    H. Suzuki, Nucl. Phys. B **585**, 471 (2000)
    M. Lüscher, JHEP **0006**, 028 (2000)

1) Mirror GW fermion model [*Poppitz et al (2006)*]

cf. using Wilson-fermions [*Montvay et al (1987)*]

$$
\begin{aligned}
S'_{\mathrm{Ov/Mi}}[\psi,\bar\psi,X^a,\bar X^a] &= \sum_{x\in\Lambda}\{\bar\psi_-(x)D\psi_-(x) + z_+\bar\psi_+(x)D\psi_+(x)\} \\
&- \sum_{x\in\Lambda}\{y\,X^a(x)\psi_+^{\mathrm{T}}(x)i\gamma_5 C_D \mathrm{T}^a P_+\psi_+(x) \\
&\qquad + \bar y\,\bar X^a(x)\bar\psi_+(x)P_- i\gamma_5 C_D \mathrm{T}^{a\dagger}\bar\psi_+(x)^T\} \\
&+ S_X[X^a].
\end{aligned}
$$

$$
\begin{aligned}
S_X[X^a] = \sum_{x\in\Lambda}\Bigg\{ &-\kappa\sum_\mu X^a(x)X^a(x+\hat\mu) + \frac{1}{2}X^a(x)X^a(x) + \frac{\lambda'}{2}(X^a(x)X^a(x)-v^2)^2 \\
&-\bar\kappa\sum_\mu \bar X^a(x)\bar X^a(x+\hat\mu) + \frac{1}{2}\bar X^a(x)\bar X^a(x) + \frac{\bar\lambda'}{2}(\bar X^a(x)\bar X^a(x)-\bar v^2)^2 \Bigg\}
\end{aligned}
$$

$$
\mathcal{D}[\psi]\mathcal{D}[\bar\psi] = \prod_x d\psi(x)d\bar\psi(x)
$$

Paramagnetic Strong-coupling (PMS) Phase (Disordered Gapped phase)

Decoupling limit of Mirror fermions!?

- Gauge symmetry preserved
- Only Massive (Gapped) Excitations in the Mirror sector
- Only Local terms left in the effective action

$$
\begin{aligned}
&y=\bar y, \qquad \frac{z_+}{\sqrt{y\bar y}} \to 0, \\
&v=\bar v = 1, \qquad \lambda' = \bar\lambda' \to \infty \\
&\kappa = \bar\kappa \to 0.
\end{aligned}
$$

Necessary condition to decouple Mirror GW fermions,
which follows from ’t Hooft anomaly-maching condition

If there exists a global continuous fermion symmetries in the Mirror-fermion sector, it must be free from the “would-be gauge anomaly”, i.e. that global symmetry must be gauged successfully without encountering gauge anomalies.

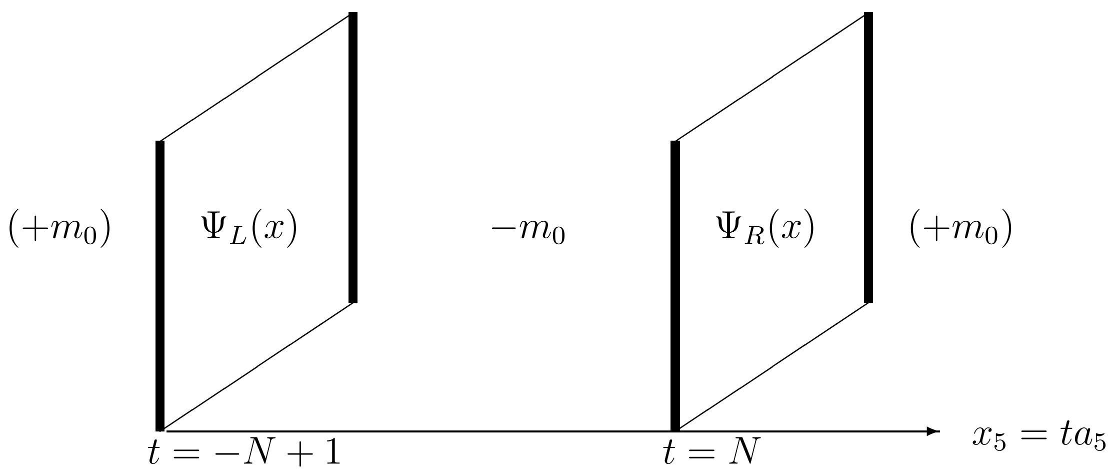
This is because the “would-be gauge anomaly” implies an IR singularity in the correlation function of the symmetry currents, and it in turn implies certain massless states in the spectrum, so that it can saturate the IR singularity.

To break the global continuous fermion symmetries in the Mirror-fermion sector, add ’t Hooft vertex operators in terms of the Mirror fermion fields.

cf. Eichten-Preskill model [*Eichten-Preskill(1986)*]

2) Domain-wall fermion model for SO(10) cf. [*Creutz et al (1997)*]

Vector-like setup



$$
\begin{aligned}
S'_{\mathrm{DW/Mi}} &= \sum_{t=1}^{L_5}\sum_{x\in\Lambda}\bar\psi(x,t)\big\{[1+a_5'(D_{4\mathrm{w}}-m_0)]\delta_{tt'}-P_-\delta_{t+1,t'}-P_+\delta_{t,t'+1}\big\}\psi(x,t') \\
&- \sum_{x\in\Lambda}\{y\,X^a(x)q_+^{\mathrm{T}}(x)i\gamma_5 C_D\mathrm{T}^a P_+ q_+(x)+\bar y\,\bar X^a(x)\bar q_+(x)P_- i\gamma_5 C_D\mathrm{T}^{a\dagger}\bar q_+(x)^T\} \\
&+ S_X[X^a]
\end{aligned}
$$

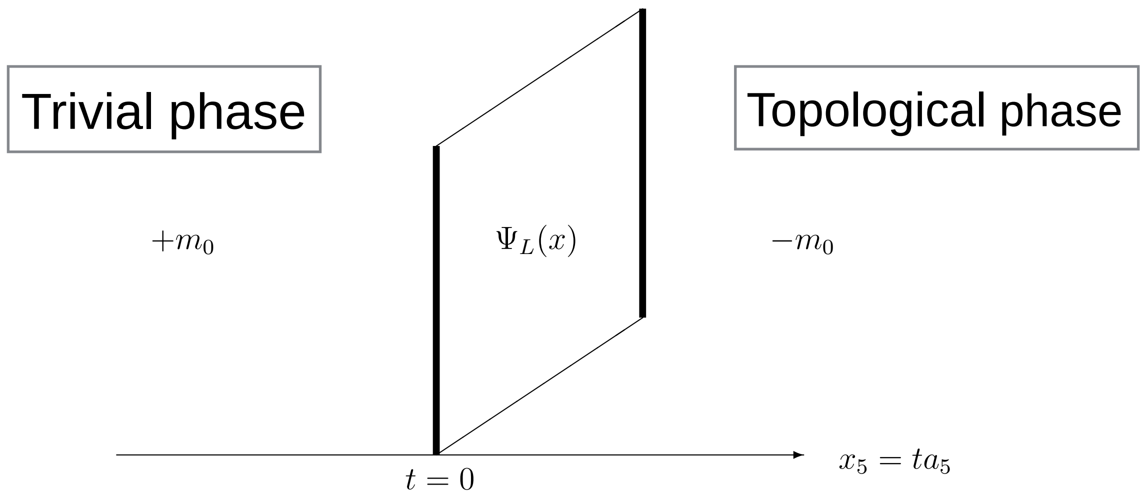
$$q(x) = \psi_-(x,1)+\psi_+(x,L_5)$$

$$\bar q(x) = \bar\psi_-(x,1)+\bar\psi_+(x,L_5)$$

3) 4D TI/TSC with Gapped boundary phase for SO(10)

[Wen(2013), You-BenTov-Xu(2014),You-Xu (2015)]

$$\hat{H}_{\rm 4DTI} = \sum_{i=1}^{\nu}\sum_{p} \hat{a}_i(p)^{\dagger}\Big\{\sum_{k=1}^{4}\alpha_k \sin(p_k) + \beta\Big(\Big[\sum_{k=1}^{4}\cos(p_k)-4\Big]+m\Big)\Big\}\hat{a}_i(p)$$



$$\hat{H}^{\rm (bd)}_{\rm 3D} = \sum_{i=1}^{\nu}\int d^3x\,\hat{\psi}_i(x)^{\dagger}\Big\{\sum_{l=1}^{3}(-i)\sigma_l\partial_l\Big\}\hat{\psi}_i(x)$$

$$\hat{H}_{\rm 3D,10} = \int d^3x\,\Big\{\hat{\psi}(x)^T i\sigma_2\check{\Gamma}^a\phi^a(x)\hat{\psi}(x) - \hat{\psi}(x)^{\dagger} i\sigma_2\check{\Gamma}^{a\dagger}\phi^a(x)\hat{\psi}(x)^{\dagger} + \mathcal{H}[\phi^a(x)]\Big\}$$

Gapped boundary phase for the case with v= 16 !?

$$\langle\phi^a(x)\rangle = 0$$

$$\phi^a(x)\phi^a(x) = M^2 \neq 0$$

$$\pi_d(S^9) = 0 \;\; (0\leq d<9)$$

- No symmetry breaking
- No vortices with massless fermion excitations
- No WZW term

Relations to other approaches/proposals

- **Mirror GW Fermion model**

  Well-defined limit of large Majorana-Yukawa couplings

  cf. Failure in 2D for 345 model (*Poppitz et al*) : C_D, *not* $i\gamma_5 C_D$

  $1^4(-1)^4$ model, $21(-1)^3$model proposed/studied - *locality OK!*

- **Domain wall fermion model with boundary Eichten-Preskill term**

  Boundary EP terms constructed explicitly (cf. *Creutz et al*)

- **4D TI/TSCs with Gapped Boundary Phases**

  Evidence for the gapped boundary phase from 4+1dim. / 3+1dim. Euclidean lattice formulation(quantization)

  cf. Proposed gapped boundary phases in 4D TI/TSC ($Z \rightarrow Z_{16}$?) (*Wen, You-BenTov-Xu, You-Xu*)

  cf. in 2D, Explicit rel. to **Gapped** 8-flavor 1D Majorana Chain (*Fidkowski-Kitaev*) / Refinement of free fermion classification of TI/TCI due to interactions ($Z \rightarrow Z_8, Z_{16}$)

  cf. Effect of gauge interaction $\rightleftarrows$ **locality Issue**

Large mass limit of GW Dirac fermions

$$S_D = \sum_{x\in\Lambda}\{\bar\psi(x)D\psi(x) + m_D\,\bar\psi(1-D)\psi(x)\}$$

$$\delta\psi(x) = \gamma_5(1-2D)\psi(x)$$

$$\delta\bar\psi(x) = \bar\psi(x)\gamma_5$$

$$\bar\psi(1-D)\psi(x)$$

$$\bar\psi i\gamma_5(1-D)\psi(x)$$

$$S_D = \sum_{x\in\Lambda}\{z\,\bar\psi(x)D\psi(x) + m\,\bar\psi(x)\psi(x)\}$$

$$z = 1 - m_D$$

$$m = m_D$$

$$z/m = (1-m_D)/m_D \to 0$$

the limit $z/M \to 0$ is well-defined

Large mass limit of GW Majorana fermions

$$
\begin{aligned}
S_M &= \sum_{x\in\Lambda}\{\bar\psi_+(x)D\psi_+(x) \\
&\qquad +m_M\,(\psi_+(x)^T C_D\psi_+(x) + \bar\psi_+(x)C_D\bar\psi_+(x)^T)\} \\
&= \sum_{x\in\Lambda}\{\bar\psi(x)P_-D\psi(x) \\
&\qquad +m_M\,(\psi^T\hat P_+^T C_D\hat P_+\psi(x) + \bar\psi P_- C_D P_-^T\bar\psi^T(x))\}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
u_j^T C_D u_k &= -\delta_{p+p',0}\frac{b(p')}{\omega(p')}\epsilon_{\sigma,\sigma'} \quad (j=\{p,\sigma\}, k=\{p',\sigma'\}) \\
\bar u_j C_D \bar u_k^T &= -\delta_{x,x'}\epsilon_{\sigma,\sigma'} \quad (j=\{x,\sigma\}, k=\{x',\sigma'\})
\end{aligned}}
$$

$$
\begin{aligned}
S_M &= \sum_{x\in\Lambda}\{z\,\bar\psi_+(x)D\psi_+(x) \\
&\qquad +M\,(\psi_+(x)^T i\gamma_5 C_D\psi_+(x) + \bar\psi_+(x)i\gamma_5 C_D\bar\psi_+(x)^T)\} \\
&= \sum_{x\in\Lambda}\{z\,\bar\psi(x)P_-D\psi(x) \\
&\qquad +M\,(\psi^T\hat P_+^T i\gamma_5 C_D\hat P_+\psi(x) + \bar\psi P_- i\gamma_5 C_D P_-^T\bar\psi^T(x))\}
\end{aligned}
$$

the limit $z/M \to 0$ is well-defined

$$
\boxed{
\begin{aligned}
u_j^T i\gamma_5 C_D u_k &= i\delta_{p+p',0}\epsilon_{\sigma,\sigma'} \quad (j=\{p,\sigma\}, k=\{p',\sigma'\}), \\
\bar u_j i\gamma_5 C_D \bar u_k^T &= i\delta_{x,x'}\epsilon_{\sigma,\sigma'} \quad (j=\{x,\sigma\}, k=\{x',\sigma'\})
\end{aligned}}
$$

$$S_{\mathrm{Ov}}[\psi,\bar{\psi},E^a,\bar{E}^a] = \sum_{x\in\Lambda} \bar{\psi}_-(x) D\psi_-(x) - \sum_{x\in\Lambda}\{E^a(x)\psi_+^{\mathrm{T}}(x) i\gamma_5 C_D \mathrm{T}^a \psi_+(x) + \bar{E}^a(x)\bar{\psi}_+(x) i\gamma_5 C_D \mathrm{T}^{a\dagger}\bar{\psi}_+(x)^T\}$$

