Highly entangled quantum spin chains and their extensions by semigroups

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Workshop "Discrete Approaches to the Dynamics of Fields and Space-Time"
Tohoku University, September 10, 2018

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Bravyi et al, Phys. Rev. Lett. **118** (2012) 207202, arXiv: 1203.5801
R. Movassagh and P. Shor, Proc. Natl. Acad. Sci. **113** (2016) 13278, arXiv: 1408.1657
F.S. and P. Padmanabhan, J. Stat. Mech. **1801** (2018) 013101, arXiv: 1710.10426
P. Padmanabhan, F.S. and V. Korepin, arXiv: 1804.00978

Outline

Introduction 1

Quantum entanglement

- Most surprising feature of quantum mechanics,
  No analog in classical mechanics

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Quantum entanglement

- Most surprising feature of quantum mechanics,
  No analog in classical mechanics
- From pure state of the full system S: $\rho = |\psi\rangle\langle\psi|$, reduced density matrix of a subsystem A: $\rho_A = \mathrm{Tr}_{S-A}\,\rho$ can become mixed states, and has nonzero entanglement entropy

$$S_A = -\mathrm{Tr}_A\left[\rho_A \ln \rho_A\right].$$

This is purely a quantum property.

Introduction 2

Area law of entanglement entropy

- Ground states of quantum many-body systems with local interactions typically exhibit the area law behavior of the entanglement entropy: $S_A \propto (\text{area of } A)$
- Gapped systems in 1D are proven to obey the area law.
  [Hastings 2007]

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- For gapless case, $(1+1)$-dimensional CFT violates logarithmically: $S_A = \frac{c}{3} \ln(\text{volume of } A)$. [Calabrese, Cardy 2009]

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- Belief for gapless case in D-dim. (over two decades) : $S_A = O(L^{D-1} \ln L)$ (L: length scale of A)
- Recently, 1D solvable spin chain model which exhibit extensive entanglement entropy have been discussed.
  - Beyond logarithmic violation: $S_A \propto \sqrt{(\text{volume of } A)}$
    [Movassagh, Shor 2014], [Salberger, Korepin 2016]
    Counterexamples of the belief!

Motzkin spin model 1

[Bravyi et al 2012]

- 1D spin chain at sites $i \in \{1, 2, \cdots, 2n\}$
- Spin-1 state at each site can be regarded as up, down and flat steps;

$$|u\rangle \Leftrightarrow \nearrow, \qquad |d\rangle \Leftrightarrow \searrow, \qquad |0\rangle \Leftrightarrow \longrightarrow$$

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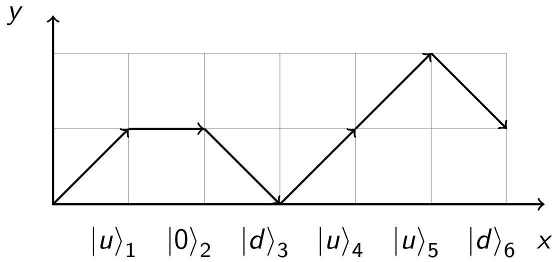
- Each spin configuration $\Leftrightarrow$ length-$2n$ walk in (x, y) plane
  Example)



Motzkin spin model 2

[Bravyi et al 2012]

Hamiltonian: $H_{Motzkin} = H_{bulk} + H_{bdy}$

- Bulk part: $H_{bulk} = \sum_{j=1}^{2n-1} \Pi_{j,j+1}$,

$$\Pi_{j,j+1} = |D\rangle_{j,j+1}\langle D| + |U\rangle_{j,j+1}\langle U| + |F\rangle_{j,j+1}\langle F|$$

(local interactions) with

$$|D\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, d\rangle - |d,\, 0\rangle\right),$$

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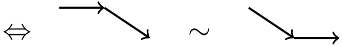

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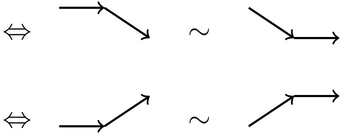

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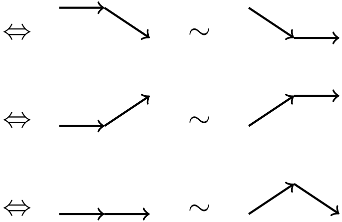

"gauge equivalence".

Motzkin spin model 3

[Bravyi et al 2012]

Hamiltonian: $H_{Motzkin} = H_{bulk} + H_{bdy}$

- Boundary part: $H_{bdy} = |d\rangle_1\langle d| + |u\rangle_{2n}\langle u|$

$$\Downarrow$$

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- $H_{Motzkin}$ is the sum of projection operators.

  $\Rightarrow$ Positive semi-definite spectrum
- We find the unique zero-energy ground state.

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  $\Rightarrow$ Positive semi-definite spectrum
- We find the unique zero-energy ground state.
  - Each projector in $H_{Motzkin}$ annihilates the zero-energy state.

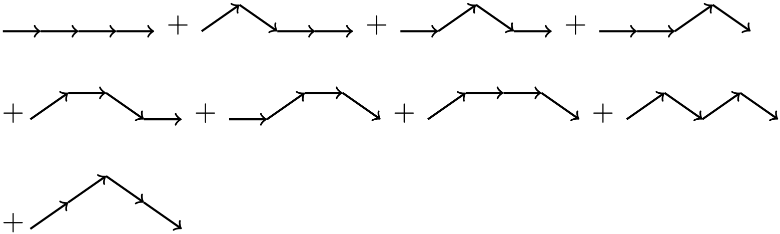
    $\Rightarrow$ Frustration free
- The ground state corresponds to randoms walks starting at $(0,0)$ and ending at $(2n,0)$ restricted to the region $y \geq 0$ (Motzkin Walks (MWs)).