Mirror GW fermion model

$$S_{\mathrm{Ov/Mi}}[\psi,\bar{\psi},X^a,\bar{X}^a] = \sum_{x\in\Lambda}\{\bar{\psi}_-(x) D\psi_-(x) + z_+\bar{\psi}_+(x) D\psi_+(x)\} - \sum_{x\in\Lambda}\{y\, X^a(x)\psi_+^{\mathrm{T}}(x) i\gamma_5 C_D \mathrm{T}^a \psi_+(x) + \bar{y}\,\bar{X}^a(x)\bar{\psi}_+(x) i\gamma_5 C_D \mathrm{T}^{a\dagger}\bar{\psi}_+(x)^T\} + S_X[X^a]$$

$$S_X[X^a] = \sum_{x\in\Lambda}\Big\{-\kappa\sum_\mu X^a(x)X^a(x+\hat{\mu}) + \frac{1}{2}X^a(x)X^a(x) + \frac{\lambda'}{2}(X^a(x)X^a(x)-v^2)^2 - \bar{\kappa}\sum_\mu \bar{X}^a(x)\bar{X}^a(x+\hat{\mu}) + \frac{1}{2}\bar{X}^a(x)\bar{X}^a(x) + \frac{\bar{\lambda}'}{2}(\bar{X}^a(x)\bar{X}^a(x)-\bar{v}^2)^2\Big\}$$

$$y=\bar{y},\quad \frac{z_+}{\sqrt{y\bar{y}}}\to 0,$$
$$v=\bar{v}=1,\quad \lambda'=\bar{\lambda}'\to\infty$$
$$\kappa=\bar{\kappa}\to 0.$$

Decoupling limit of Mirror fermions ?!

Leave only local terms ?!

Partition function

$$Z \equiv \int \mathcal{D}[U]\, \mathrm{e}^{-S_G[U]+\Gamma_W[U]}$$

$$\begin{aligned}
\mathrm{e}^{\Gamma_W[U]} &\equiv \int \mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-]\, \mathrm{e}^{-S_W[\psi_-,\bar{\psi}_-]} \\
&= \int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] \prod_{x\in\Lambda} F(T_+(x)) \prod_{x\in\Lambda} F(\bar{T}_+(x))\, \mathrm{e}^{-S_W[\psi_-,\bar{\psi}_-]} \\
&= \int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\, \mathcal{D}[E]\mathcal{D}[\bar{E}]\; \mathrm{e}^{-S_W[\psi_-,\bar{\psi}_-]+\sum_{x\in\Lambda}\{E^a(x)V_+^a(x)+\bar{E}^a(x)\bar{V}_+^a(x)\}[\psi_+,\bar{\psi}_+]}
\end{aligned}$$

$$\mathcal{D}[E] = \prod_{x\in\Lambda}(\pi^5/12)^{-1} \prod_{a=1}^{10} dE^a(x)\delta(\sqrt{E^b(x)E^b(x)}-1)$$

$$\mathcal{D}[\bar{E}] = \prod_{x\in\Lambda}(\pi^5/12)^{-1} \prod_{a=1}^{10} d\bar{E}^a(x)\delta(\sqrt{\bar{E}^b(x)\bar{E}^b(x)}-1)$$

$$F(w)\Big|_{w=(1/2)u^a u^a} = (\pi^5/12)^{-1} \int \prod_{a=1}^{10} de^a \delta(\sqrt{e^b e^b}-1)\, \mathrm{e}^{e^c u^c}$$

Domain wall fermion model with boundary Eichten-Preskill term

$$
\begin{aligned}
S_{\mathrm{DW/Ov}} = &\sum_{t=1}^{L_5}\sum_{x\in\Lambda}\bar\psi(x,t)\big\{[1+a_5'(D_{4\mathrm{w}}-m_0)]\delta_{tt'}-P_-\delta_{t+1,t'}-P_+\delta_{t,t'+1}\big\}\psi(x,t') \\
&-\sum_{x\in\Lambda}\bar\psi(x,L_5)P_-[1+a_5'(D_{4\mathrm{w}}-m_0)]\psi(x,L_5) \\
&-\sum_{x\in\Lambda}\{E^a(x)\psi^{\mathrm{T}}(x,L_5)i\gamma_5 C_D \mathrm{T}^a\psi(x,L_5) \\
&\qquad\qquad +\bar E^a(x)\bar\psi(x,L_5)P_- i\gamma_5 C_D \mathrm{T}^{a\dagger}\bar\psi(x,L_5)^T\}
\end{aligned}
$$

$$
\psi(x,L_5)=(-\gamma_5)\big(1+e^{a_5L_5\widetilde H}\big)^{-1}\psi(x) \quad \text{—>} \quad \psi_+(x)=\hat P_+\psi(x) \qquad L_5\to\infty \ (\text{plus } a_5\to 0)
$$

cf. $q(x) = \psi_-(x,1)+\psi_+(x,L_5) \qquad \bar q(x) = \bar\psi_-(x,1)+\bar\psi_+(x,L_5)$

$$
\begin{aligned}
S_{\mathrm{DW/Mi}} = &\sum_{t=1}^{L_5}\sum_{x\in\Lambda}\bar\psi(x,t)\big\{[1+a_5'(D_{4\mathrm{w}}-m_0)]\delta_{tt'}-P_-\delta_{t+1,t'}-P_+\delta_{t,t'+1}\big\}\psi(x,t') \\
&+\sum_{x\in\Lambda}(z_+-1)\,\bar\psi(x,L_5)P_-[1+a_5'(D_{4\mathrm{w}}-m_0)]\psi(x,L_5) \\
&-\sum_{x\in\Lambda}\{y\,X^a(x)\psi^{\mathrm{T}}(x,L_5)i\gamma_5 C_D \mathrm{T}^a\psi(x,L_5) \\
&\qquad\qquad +\bar y\,\bar X^a(x)\bar\psi(x,L_5)P_- i\gamma_5 C_D \mathrm{T}^{a\dagger}\bar\psi(x,L_5)^T\} \\
&+\ S_X[X^a],
\end{aligned}
$$

$$
y=\bar y, \quad \frac{z_+}{\sqrt{y\bar y}}\to 0, \qquad v=\bar v=1, \quad \lambda'=\bar\lambda'\to\infty \qquad \kappa=\bar\kappa\to 0.
$$

Domain wall fermion model with boundary Eichten-Preskill term (con't)

$$\begin{aligned}
\langle 1 \rangle_F &= \int \mathcal{D}[\psi]\mathcal{D}[\bar\psi]\,\mathcal{D}[E]\mathcal{D}[\bar E]\ \mathrm{e}^{-S_{\mathrm{Ov}}[\psi,\bar\psi,E^a,\bar E^a]} \\
&= \frac{\int \prod_{x,t} d\bar\psi(x,t) d\psi(x,t)\ \mathcal{D}[E]\mathcal{D}[\bar E]\ \mathrm{e}^{-S_{\mathrm{DW/Ov}}[\psi,\bar\psi,E^a,\bar E^a]}\Big|_{\mathrm{Dir}}}{\int \prod_{x,t} d\bar\psi(x,t) d\psi(x,t)\ \ \mathrm{e}^{-S_{\mathrm{DW}}[\psi,\bar\psi]}\Big|_{\mathrm{AP}}} \qquad [L_5 \to \infty\,(a_5' \to 0)] \\
&= \frac{\left\langle \mathrm{pf} \begin{pmatrix} -i\gamma_5 C_D \mathrm{T}^a E^a \delta_{tL_5}\delta_{t'L_5} & -a_5'(D_{5\mathrm{w}} - m_0)'^T/2 \\ a_5'(D_{5\mathrm{w}} - m_0)'/2 & -i\gamma_5 C_D P_- \mathrm{T}^{a\dagger} \bar E^a \delta_{tL_5}\delta_{t'L_5} \end{pmatrix} \Big|_{\mathrm{Dir}} \right\rangle_E'}{\det a_5'(D_{5\mathrm{w}} - m_0)\Big|_{\mathrm{AP}}}
\end{aligned}$$

$$P_+\psi(x,0) = 0, \quad \bar\psi(x,0)P_- = 0 \quad ; \quad P_-\psi(x,L_5+1) = 0, \quad \bar\psi(x,L_5+1)P_+ = 0.$$

4D TI/TSC with the gapped boundary phases

4D TI/TSC in the proposals by Wen, You-BenTov-Xu, You-Xu = DW fermion by Kaplan

$$\hat{H}_{\mathrm{4DTI}} = \sum_{i=1}^{\nu} \sum_{p} \hat{a}_i(p)^\dagger \Big\{ \sum_{k=1}^{4} \alpha_k \sin(p_k) + \beta \Big(\big[\sum_{k=1}^{4} \cos(p_k) - 4 \big] + m \Big) \Big\} \hat{a}_i(p)$$

5dim. Euclidean formulation of the proposals by Wen, You-BenTov-Xu, You-Xu

$$Z_{\mathrm{4DTI}/\nu=16} \equiv \int \prod_{t=-L_5+1}^{L_5} \mathcal{D}[\psi(t)]\mathcal{D}[\bar{\psi}(t)]\, \mathcal{D}[E]\mathcal{D}[\bar{E}] \; \mathrm{e}^{-S_{\mathrm{DW/Mi}}[\psi,\bar{\psi},E^a,\bar{E}^a]}\Big|_{\mathrm{Dir}}$$

$$= \det\, a_5'(D_{5\mathrm{w}} - m_0)\big|_{\mathrm{AP}}\, Z_{\mathrm{Ov/Mi}}$$

$$[L_5 \to \infty\, (a_5' \to 0)]$$

$$E^a(x)E^a(x) = 1, \quad \bar{E}^a(x)\bar{E}^a(x) = 1$$

$$y = \bar{y}, \quad \frac{z_+}{\sqrt{y\bar{y}}} \to 0$$

$$Z_{\mathrm{4DTI}/\nu=16} = \det\, a_5'(D_{5\mathrm{w}} - m_0)\big|_{\mathrm{AP}}\, Z_{\mathrm{Ov}}$$

$$= \det\, a_5'(D_{5\mathrm{w}} - m_0)\big|_{\mathrm{AP}}\;\; \det(\bar{v}Dv)\; \big\langle \mathrm{pf}(u^T i\gamma_5 C_{\mathrm{D}} \mathrm{T}^a E^a u) \big\rangle_E'$$

Partition func. of the boundary phase

positive definite without singularity at $m_0 = 0$

Attempts/Approaches to "exactly gauge-invariant formulation"

1. To solve the local cohomology problem [*Luscher (2000)*]

   - to determine the measure term $\Sigma_i(v_i, \delta_\eta v_i)$, based on the topological properties of the measure term (gauge anomaly) in 4+2 dim.
     cf. Wess-Zumino's Descent relation through 4+2, 4+1, 4 dim.

2. To decouple the mirror degrees of freedom out of Overlap Dirac fields

   1) Mirror Ginsparg-Wilson fermions [*Poppitz et al (2006)*]

      - by Multi-fermion interactions or Yukawa-interactions

   2) Mirror modes of Domain-wall fermion [*Creutz et al (1997)*]

      - by boundary interactions

   3) 4D TI / TSC with Gapped Boundary Phases

      [*Wen(2013), You-BenTov-Xu(2014), You-Xu (2015)*]

      - by boundary/bulk interactions

3. To Saturate the r.h.p. of Dirac-measure by 't Hooft vertices [*YK(2017): SO(10)*]

"All things merge into one, and a river runs through it"

1. A gauge invariant path-integral measure for the overlap Weyl fermions in $\underline{16}$ of SO(10)

A gauge invariant path-integral measure for the overlap Weyl fermions in 16 of SO(10)

$$\mathcal{D}[U] \equiv \prod_{x\in\Lambda}\prod_{\mu=0}^{3} dU(x,\mu)$$

$$\mathcal{D}[\psi_-]\mathcal{D}[\bar\psi_-] \equiv \mathcal{D}[\psi]\mathcal{D}[\bar\psi] \prod_{x\in\Lambda} F(T_+(x)) \prod_{x\in\Lambda} F(\bar T_+(x))$$

$$\mathcal{D}[\psi]\mathcal{D}[\bar\psi] \equiv \prod_{x\in\Lambda}\prod_{\alpha=1}^{4}\prod_{s=1}^{16} d\psi_{\alpha s}(x) \prod_{x\in\Lambda}\prod_{\alpha=1}^{4}\prod_{s=1}^{16} d\bar\psi_{\alpha s}(x)$$

$$T_+(x) = \frac{1}{2} V_+^a(x) V_+^a(x), \quad V_+^a(x) = \psi_+(x)^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a \psi_+(x) \qquad \mathrm{T}^a = \mathrm{C}\Gamma^a$$

$$\bar T_+(x) = \frac{1}{2} \bar V_+^a(x) \bar V_+^a(x), \quad \bar V_+^a(x) = \bar\psi_+(x) i\gamma_5 C_D \mathrm{T}^a \bar\psi_+(x)^{\mathrm{T}} \qquad \mathrm{T}^{aT} = \mathrm{T}^a$$

$$F(w) \equiv 4!\,(z/2)^{-4} I_4(z)\Big|_{(z/2)^2=w} = 4! \sum_{k=0}^{\infty} \frac{w^k}{k!(k+4)!}$$

$$F(w)\Big|_{w=(1/2)u^a u^a} = (\pi^5/12)^{-1} \int \prod_{a=1}^{10} de^a \delta(\sqrt{e^b e^b} - 1)\, \mathrm{e}^{e^c u^c}$$

Partition function

$$Z \equiv \int \mathcal{D}[U]\, \mathrm{e}^{-S_G[U]+\Gamma_W[U]}$$

$$\begin{aligned}
\mathrm{e}^{\Gamma_W[U]} &\equiv \int \mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-]\, \mathrm{e}^{-S_W[\psi_-,\bar{\psi}_-]} \\
&= \int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] \prod_{x\in\Lambda} F(T_+(x)) \prod_{x\in\Lambda} F(\bar{T}_+(x))\, \mathrm{e}^{-S_W[\psi_-,\bar{\psi}_-]} \\
&= \int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\, \mathcal{D}[E]\mathcal{D}[\bar{E}]\; \mathrm{e}^{-S_W[\psi_-,\bar{\psi}_-]+\sum_{x\in\Lambda}\{E^a(x)V_+^a(x)+\bar{E}^a(x)\bar{V}_+^a(x)\}[\psi_+,\bar{\psi}_+]}
\end{aligned}$$

$$\mathcal{D}[E] = \prod_{x\in\Lambda}(\pi^5/12)^{-1} \prod_{a=1}^{10} dE^a(x)\delta(\sqrt{E^b(x)E^b(x)}-1)$$

$$\mathcal{D}[\bar{E}] = \prod_{x\in\Lambda}(\pi^5/12)^{-1} \prod_{a=1}^{10} d\bar{E}^a(x)\delta(\sqrt{\bar{E}^b(x)\bar{E}^b(x)}-1)$$

$$F(w)\Big|_{w=(1/2)u^a u^a} = (\pi^5/12)^{-1} \int \prod_{a=1}^{10} de^a \delta(\sqrt{e^b e^b}-1)\, \mathrm{e}^{e^c u^c}$$

The measure in the chiral bases $n = 4 \times 16 \times L^4$

$$\mathrm{P}_+ \otimes \hat{P}_- v_i(x) = v_i(x) \quad (i = 1, \cdots, n/2 + 8Q); \quad (v_i, v_j) = \delta_{ij}$$
$$\bar{v}_k(x) P_+ \otimes \mathrm{P}_+ = \bar{v}_k(x) \quad (k = 1, \cdots, n/2); \quad (\bar{v}_k, \bar{v}_l) = \delta_{kl}$$

$$\mathrm{P}_+ \otimes \hat{P}_+ u_i(x) = u_i(x) \quad (i = 1, \cdots, n/2 - 8Q); \quad (u_i, u_j) = \delta_{ij}$$
$$\bar{u}_k(x) P_- \otimes \mathrm{P}_+ = \bar{u}_k(x) \quad (k = 1, \cdots, n/2); \quad (\bar{u}_k, \bar{u}_l) = \delta_{kl}$$

$$\psi_-(x) = \sum_j v_j(x) c_j, \quad \bar{\psi}_-(x) = \sum_k \bar{c}_k \bar{v}_k(x)$$

$$\psi_+(x) = \sum_j u_j(x) b_j, \quad \bar{\psi}_+(x) = \sum_k \bar{b}_k \bar{u}_k(x)$$

$$\mathcal{D}_\star[\psi_-]\mathcal{D}_\star[\bar{\psi}_-]\mathcal{D}_\star[\psi_+]\mathcal{D}_\star[\bar{\psi}_+] = \prod_{j=1}^{n/2+8Q} dc_j \prod_{k=1}^{n/2} d\bar{c}_k \prod_{j=1}^{n/2-8Q} db_j \prod_{k=1}^{n/2} d\bar{b}_k$$

$$\mathcal{D}_\star[\psi_-]\mathcal{D}_\star[\bar{\psi}_-]\mathcal{D}_\star[\psi_+]\mathcal{D}_\star[\bar{\psi}_+] = \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] \qquad \sum_j (u_j, \delta_\eta u_j) + \sum_j (v_j, \delta_\eta v_j) = 0$$

The measure in the chiral bases (con't)

$$S_W[\psi_-, \bar{\psi}_-] = \sum_{k,i} \bar{c}_k (\bar{v} D v)_{ki} c_i$$

$$\sum_{x\in\Lambda} \{E^a(x) V_+^a(x)\}[\psi_+] = \sum_{i,j} b_i (u^{\mathrm{T}}\, i\gamma_5 C_D \mathrm{T}^a E^a u)_{ij} b_j$$

$$\sum_{x\in\Lambda} \{\bar{E}^a(x) \bar{V}_+^a(x)\}[\bar{\psi}_+] = \sum_{k,l} \bar{b}_k (\bar{u}\, i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a \bar{u}^{\mathrm{T}})_{kl} \bar{b}_l$$

$$(\bar{v} D v)_{ki} = \sum_{x\in\Lambda} \bar{v}_k(x) D v_i(x) \qquad (k = 1, \cdots, n/2; i = 1, \cdots, n/2 + 8Q)$$

$$\left(u^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a u\right)_{ij} \equiv \sum_{x\in\Lambda} u_i(x)^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a(x) u_j(x)$$

$$(i, j = 1, \cdots, n/2 - 8Q)$$

$$\left(\bar{u} i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a \bar{u}^{\mathrm{T}}\right)_{kl} \equiv \sum_{x\in\Lambda} \bar{u}_k(x) i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a(x) \bar{u}_l(x)^{\mathrm{T}}$$

$$(k, l = 1, \cdots, n/2)$$

The measure in the chiral bases (con't)

$$\mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-] = \mathcal{D}_\star[\psi_-]\mathcal{D}_\star[\bar{\psi}_-] \times$$
$$\mathcal{D}_\star[\psi_+]\mathcal{D}_\star[\bar{\psi}_+] \int \mathcal{D}[E]\mathcal{D}[\bar{E}] \; \mathrm{e}^{\sum_{x\in\Lambda}\{E^a(x)V_+^a(x)+\bar{E}^a(x)\bar{V}_+^a(x)\}[\psi_+,\bar{\psi}_+]}$$

$$= \mathcal{D}_\star[\psi_-]\mathcal{D}_\star[\bar{\psi}_-] \times$$
$$\int \mathcal{D}[E]\,\mathrm{pf}\left(u^{\mathrm{T}}\, i\gamma_5 C_D \mathrm{T}^a E^a u\right) \int \mathcal{D}[\bar{E}]\,\mathrm{pf}\left(\bar{u}\, i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a \bar{u}^{\mathrm{T}}\right)$$

$$\left(u^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a u\right)_{ij} \equiv \sum_{x\in\Lambda} u_i(x)^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a(x) u_j(x)$$
$$(i,j = 1,\cdots,n/2-8Q),$$

$$\left(\bar{u} i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a \bar{u}^{\mathrm{T}}\right)_{kl} \equiv \sum_{x\in\Lambda} \bar{u}_k(x) i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a(x) \bar{u}_l(x)^{\mathrm{T}}$$
$$(k,l = 1,\cdots,n/2),$$

the pfaffians of these matrices do not vanish identically in general !

The measure in the chiral bases (con't)

$$\begin{aligned}
e^{\Gamma_W[U]} &\equiv \int \mathcal{D}[\psi_-]\mathcal{D}[\bar\psi_-]\, e^{-S_W[\psi_-,\bar\psi_-]} \\
&= \int \mathcal{D}_\star[\psi_-]\mathcal{D}_\star[\bar\psi_-]\, e^{-S_W[\psi_-,\bar\psi_-]} \times \\
&\quad \int \mathcal{D}[E]\, \mathrm{pf}\big(u^{\mathrm{T}}\, i\gamma_5 C_D \mathrm{T}^a E^a u\big) \int \mathcal{D}[\bar E]\, \mathrm{pf}\big(\bar u\, i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar E^a \bar u^{\mathrm{T}}\big), \\
&= \det(\bar v D v) \times \int \mathcal{D}[\bar E]\, \mathrm{pf}(u^T i\gamma_5 C_D T^a E^a u) \int \mathcal{D}[\bar E]\, \mathrm{pf}\big(\bar u\, i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar E^a \bar u^{\mathrm{T}}\big)
\end{aligned}$$

$$\sum_j (u_j, \delta_\eta u_j) + \sum_j (v_j, \delta_\eta v_j) = 0$$

cf. dependence on the basis *Luscher(98)*

- change of the chiral basis by a unitary transformation

$$\tilde v_i(x) = v_j(x)\left(\tilde Q^{-1}\right)_{ji} \qquad \tilde c_i = \tilde Q_{ij} c_j$$

$$\mathcal{D}[\psi_-]\mathcal{D}[\bar\psi_-] \Longrightarrow \mathcal{D}[\psi_-]\mathcal{D}[\bar\psi_-]\, \det \tilde Q\, [U_\mu(x)]$$

The saturation of the Right-handed measures due to ’t Hooft vertices (the anti-fields)

the part of the anti-field: $\mathcal{D}_\star[\bar\psi_+]$

$$\int \mathcal{D}[\bar E]\,\mathrm{pf}\big(\bar u\, i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar E^a \bar u^{\mathrm{T}}\big) = 1$$

for all topological sectors

$$(\bar u i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar E^a \bar u^{\mathrm{T}})_{kl} = i\,\epsilon_{\sigma\sigma'}\delta_{xx'}\big(\mathrm{T}^{a\dagger}\mathrm{P}_+\big)_{tt'}\bar E^a(x')$$

$$\begin{aligned}\mathrm{pf}\big(\bar u i\gamma_5 C_D \mathrm{T}^{a} \bar E^a \bar u^{\mathrm{T}}\big) &= \prod_x \det\big(\mathrm{P}_- + \mathrm{P}_+ i\,\mathrm{T}^{a\dagger}\bar E^a(x)\big) \\ &= \prod_x \det\big(i\,\check{\mathrm{T}}^{a\dagger}\bar E^a(x)\big) \\ &= \prod_x \det\big(i\check C^\dagger[E^{10}(x) + i\check\Gamma^{a'}\bar E^{a'}(x)]\big) \\ &= 1.\end{aligned}$$

$$\begin{aligned}1 &= \int \mathcal{D}_\star[\bar\psi_+] F\big(\bar T_+(x)[\bar\psi_+]\big) \\ &= \int \prod_{x\in\Lambda}\prod_{\alpha=3}^{4}\prod_{s=1}^{16} d\bar\psi_{\alpha s}(x)\,\prod_{x\in\Lambda}\frac{4!}{8!12!}\left\{\frac{1}{2}\,\bar\psi(x)P_- i\gamma_5 C_D \mathrm{T}^a\bar\psi(x)^{\mathrm{T}}\,\bar\psi(x)P_- i\gamma_5 C_D \mathrm{T}^a\bar\psi(x)^{\mathrm{T}}\right\}^8\end{aligned}$$

The saturation of the Right-handed measures due to ’t Hooft vertices (the fields)

the part of the field: $\mathcal{D}_\star[\psi_+]$

$$\int \mathcal{D}[E]\,\mathrm{pf}\left(u^{\mathrm{T}}\, i\gamma_5 C_D \mathrm{T}^a E^a u\right) = c\,[U(x,\mu)] \neq 0$$

for all topological sectors

$$\left(u^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a u\right)_{ij} \equiv \sum_{x\in\Lambda} u_i(x)^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a(x) u_j(x)$$

$$(i, j = 1, \cdots, n/2 - 8Q)$$

the pfaffian of the matrix does not vanish identically in general !

More on the saturation of the Right-handed measures due to 't Hooft vertices

the trivial link field (in the weak gauge-coupling limit)

$$U(x,\mu) = 1$$

the SU(2) link fields representing the topological sectors $\mathfrak{U}[Q]$

$$U(x,\mu) = \mathrm{e}^{i\theta_{12}(x,\mu)\Sigma^{12}} \quad \textbf{in} \quad \mathfrak{U}[Q] \quad \textbf{with} \quad Q = 2\, m_{01} m_{23} \; (m_{01}, m_{23} \in \mathbb{Z}):$$

where

$$\theta_{12}(x,0) = \begin{cases} 0 & (x_0 < L-1) \\ -F_{01} L x_1 & (x_0 = L-1) \end{cases}, \qquad \theta_{12}(x,1) = F_{01}\, x_0$$

$$\theta_{12}(x,3) = \begin{cases} 0 & (x_2 < L-1) \\ -F_{23} L x_3 & (x_2 = L-1) \end{cases}, \qquad \theta_{12}(x,4) = F_{23}\, x_2$$

$$F_{01} = \frac{4\pi m_{01}}{L^2}, \qquad F_{23} = \frac{4\pi m_{23}}{L^2}$$

More on the saturation of the Right-handed measures due to 't Hooft vertices (con't)

as long as the link field $U(x,\mu)$ is in SO(9) subgroup

$$u_j(x)^T i\gamma_5 C_D \, \mathrm{C}\Gamma^{10} = \mathcal{C}_{jk} \, u_k(x)^\dagger$$
$$\mathcal{C}^{-1} = \mathcal{C}^\dagger = \mathcal{C}^T = -\mathcal{C}.$$

$$\mathcal{C}_{jk} = (u^T i\gamma_5 C_D \, \mathrm{C}\Gamma^{10} u)_{jk}$$

$$\begin{aligned}(u^{\mathrm{T}} \, i\gamma_5 C_D \mathrm{T}^a E^a u) &= \mathcal{C} \times (u^\dagger \Gamma^{10} \Gamma^a E^a u) \\ &= (u^\dagger \Gamma^{10} \Gamma^a E^a u)^T \times \mathcal{C}\end{aligned}$$

$$\mathrm{pf}(u^{\mathrm{T}} \, i\gamma_5 C_D \mathrm{T}^a E^a u) = \mathrm{pf}(u^T i\gamma_5 C_D \, \mathrm{C}\Gamma^{10} u) \times \prod_{i=1}^{n/4-4Q} \lambda_i$$

More on the saturation of the Right-handed measures due to 't Hooft vertices (con't)

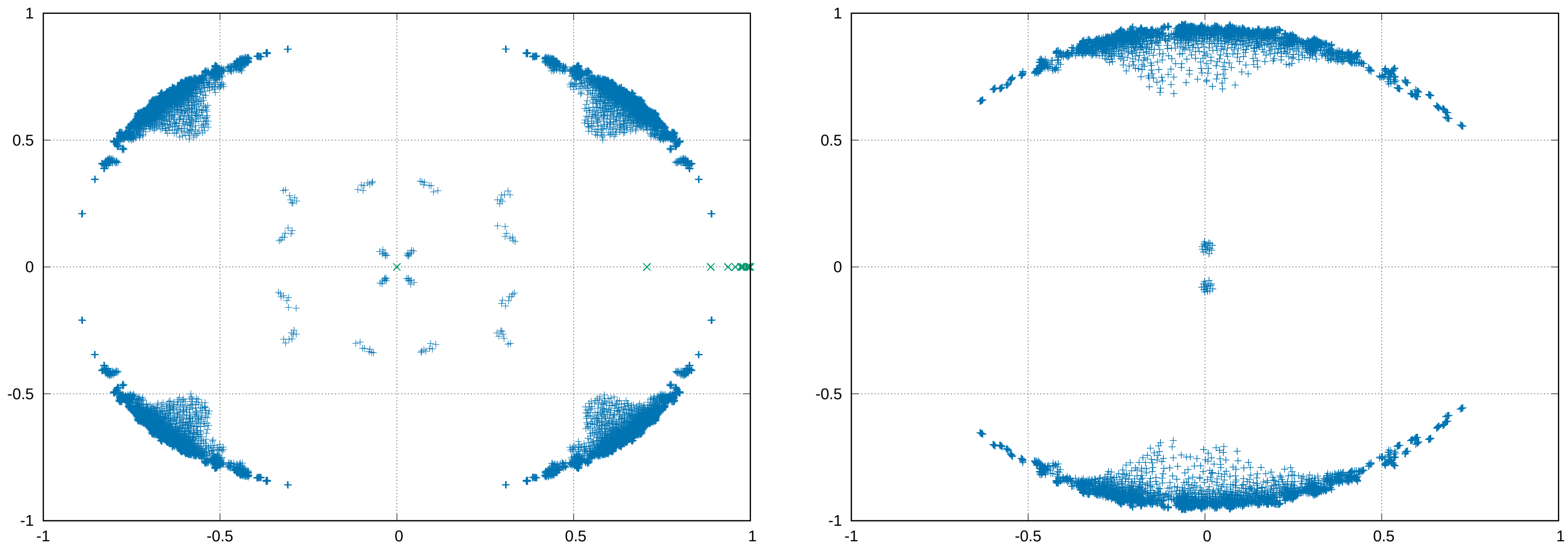


Figure 2. The eigenvalue spectra of the matrices $\left(u^{\mathrm{T}}\, i\gamma_5 C_D \mathrm{T}^a E^a u\right)$ and $\left(u^{\dagger}\Gamma^{10}\Gamma^a E^a u\right)$ with a randomly generated spin-field configuration for the case of the trivial link field. The lattice size is $L = 4$ and the boundary condition for the fermion field is periodic. For reference, the eigenvalue spectrum of the matrix $(\bar{v}_k D v_i)$ is also shown with green x symbol for the same boundary condition.

More on the saturation of the Right-handed measures due to 't Hooft vertices (con't)

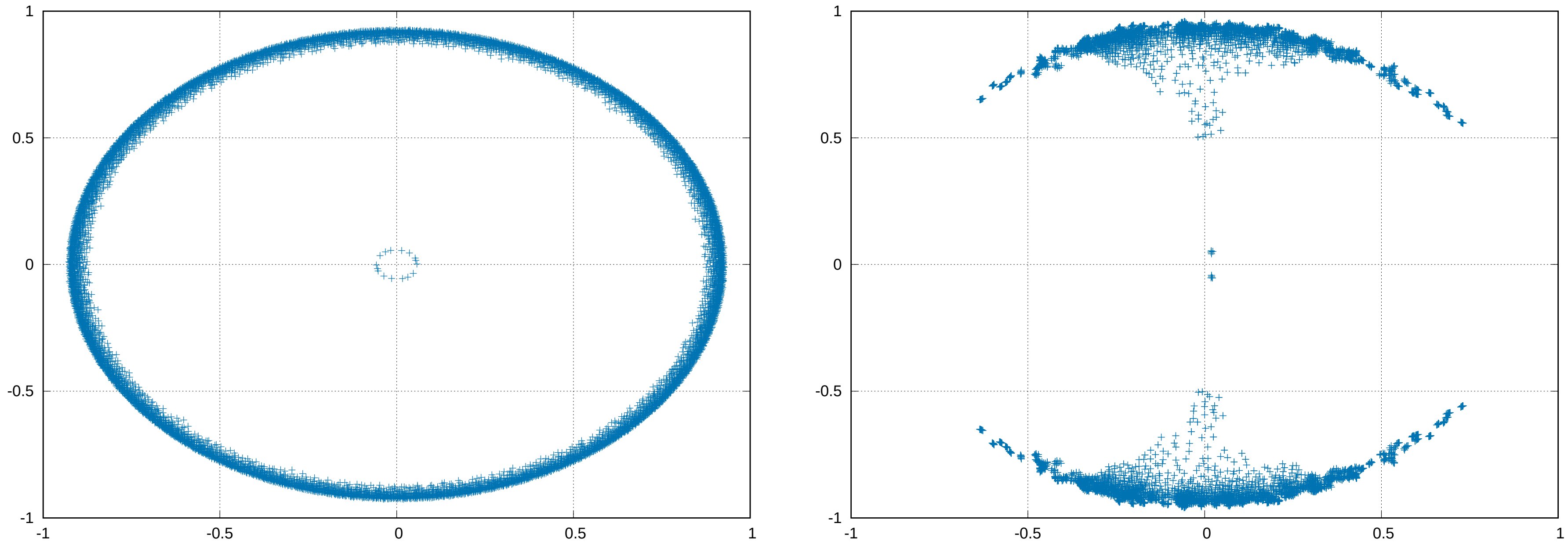


Figure 5. The eigenvalue spectra of the matrices $\left(u^{\mathrm{T}}\, i\gamma_5 C_D \mathrm{T}^a E^a u\right)$ and $\left(u^{\dagger}\Gamma^{10}\Gamma^a E^a u\right)$ with a randomly generated spin-field configuration for the case of the representative SU(2) link field of the topological sector with $Q = -2$. The lattice size is $L = 4$ and the boundary condition for the fermion field is periodic.

More on the saturation of the Right-handed measures due to ’t Hooft vertices (con’t)

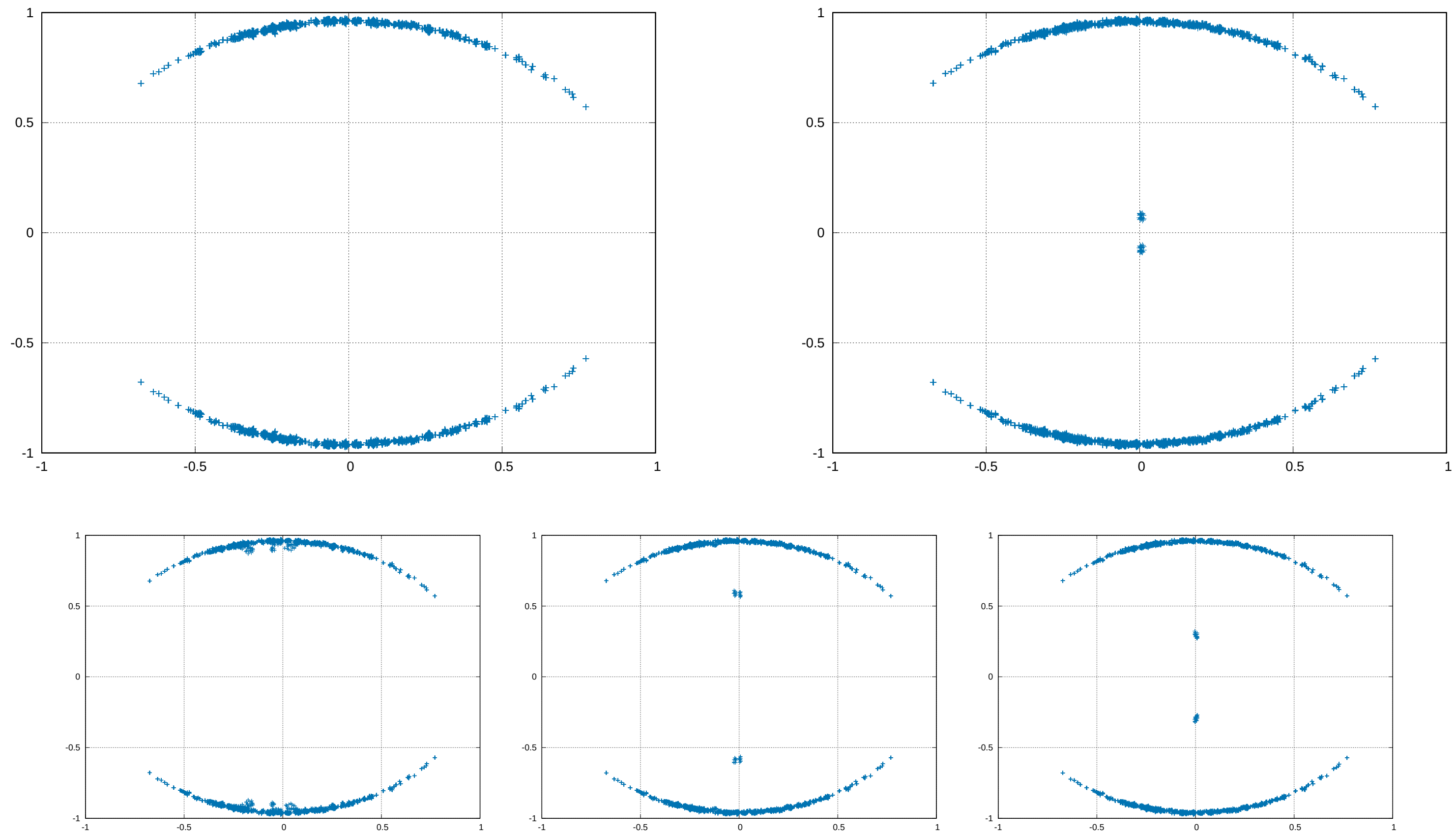


Figure 6. The eigenvalue spectra of $\left(u^\dagger \Gamma^{10} \Gamma^a E^a u\right)$ in the limit $m_0 \to \mp 0$ with a randomly generated spin configuration for the trivial link field. The interpolation parameter θ_α is chosen as $\theta_\alpha = 0, 3\pi/12, 4\pi/12, 5\pi/12, \pi/2$ for the top-left, bottom-left, bottom-middle, bottom-right, top-right figures, respectively. The lattice size is $L = 4$ and the boundary condition for the fermion field is periodic.

Partition function of the SO(10) chiral lattice gauge theory

$$Z \equiv \int \mathcal{D}[U]\, \mathrm{e}^{-S_G[U]+\Gamma_W[U]}$$

$$\begin{aligned}
\mathrm{e}^{\Gamma_W[U]} &\equiv \int \mathcal{D}[\psi_-]\mathcal{D}[\bar\psi_-]\, \mathrm{e}^{-S_W[\psi_-,\bar\psi_-]} \\
&= \int \mathcal{D}[\psi]\mathcal{D}[\bar\psi] \prod_{x\in\Lambda} F(T_+(x)) \prod_{x\in\Lambda} F(\bar T_+(x))\, \mathrm{e}^{-S_W[\psi_-,\bar\psi_-]} \\
&= \int \mathcal{D}[\psi]\mathcal{D}[\bar\psi]\, \mathcal{D}[E]\mathcal{D}[\bar E]\ \mathrm{e}^{-S_W[\psi_-,\bar\psi_-]+\sum_{x\in\Lambda}\{E^a(x)V_+^a(x)+\bar E^a(x)\bar V_+^a(x)\}[\psi_+,\bar\psi_+]} \\
&= \det(\bar v D v) \int \mathcal{D}[E]\, \mathrm{pf}(u^T i\gamma_5 C_D T^a E^a u)
\end{aligned}$$

for all topological sectors

cf. $$\begin{aligned} S_{\mathrm{Ov}}[\psi,\bar\psi,E^a,\bar E^a] &= \sum_{x\in\Lambda} \bar\psi_-(x) D \psi_-(x) \\
&- \sum_{x\in\Lambda}\{E^a(x)\psi_+^{\mathrm{T}}(x) i\gamma_5 C_D \mathrm{T}^a \psi_+(x) + \bar E^a(x)\bar\psi_+(x) i\gamma_5 C_D \mathrm{T}^{a\dagger}\bar\psi_+(x)^T\}\end{aligned}$$

SO(10) lattice gauge theory with Weyl fermions in 16 in the framework of overlap fermion/the Ginsparg-Wilson rel.

- manifestly gauge-invariant by using full Dirac-field measure, but **saturating the right-handed part with 't Hooft vertices completely !**

- **without solving the local cohomology problem**

  cf. U(1) chiral gauge theory with exact gauge invariance
  [*Luscher (1999)*]

  cf. $SU(2)_L \times U(1)_Y$ GWS model
  [*Kadoh-Kikukawa (2008), Nakayama-Kikukawa(2001)*]

  cf. All orders in the weak gauge-coupling expansion
  [*Suzuki (2000), Luscher (2000)*]

- **all possible topological sectors**
- **CP invariance** $\Gamma_W[U^{\mathrm{CP}}] = \Gamma_W[U]$
- Issues of **locality/smoothness** remaining

  Testable: To see if it works, examine $\langle \psi_+(y) \left[\psi_+^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a \hat{P}_-(x) \right] \rangle_F$

  MC studies in weak gauge-coupling limit feasible without sign problem

  Analytic studies desirable

4. Discussions

Gauge-field dependence of the Weyl fermion measures

variation of effective action $\delta_\eta U(x,\mu) = i\eta_\mu(x) U(x,\mu)$

cf. $$S_{\mathrm{Ov}}[\psi,\bar\psi,E^a,\bar E^a] = \sum_{x\in\Lambda} \bar\psi_-(x) D\psi_-(x) - \sum_{x\in\Lambda} \{E^a(x)\psi_+^{\mathrm{T}}(x) i\gamma_5 C_D \mathrm{T}^a \psi_+(x) + \bar E^a(x)\bar\psi_+(x) i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar\psi_+(x)^T\}$$

$$\delta_\eta \Gamma_W[U] = \Big\langle -\sum_{x\in\Lambda} \bar\psi(x) P_+ \delta_\eta D \psi(x) + 2\sum_{x\in\Lambda} \psi^{\mathrm{T}}(x) \hat P_+^T i\gamma_5 C_D \mathrm{T}^a E^a \delta_\eta \hat P_+ \psi(x) \Big\rangle_F / \langle 1\rangle_F$$
$$= \mathrm{Tr}\{P_+\delta_\eta D \langle \psi_-\bar\psi_-\rangle_F\} / \langle 1\rangle_F - 2\,\mathrm{Tr}\{\delta_\eta \hat P_+ \langle \psi_+ [\psi_+^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a]\rangle_F\} / \langle 1\rangle_F$$

$$\mathrm{Tr}\{P_+\delta_\eta D \langle \psi_-\bar\psi_-\rangle_F\} / \langle 1\rangle_F = \mathrm{Tr}\{P_+\delta_\eta D D^{-1}\}$$ **[Left-handed Weyl field]**

$$-i\mathfrak{T}_\eta \equiv -2\,\mathrm{Tr}\{\delta_\eta \hat P_+ \langle \psi_+ [\psi_+^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a]\rangle_F\} / \langle 1\rangle_F$$ **[Right-handed ’t Hooft ops.]**

cf. the measure term *Luscher(98)*

$$\Gamma_{\mathrm{eff}} = \ln\det(\bar v_k D v_j)$$

$$\delta_\eta \Gamma_{\mathrm{eff}} = \mathrm{Tr}\left\{(\delta_\eta D)\hat P_- D^{-1} P_+\right\} + \sum_i (v_i, \delta_\eta v_i)$$

$$-i\mathfrak{L}_\eta = \sum_j (v_j, \delta_\eta v_j)$$ **(=> a smooth/local functional of the link field!)**

Locality / Smoothness of the Weyl fermion measures

The term from the right-handed 't Hooft ops.

$$
\begin{aligned}
-i\mathfrak{T}_\eta &\equiv -2\,\mathrm{Tr}\{\delta_\eta \hat{P}_+ \langle \psi_+ [\psi_+^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a] \rangle_F\} / \langle 1 \rangle_F \\
&= \langle \mathrm{Tr}\{(u^T i\gamma_5 C_D T^a E^a \delta_\eta \hat{P}_+ u)(u^T i\gamma_5 C_D T^a E^a u)^{-1}\} \rangle_E / \langle 1 \rangle_E
\end{aligned}
$$

should be a smooth/local functional of the link field!

$$
-i\mathfrak{T}_\eta = \sum_{m=0}^{\infty} \frac{(ig)^{1+m}}{V^{1+m}\, m!} \sum_{k,p_1,\cdots,p_m} \tilde{\eta}_\mu^{ab}(-k)\, \mathfrak{C}^{aba_1b_1\cdots a_mb_m}_{\mu\nu_1\cdots\nu_m}(k,p_1,\cdots,p_m) \times \tilde{A}^{a_1b_1}_{\nu_1}(p_1)\cdots\tilde{A}^{a_mb_m}_{\nu_m}(p_m)
$$

(regular vertex functions!)

A necessary and sufficient condition

$$
\left\| \langle \psi_+(x) [\psi_+^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a(y)] \rangle_F / \langle 1 \rangle_F \right\| \;\le\; C\,|x-y|^\sigma\, \mathrm{e}^{-|x-y|/\xi}
$$

and the similar bounds for the variations w.r.t. the link field

$$
\left\langle \psi_+(y) \left[\psi_+^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a \hat{P}_+(x)\right] \right\rangle_F = -\frac{1}{2}\hat{P}_+(y,x)\langle 1 \rangle_F
$$

Indeed satisfied by the SD eqs.

$$
\left\langle \psi_+(y) \left[\psi_+^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a \hat{P}_-(x)\right] \right\rangle_F
$$

Unknown yet!
Need further studies → OK in 2D!

The Schwinger-Dyson equations

$$\left\langle \left[-\delta_\eta S_G[U] - \sum_{x\in\Lambda} \bar\psi(x) P_+ \delta_\eta D \psi(x) + 2\sum_{x\in\Lambda} \psi^{\mathrm{T}} \hat{P}_+^T i\gamma_5 C_D \mathrm{T}^a E^a \delta_\eta \hat{P}_+ \psi(x) \right] \right\rangle = 0$$

$$\begin{aligned}
\left\langle \psi(y) \left[\bar\psi P_+ D(x) - 2\psi^{\mathrm{T}} \hat{P}_+^T i\gamma_5 C_D \mathrm{T}^a E^a \hat{P}_+(x) \right] \right\rangle_F &= \delta_{xy} \langle 1 \rangle_F \\
\left\langle \left[P_+ D\psi(x) - 2P_- i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a {P_-}^T \bar\psi^{\mathrm{T}}(x) \right] \bar\psi(y) \right\rangle_F &= \delta_{xy} \langle 1 \rangle_F \\
\left\langle \psi^{\mathrm{T}} \hat{P}_+^T i\gamma_5 C_D \mathrm{C}[\Sigma_{bc}, \Gamma^a] E^a(x) \hat{P}_+ \psi \right\rangle_F &= 0.
\end{aligned}$$

Fermion two-point correlation functions

$$\begin{aligned}
\left\langle \psi_-(x)\, \bar\psi_-(y) \right\rangle_F &= \hat{P}_- D^{-1} P_+(x,y) \langle 1 \rangle_F, \\
\left\langle \psi_+(y) \left[\psi_+^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a \hat{P}_+(x) \right] \right\rangle_F &= -\frac{1}{2} \hat{P}_+(y,x) \langle 1 \rangle_F, \\
\left\langle \left[P_- i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a \bar\psi_+^{\mathrm{T}}(x) \right] \bar\psi_+(y) \right\rangle_F &= -\frac{1}{2} P_- \delta_{xy} \langle 1 \rangle_F,
\end{aligned}$$

Locality / Smoothness of the Weyl fermion measures
— Case of 2dim. U(1) chiral gauge theories

The term from the right-handed 't Hooft ops.

$$
\begin{aligned}
-i\mathfrak{T}_\eta &\equiv -2\,\mathrm{Tr}\left\{\delta_\eta \hat{P}_+ \left\langle \psi_+ \left[\psi_+^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a\right] \right\rangle_F \right\} / \left\langle 1 \right\rangle_F \\
&= \left\langle \mathrm{Tr}\{(u^T i\gamma_5 C_D T^a E^a \delta_\eta \hat{P}_+ u)(u^T i\gamma_5 C_D T^a E^a u)^{-1}\} \right\rangle_E / \left\langle 1 \right\rangle_E
\end{aligned}
$$

should be a smooth/local functional of the link field!

$$
-i\mathfrak{T}_\eta = \sum_{m=0}^{\infty} \frac{(ig)^{1+m}}{V^{1+m}\, m!} \sum_{k,p_1,\cdots,p_m} \tilde{\eta}_\mu^{ab}(-k)\, \mathfrak{C}^{aba_1b_1\cdots a_mb_m}_{\mu\nu_1\cdots\nu_m}(k,p_1,\cdots,p_m) \times \tilde{A}^{a_1b_1}_{\nu_1}(p_1)\cdots\tilde{A}^{a_mb_m}_{\nu_m}(p_m)
$$

A necessary and sufficient condition

$$
\left\| \left\langle \psi_+(x) \left[\psi_+^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a(y)\right] \right\rangle_F / \left\langle 1 \right\rangle_F \right\| \;\le\; C\,|x-y|^\sigma\, \mathrm{e}^{-|x-y|/\xi}
$$

and the similar bounds for the variations w.r.t. the link field

$$
\left\langle \psi_+(y) \left[\psi_+^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a \hat{P}_+(x)\right] \right\rangle_F = -\frac{1}{2}\hat{P}_+(y,x)\langle 1 \rangle_F
$$

Indeed satisfied by the SD eqs.

$$
\left\langle \psi_+(y) \left[\psi_+^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a \hat{P}_-(x)\right] \right\rangle_F
$$

Unknown yet!
Need further studies → Study in 2D

1^4(-1)^4 axial gauge model with Spin(6)(SU(4)) symmetry

$$
\begin{aligned}
Q &= \mathrm{diag}(q_1, q_2, q_3, q_4) = \mathrm{diag}(+1, +1, +1, +1) \\
Q' &= \mathrm{diag}(q'_1, q'_2, q'_3, q'_4) = \mathrm{diag}(-1, -1, -1, -1)
\end{aligned}
$$

$$
\begin{aligned}
S_W &= \sum_{x \in \Gamma} \bar{\psi}(x) P_+ D \psi(x) + \sum_{x \in \Gamma} \bar{\psi}'(x) P_- D' \psi'(x) \\
S_M &= \sum_{x} z \left\{ \bar{\psi}_+(x) D_{+1} \psi_+(x) + \bar{\psi}'_-(x) D_{-1} \psi'_-(x) \right\} \\
&\quad + \sum_{x} h \left\{ \psi_+(x)^T i \gamma_3 c_D \mathrm{T}^a E^a(x) \psi'_-(x) + \bar{\psi}_+(x) i \gamma_3 c_D \mathrm{T}^{a\dagger} \bar{E}^a(x) \bar{\psi}'_-(x)^T \right\} \\
&\quad + \sum_{x,\mu} \kappa \left\{ E^a(x) E^a(x+\hat{\mu}) + \bar{E}^a(x) \bar{E}^a(x+\hat{\mu}) \right\} .
\end{aligned}
$$

$$
\begin{aligned}
O_V(x) &= \frac{1}{2} \psi_+(x)^T i \gamma_3 c_D \mathrm{T}^a \psi'_-(x) \, \psi_+(x)^T i \gamma_3 c_D \mathrm{T}^a \psi'_-(x), \\
\bar{O}_V(x) &= \frac{1}{2} \bar{\psi}_+(x) i \gamma_3 c_D \mathrm{T}^{a\dagger} \bar{\psi}'_-(x)^T \, \bar{\psi}_+(x) i \gamma_3 c_D \mathrm{T}^{a\dagger} \bar{\psi}'_-(x)^T,
\end{aligned}
$$

	+	−	gauge anomaly	chiral anomaly
$\mathrm{U}(1)_g$	1	-1	matched (gauged)	—
Spin(6)/SU(4)	$\underline{4}$	$\underline{4}$	matched (can be gauged)	anomaly free
$\mathrm{U}(1)_V$	1	1	not matched	anomalous

Table 3. Fermionic continuous symmetries in the mirror sector of the 1^4(-1)^4 model and their would-be gauge anomalies

21(-1)^3 chiral gauge model

$$Q = +\frac{1}{2} + \Sigma^{12} + \Sigma^{34} + \Sigma^{56}$$
$$Q' = -\frac{1}{2} + \Sigma^{12} + \Sigma^{34} + \Sigma^{56}$$

$$Q = \mathrm{diag}(q_1, q_2, q_3, q_4) = \mathrm{diag}(+2,\ \ 0,\ \ 0,\ \ 0),$$
$$Q' = \mathrm{diag}(q'_1, q'_2, q'_3, q'_4) = \mathrm{diag}(+1, -1, -1, -1).$$

$$S_W = \sum_{x\in\Gamma} \bar{\psi}(x) P_+ D \psi(x) + \sum_{x\in\Gamma} \bar{\psi}'(x) P_- D' \psi'(x)$$

$$S_M = \sum_{x\in\Gamma} \left\{ \psi_+(x)^T i\gamma_3 c_D \mathrm{T}^a E^a(x) \psi'_-(x) + \bar{\psi}_+(x) i\gamma_3 c_D \mathrm{T}^{a\dagger} \bar{E}^a(x) \bar{\psi}'_-(x)^T \right\}$$

	+	+	−	−	(mixed) gauge anomaly	chiral anomaly
$\mathrm{U}(1)_g$	2	0	1	-1	matched (gauged)	—
SU(3)	$\underline{1}$	$\underline{3}$	$\underline{1}$	$\underline{3}$	matched (can be gauged)	anomaly free
$\mathrm{U}(1)_b$	0	1	0	1	not matched	anomalous
$\mathrm{U}(1)_a$	0	1	0	-1	not matched	anomalous
$\mathrm{U}(1)_{b-3l}$	-3	1	-3	1	matched (can be gauged)	anomaly free

Table 4. Fermionic continuous symmetries in the mirror sector of the 21(-1)^3 model and their would-be gauge anomalies

$$\begin{aligned} S_E[E^a] &= -\ln\det(u^T\, i\gamma_3 c_D \check{\Gamma}^a E^a v') \\ &= -\ln\det(u^\dagger \Gamma^6 \Gamma^a E^a u). \end{aligned}$$

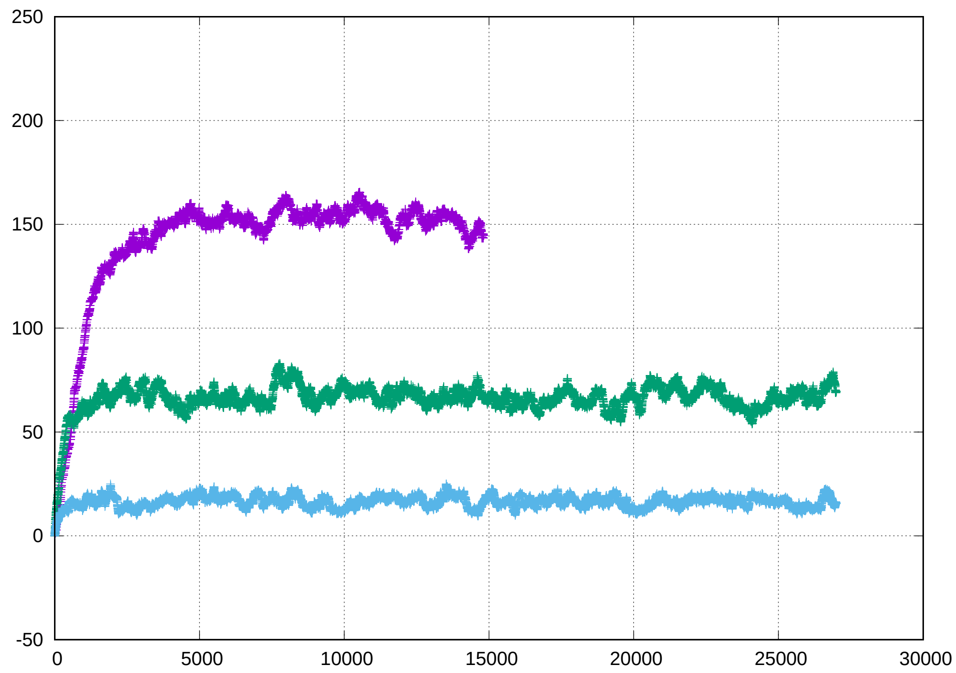


Figure 5. Monte Carlo histories of the effective action $S_E[E^a]$ fot the lattice sizes $L = 4, 8, 12$. The periodic boundary condition is used for the fermion fields.

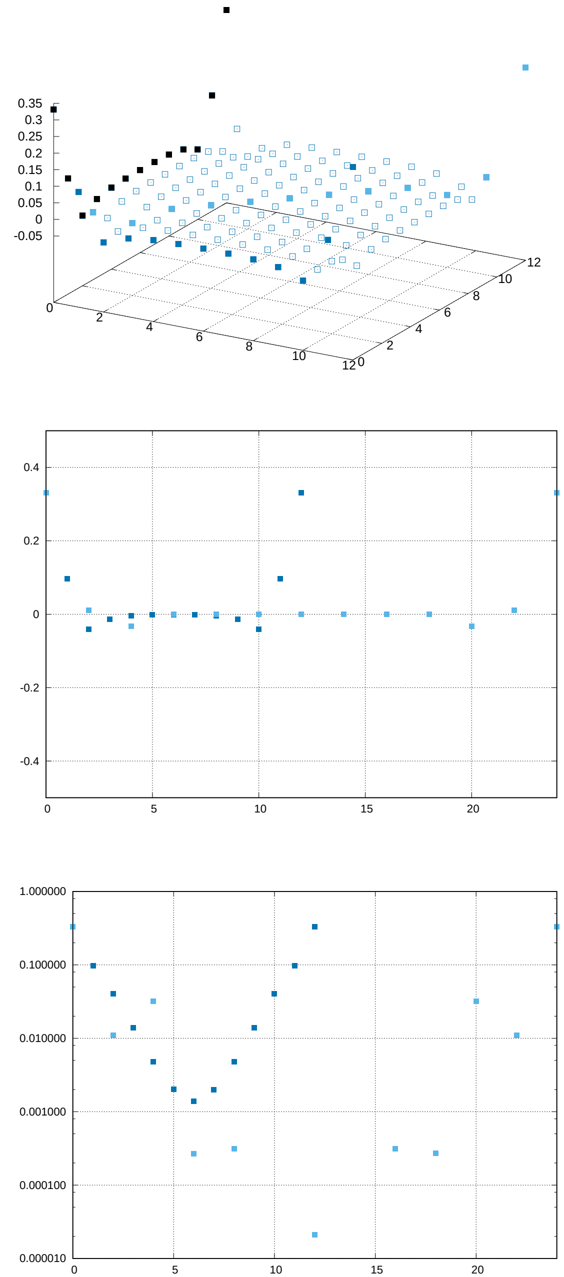


Figure 7. $\sum_{s=1}^{4}\{G_{\psi'\psi E}\hat{P}_{+}\}_{00,ss}(x)$ vs. $x = (x_0, x_1)$ [top] and $|x|_1 \equiv |x_0| + |x_1|$ [middle, bottom]. The lattice size is $L = 8$. The blue-symbol and black-symbol plots are along the spacial axis ($x_0 = 0$) and temporal axis ($x_1 = 0$), respectively, while the lightblue-symbol plot is along the diagonal axis ($x_0 = x_1$). 1,100 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones.

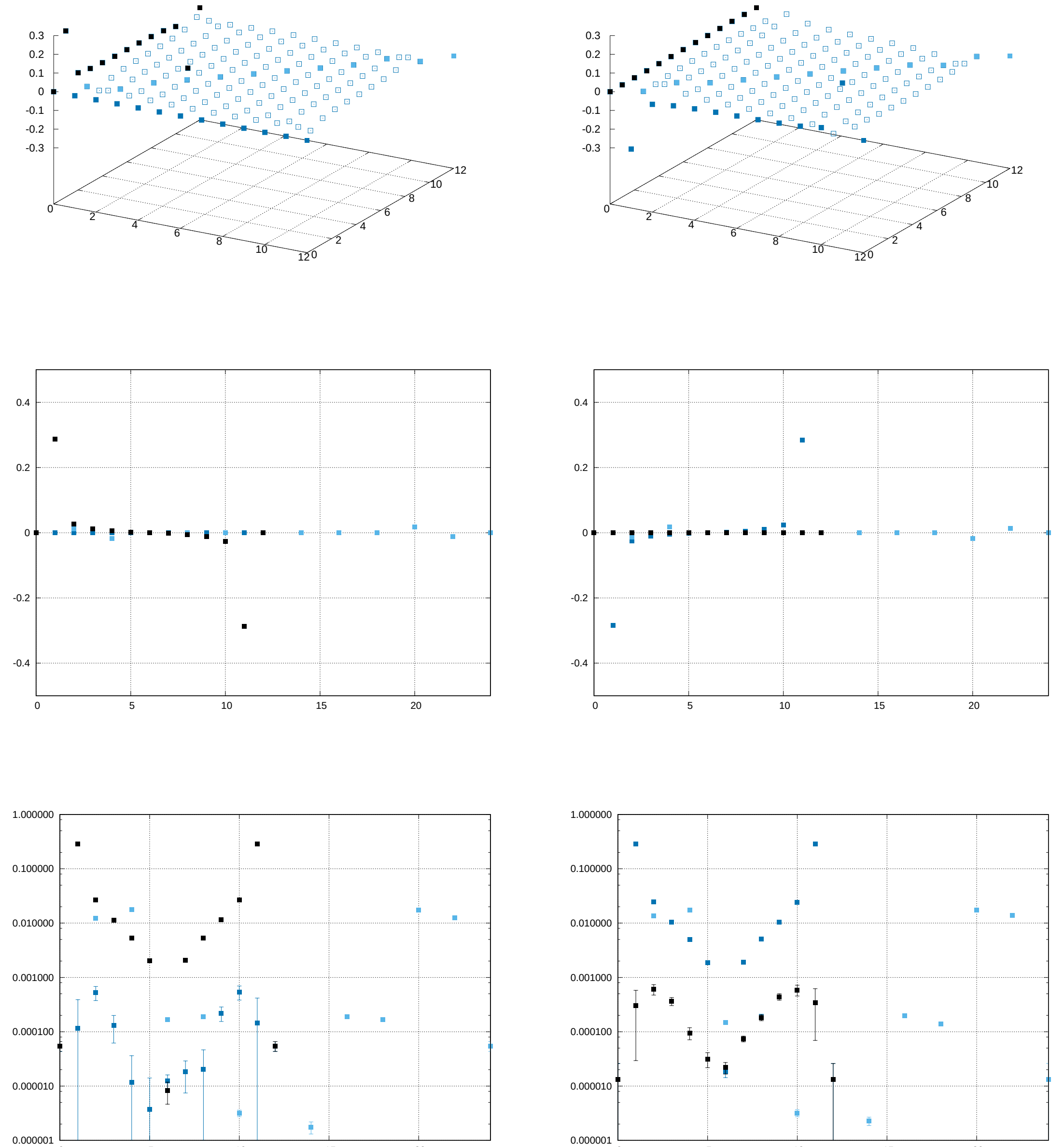


Figure 8. The real [left] and imaginary [right] parts of $\sum_{s=1}^{4}\{G_{\psi'\psi E}\hat{P}_+\}_{01,ss}(x)$ vs. $x = (x_0, x_1)$ [top] and $|x|_1 \equiv |x_0| + |x_1|$ [middle, bottom]. The lattice size is $L = 12$. The blue-symbol and black-symbol plots are along the spacial axis ($x_0 = 0$) and temporal axis ($x_1 = 0$), respectively, while the light-blue-symbol plot is along the diagonal axis ($x_0 = x_1$). 1,100 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones.

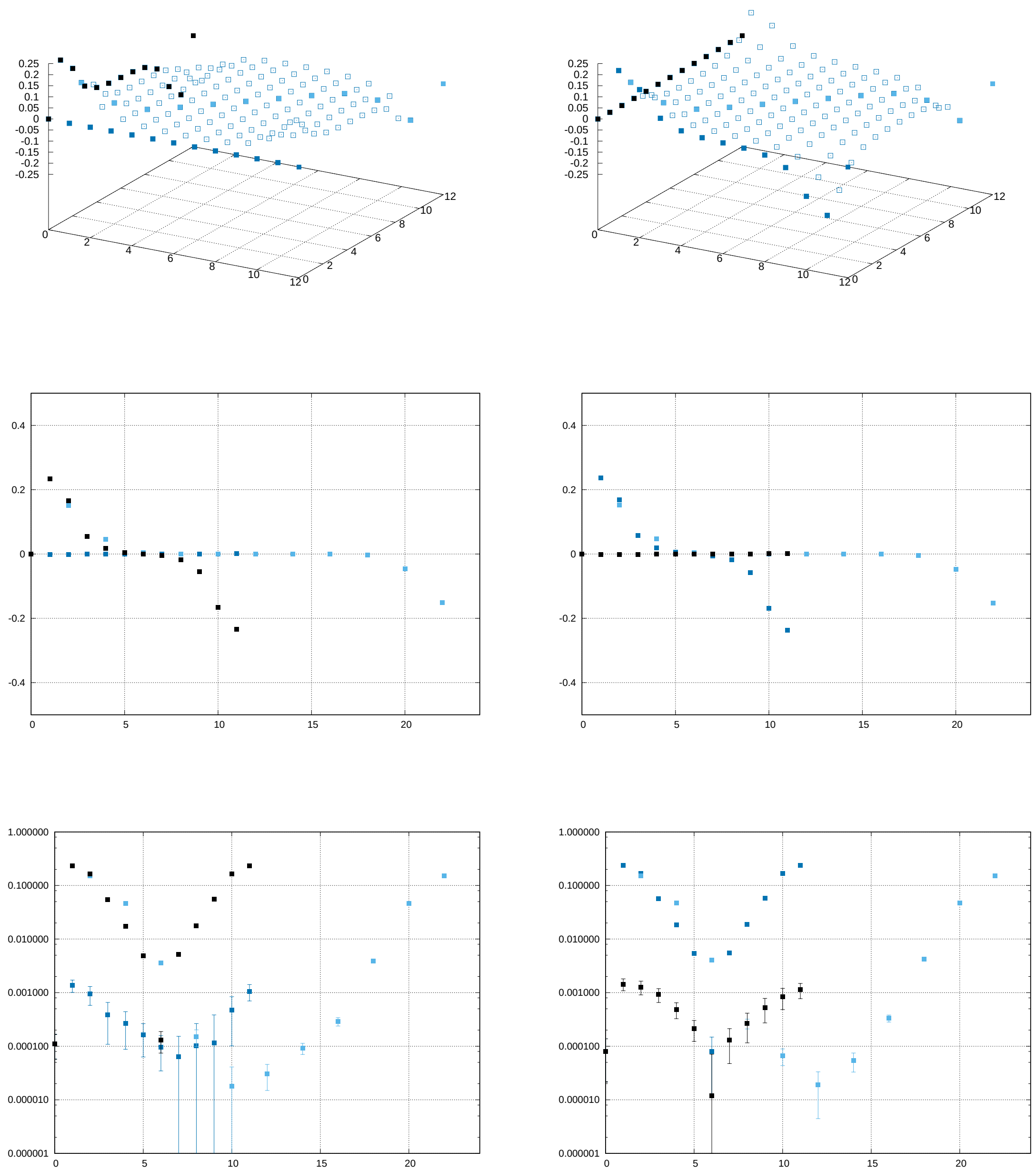


Figure 9. The real [left] and imaginary [right] parts of $\sum_{s=1}^{4}\{G_{\psi'\psi E}\hat{P}_{+}\}_{10,ss}(x)$ vs. $x=(x_0,x_1)$ [top] and $|x|_1 \equiv |x_0|+|x_1|$ [middle, bottom]. The lattice size is $L=12$. The blue-symbol and black-symbol plots are along the spacial axis ($x_0=0$) and temporal axis ($x_1=0$), respectively, while the light-blue-symbol plot is along the diagonal axis ($x_0=x_1$). 1,100 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones.

$$\frac{1}{L^2}\sum_k \tilde{\eta}_\mu(-k)\,\tilde{\Pi}'_{\mu\nu}(k)\,\tilde{\zeta}_\nu(k) = \delta_\zeta\left[\big\langle -\delta_\eta S_M\big\rangle_M/\big\langle 1\big\rangle_M\right]\Big|_{U(x,\mu)\to 1}$$

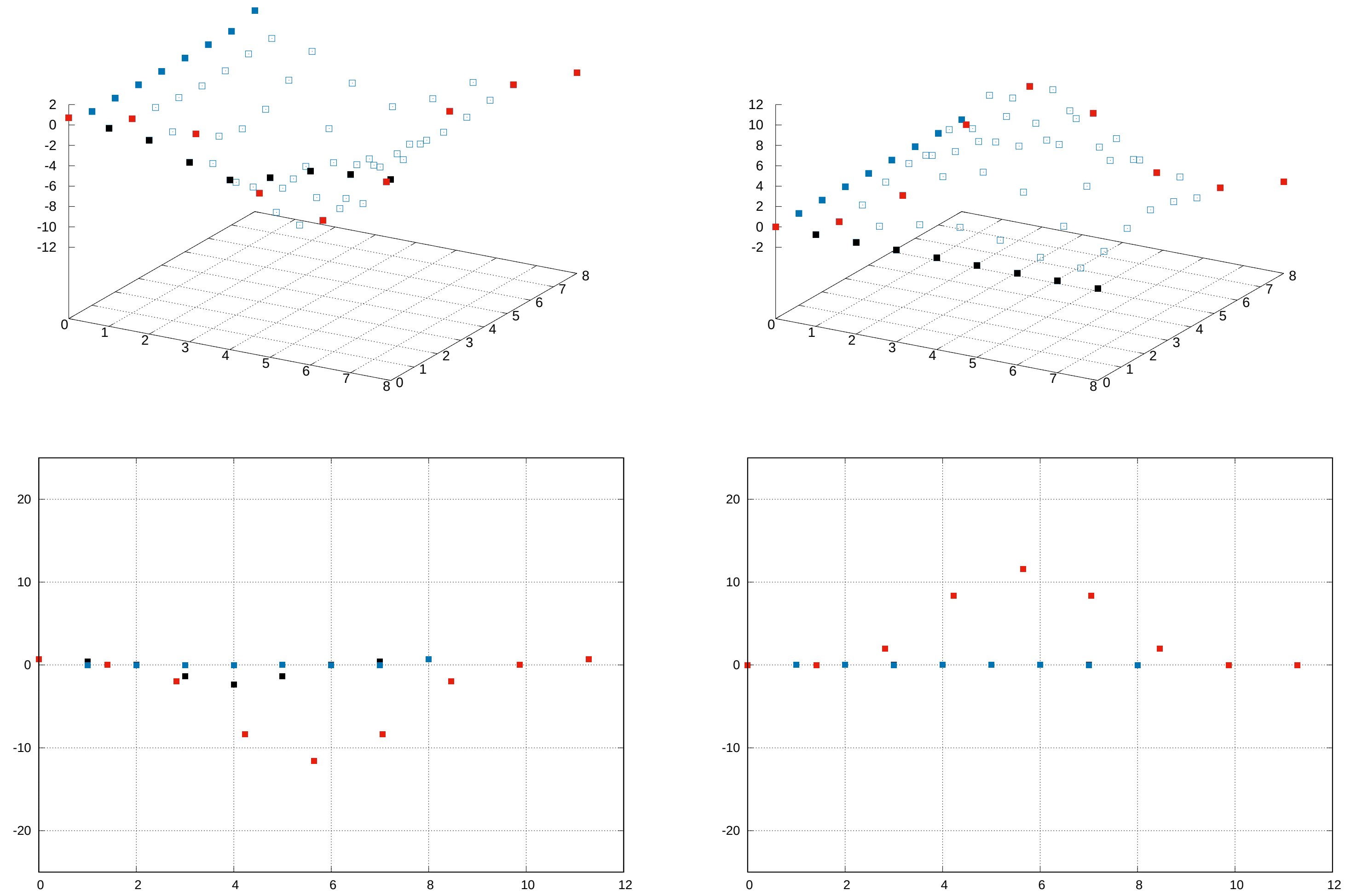


Figure 10. $L^2\,\tilde{\Pi}'_{00}(k)$ [left] and $L^2\,\tilde{\Pi}'_{01}(k)$ [right] vs. $|k|_2 \equiv \sqrt{k_0^2+k_1^2}$. The lattice size is $L=8$. The periodic boundary condition is assumed for the fermion fields. The black-, blue-, red-symbol plots are along the spacial momentum axis ($k_0=0$), the temporal momentum (energy) axis ($k_1=0$) and the diagonal momentum axis ($k_0=k_1$), respectively. 5,000 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones.

$$\frac{1}{L^2}\sum_k \tilde{\eta}_\mu(-k)\,\tilde{\Pi}'_{\mu\nu}(k)\,\tilde{\zeta}_\nu(k) = \delta_\zeta\left[\left\langle -\delta_\eta S_M\right\rangle_M / \left\langle 1\right\rangle_M\right]\Big|_{U(x,\mu)\to 1}$$

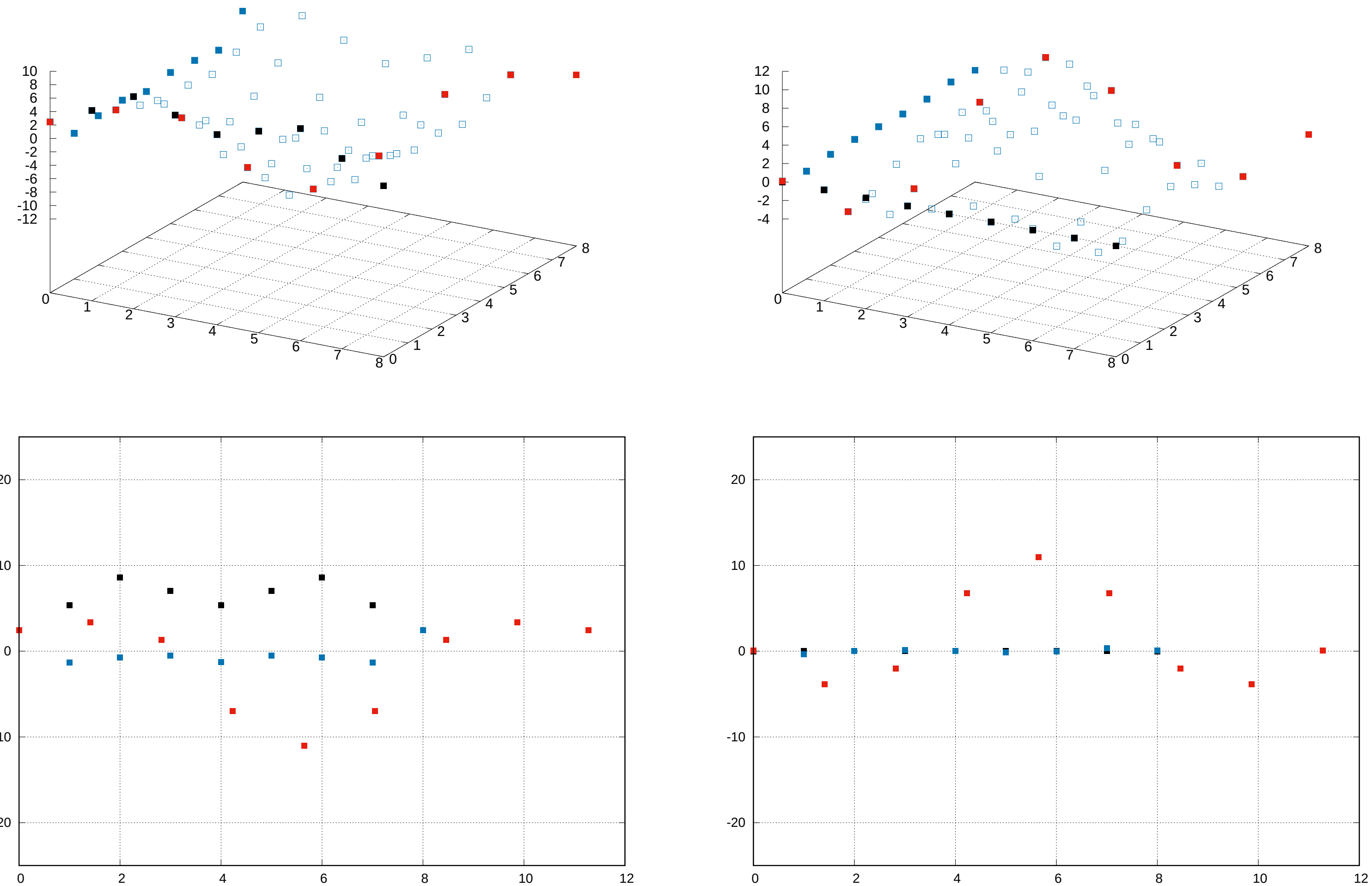


Figure 15. $L^2\,\tilde{\Pi}^{\hat{Q}\hat{Q}}_{00}(k)$ [left] and $L^2\,\tilde{\Pi}^{\hat{Q}\hat{Q}}_{01}(k)$ [right] vs. $|k|_2 \equiv \sqrt{k_0^2+k_1^2}$. The lattice size is $L=8$. The periodic boundary condition is assumed for the fermion fields. The black-, blue-, red-symbol plots are along the spacial momentum axis ($k_0=0$), the temporal momentum (energy) axis ($k_1=0$) and the diagonal momentum axis ($k_0=k_1$), respectively. 5,000 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones.

$$\frac{1}{L^2}\sum_k \tilde{\eta}_\mu(-k)\,\tilde{\Pi}_{\mu\nu}(k)\,\tilde{\zeta}_\nu(k) = \delta_\zeta\left[\mathrm{Tr}\{P_+\delta_\eta DD^{-1}\} + \mathrm{Tr}\{P_-\delta_\eta D'D'^{-1}\}\right]\Big|_{U(x,\mu)\to 1}$$

$$= \left[\mathrm{Tr}\{\delta_\zeta\delta_\eta DD^{-1}\} - \mathrm{Tr}\{\delta_\eta DD^{-1}\delta_\zeta DD^{-1}\}\right]\Big|_{U(x,\mu)\to 1}$$

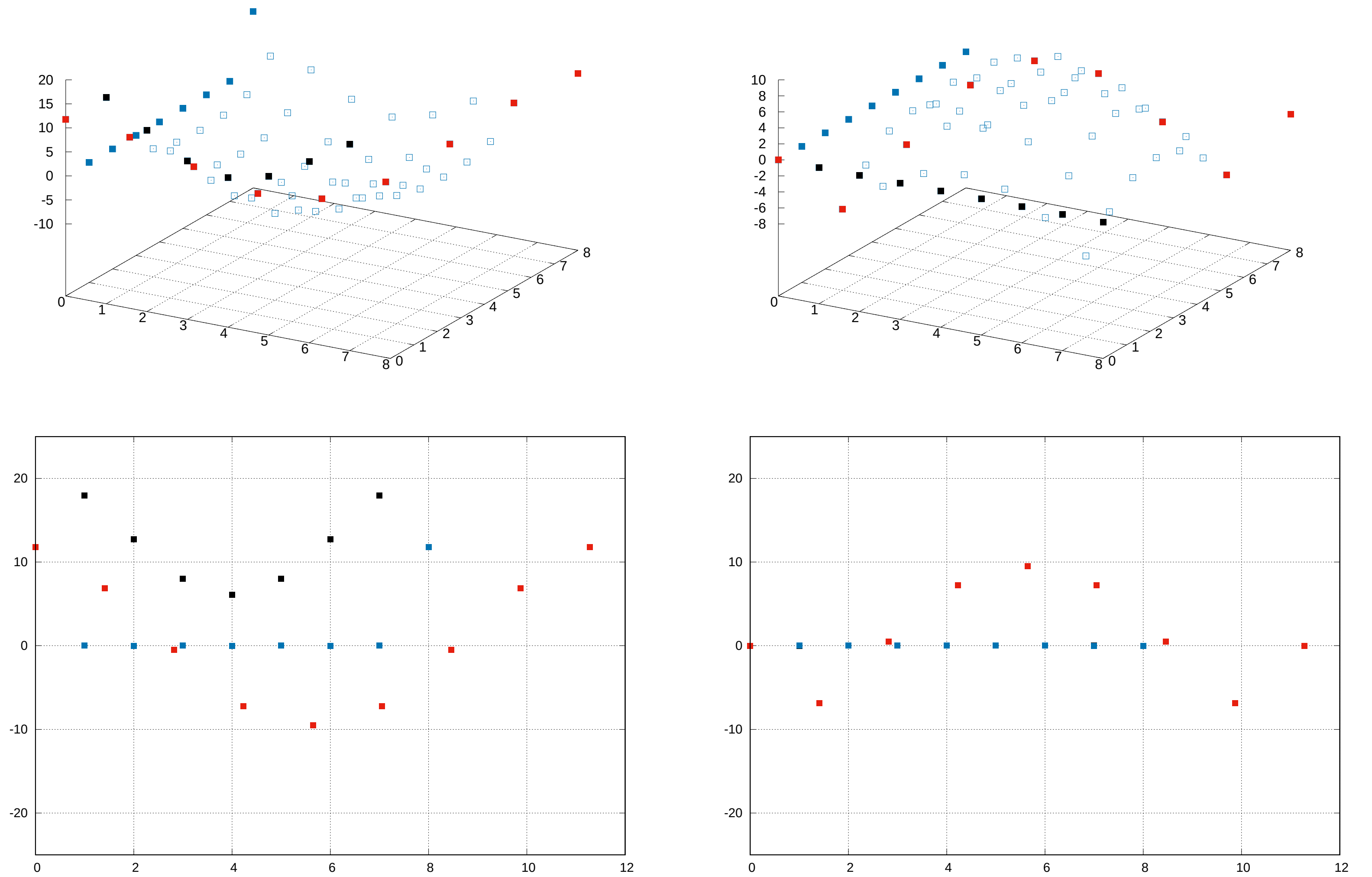


Figure 12. $(1/4)L^2\,\tilde{\Pi}_{00}(k)$ [left] and $(1/4)L^2\,\tilde{\Pi}_{01}(k)$ [right] vs. $|k|_2 \equiv \sqrt{k_0^2+k_1^2}$. The lattice size is $L=8$. The anti-periodic boundary condition is assumed for the fermion fields. The black-, blue-, red-symbol plots are along the spacial momentum axis ($k_0=0$), the temporal momentum axis ($k_1=0$) and the diagonal momentum axis ($k_0=k_1$), respectively.

$$\frac{1}{L^2}\sum_k \tilde{\eta}_\mu(-k)\,\tilde{\Pi}_{\mu\nu}(k)\,\tilde{\zeta}_\nu(k) = \delta_\zeta\left[\mathrm{Tr}\{P_+\delta_\eta D D^{-1}\} + \mathrm{Tr}\{P_-\delta_\eta D' D'^{-1}\}\right]\Big|_{U(x,\mu)\to 1}$$

$$= \left[\mathrm{Tr}\{\delta_\zeta\delta_\eta D D^{-1}\} - \mathrm{Tr}\{\delta_\eta D D^{-1}\delta_\zeta D D^{-1}\}\right]\Big|_{U(x,\mu)\to 1}$$

$$\tilde{\Pi}_{\mu\nu}(k) \simeq [4\times 1^2 + 4\times(-1)^2]\,\frac{1}{2\pi}\frac{\delta_{\mu\nu}k^2 - k_\mu k_\nu}{k^2}$$

$$\tilde{\Pi}_{00}(k) \simeq \frac{4}{\pi}\frac{k_1^2}{k_0^2+k_1^2} \longrightarrow \frac{4}{\pi}\times\begin{cases} 0 & k_\mu = (|k|,0) \\ 1/2 & k_\mu = (|k|,|k|)/\sqrt{2} \\ 1 & k_\mu = (0,|k|) \end{cases}$$

$$\tilde{\Pi}_{01}(k) \simeq \frac{4}{\pi}\frac{-k_0 k_1}{k_0^2+k_1^2} \longrightarrow \frac{4}{\pi}\times\begin{cases} 0 & k_\mu = (|k|,0) \\ -1/2 & k_\mu = (|k|,|k|)/\sqrt{2} \\ 0 & k_\mu = (0,|k|) \end{cases}$$

CP invariance of the effective action

$$
\begin{aligned}
U(x,\mu) &\longrightarrow U(x,\mu)^{CP} = \Big(U(x^P,0)^*, U(x^P-\hat{k},k)^{*\,-1}\Big), \\
\psi(x) &\longrightarrow \psi(x)^{CP} \quad = \quad +(\mathcal{P}\gamma_0)^{-1} C_D^{-1}\,\bar{\psi}(x)^T, \\
\bar{\psi}(x) &\longrightarrow \bar{\psi}(x)^{CP} \quad = \quad -\psi(x)^T\, C_D \mathcal{P}\gamma_0, \\
E^a(x) &\longrightarrow E^a(x)^{CP} \quad = \quad (-1)^a\, \bar{E}^a(x^P) \qquad (a = 0,1,\cdots,9).
\end{aligned}
$$

$$
\begin{aligned}
D[U^{CP}] &= (\mathcal{P}\gamma_0)^{-1} C_D^{-1} D[U]^T C_D\, \mathcal{P}\gamma_0 \\
&= (\mathcal{P}\gamma_0)^{-1} (\gamma_5 C_D)^{-1} D[U]^* \gamma_5 C_D\, \mathcal{P}\gamma_0
\end{aligned}
\qquad
\begin{aligned}
\hat{P}_\pm[U^{CP}] &= (\mathcal{P}\gamma_0)^{-1} (\gamma_5 C_D)^{-1} \hat{P}_\mp[U]^T \gamma_5 C_D \mathcal{P}\gamma_0 \\
P_\pm &= (\mathcal{P}\gamma_0)^{-1} (\gamma_5 C_D)^{-1} P_\mp^T \gamma_5 C_D \mathcal{P}\gamma_0.
\end{aligned}
$$

$$
S_{\mathrm{W}} = \sum_{x\in\Lambda} \bar{\psi}(x) P_+ D \psi(x) \quad \longrightarrow \quad S'_{\mathrm{W}} = \sum_{x\in\Lambda} \bar{\psi}(x) D P_- \psi(x)
$$

$$
\begin{aligned}
T_+(x) &= \frac{1}{2}\, V_+^a(x) V_+^a(x), \qquad V_+^a(x) = \psi^{\mathrm{T}}(x) \hat{P}_+^T i\gamma_5 C_D \mathrm{T}^a \hat{P}_+ \psi(x) \\
\longrightarrow T'_+(x) &= \frac{1}{2}\, {V'}_+^a(x) {V'}_+^a(x), \quad {V'}_+^a(x) = (-1)^a\, \bar{\psi}(x) \{\gamma_5 \hat{P}_- \gamma_5\} i\gamma_5 C_D \mathrm{T}^{a\dagger} \{\gamma_5 \hat{P}_- \gamma_5\}^T \bar{\psi}(x)^T
\end{aligned}
$$

$$
\begin{aligned}
\bar{T}_+(x) &= \frac{1}{2}\, \bar{V}_+^a(x) \bar{V}_+^a(x), \qquad \bar{V}_+^a(x) = \bar{\psi}(x) P_- i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a P_-^T \bar{\psi}(x)^T \\
\longrightarrow \bar{T}'_+(x) &= \frac{1}{2}\, \bar{V}_+^{\prime a}(x) \bar{V}'_+ a(x), \quad \bar{V}_+^{\prime a}(x) = (-1)^a\, \psi^{\mathrm{T}}(x) {P_+}^T i\gamma_5 C_D \mathrm{T}^a P_+ \psi(x)
\end{aligned}
$$

$$\begin{aligned}
\langle 1 \rangle_F [U] \quad &= \det(\bar{v} D v) \int \mathcal{D}[E] \, \mathrm{pf}(u^T i\gamma_5 C_D T^a E^a u) \\
\longrightarrow \langle 1 \rangle_F [U^{CP}] &= \det(u^\dagger \gamma_5 D \gamma_5 \bar{u}^\dagger) \int \mathcal{D}[\bar{E}] \, \mathrm{pf}(v^\dagger \gamma_5 i\gamma_5 C_D T^{a\dagger} \bar{E}^a \gamma_5 v^*) \\
&= \left\{ \det(\bar{u} D u) \int \mathcal{D}[E] \, \mathrm{pf}(v^T i\gamma_5 C_D T^a E^a v) \right\}^* .
\end{aligned}$$

$$\langle 1 \rangle_F [U^{CP}] = \langle 1 \rangle_F [U]$$

<=>

$$\det(\bar{v} D v) \int \mathcal{D}[E] \, \mathrm{pf}(u^T i\gamma_5 C_D T^a E^a u) = \left\{ \det(\bar{u} D u) \int \mathcal{D}[E] \, \mathrm{pf}(v^T i\gamma_5 C_D T^a E^a v) \right\}^*$$

$$\det(\bar{v}Dv)\ \int \mathcal{D}[E]\,\mathrm{pf}(u^T i\gamma_5 C_D T^a E^a u) = \Big\{\det(\bar{u}Du)\ \int \mathcal{D}[E]\,\mathrm{pf}(v^T i\gamma_5 C_D T^a E^a v)\Big\}^*$$

$$\begin{pmatrix} (\bar{u}u) & (\bar{u}v) \\ (\bar{v}u) & (\bar{v}v) \end{pmatrix},$$

$$\begin{pmatrix} \big(u^T i\gamma_5 C_D T^a E^a u\big) & \big(u^T i\gamma_5 C_D T^a E^a v\big) \\ \big(v^T i\gamma_5 C_D T^a E^a u\big) & \big(v^T i\gamma_5 C_D T^a E^a v\big) \end{pmatrix}$$

$$\begin{aligned} \det(\bar{v}Dv) &= \Big\{\det(\bar{u}Du)\Big\}^*, \\ \mathrm{pf}(u^T i\gamma_5 C_D T^a E^a u) &= \pm\Big\{\mathrm{pf}(v^T i\gamma_5 C_D T^a E^a v)\Big\}^* \end{aligned}$$

$$U = \begin{pmatrix} N & O \\ P & M \end{pmatrix} \qquad \begin{aligned} \det U &= \det N \times \det\big(M - PN^{-1}O\big) \\ &= \det N / \det M^\dagger. \end{aligned}$$

I hope it is not...

rice cake in a drawing

絵に描いた餅 in japanese

What is the sound of one hand clapping?

両手の鳴る音は知る。
片手の鳴る音はいかに？
ー 禅の公案 ー