Motzkin spin model 4

[Bravyi et al 2012]

Example) $2n = 4$ case,

MWs:



$$\Updownarrow$$

Ground state:

$$\begin{aligned}|P_4\rangle \;\; = \;\; \frac{1}{\sqrt{9}} & \left[|0000\rangle + |ud00\rangle + |0ud0\rangle + |00ud\rangle\right. \\ & + |u0d0\rangle + |0u0d\rangle + |u00d\rangle + |udud\rangle \\ & \left. + |uudd\rangle\right].\end{aligned}$$

Motzkin spin model 5

[Bravyi et al 2012]

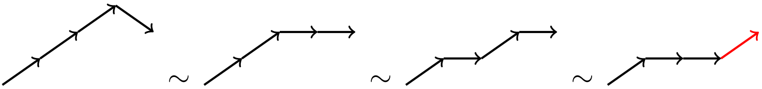
Note

Forbidden paths for the ground state

1. Path entering $y < 0$ region

Forbidden by H_{bdy}

2. Path ending at nonzero height



Forbidden by H_{bdy}

Motzkin spin model 6

[Bravyi et al 2012]

Entanglement entropy of the subsystem $A = \{1, 2, \cdots, n\}$:

- Normalization factor of the ground state $|P_{2n}\rangle$ is given by the number of MWs of length $2n$: $M_{2n} = \sum_{k=0}^{n} C_k \binom{2n}{2k}$.

$C_k = \frac{1}{k+1}\binom{2k}{k}$: Catalan number

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- Consider to trace out the density matrix $\rho = |P_{2n}\rangle\langle P_{2n}|$ w.r.t. the subsystem $B = \{n+1, \cdots, 2n\}$.
  Schmidt decomposition:

$$|P_{2n}\rangle = \sum_{h\geq 0} \sqrt{p_{n,n}^{(h)}} \left| P_n^{(0\to h)} \right\rangle \otimes \left| P_n^{(h\to 0)} \right\rangle$$

with $p_{n,n}^{(h)} \equiv \frac{\left(M_n^{(h)}\right)^2}{M_{2n}}$.

↑
Paths from $(0,0)$ to (n,h)

Motzkin spin model 7

[Bravyi et al 2012]

- $M_n^{(h)}$ is the number of paths in $P_n^{(0\to h)}$.
  For $n \to \infty$, Gaussian distribution

$$p_{n,n}^{(h)} \sim \frac{3\sqrt{6}}{\sqrt{\pi}} \frac{(h+1)^2}{n^{3/2}} e^{-\frac{3}{2}\frac{(h+1)^2}{n}} \times [1 + O(1/n)] \,.$$

- Reduced density matrix

$$\rho_A = \mathrm{Tr}\,_B \rho = \sum_{h\geq 0} p_{n,n}^{(h)} \left| P_n^{(0\to h)} \right\rangle \left\langle P_n^{(0\to h)} \right|$$

- Entanglement entropy

$$\begin{aligned} S_A &= -\sum_{h\geq 0} p_{n,n}^{(h)} \ln p_{n,n}^{(h)} \\ &= \frac{1}{2}\ln n + \frac{1}{2}\ln\frac{2\pi}{3} + \gamma - \frac{1}{2} \qquad (\gamma\text{: Euler constant}) \end{aligned}$$

up to terms vanishing as $n \to \infty$.

Motzkin spin model 8

[Bravyi et al 2012]

Notes

- The system is critical (gapless).
  S_A is similar to the $(1+1)$-dimensional CFT with $c = 3/2$.

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  The system cannot be described by relativistic CFT.

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- The system is critical (gapless).
  S_A is similar to the $(1+1)$-dimensional CFT with $c = 3/2$.
- But, gap scales as $O(1/n^z)$ with $z \geq 2$.
  The system cannot be described by relativistic CFT.
- Excitations have not been much investigated.

Colored Motzkin spin model 1

[Movassagh, Shor 2014]

- Introducing color d.o.f. $k = 1, 2, \cdots, s$ to up and down spins as

$$\left|u^k\right\rangle \Leftrightarrow \overset{k}{\nearrow}, \qquad \left|d^k\right\rangle \Leftrightarrow \overset{k}{\searrow}, \qquad |0\rangle \Leftrightarrow \longrightarrow$$

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Color d.o.f. decorated to Motzkin Walks

- Hamiltonian $H_{cMotzkin} = H_{bulk} + H_{bdy}$
  - Bulk part consisting of local interactions:

$$H_{bulk} = \sum_{j=1}^{2n-1} \left(\Pi_{j,j+1} + \Pi^{cross}_{j,j+1}\right),$$

$$\Pi_{j,j+1} = \sum_{k=1}^{s} \left[\left|D^k\right\rangle_{j,j+1}\left\langle D^k\right| + \left|U^k\right\rangle_{j,j+1}\left\langle U^k\right| + \left|F^k\right\rangle_{j,j+1}\left\langle F^k\right|\right]$$

with

Colored Motzkin spin model 2

[Movassagh, Shor 2014]

$$\left|D^k\right\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|0,\, d^k\right\rangle - \left|d^k,\, 0\right\rangle\right),$$

$$\left|U^k\right\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|0,\, u^k\right\rangle - \left|u^k,\, 0\right\rangle\right),$$

$$\left|F^k\right\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|0,\, 0\right\rangle - \left|u^k,\, d^k\right\rangle\right),$$

and

$$\Pi^{cross}_{j,j+1} = \sum_{k \neq k'} \left|u^k,\, d^{k'}\right\rangle_{j,j+1}\left\langle u^k,\, d^{k'}\right|.$$

$\Rightarrow$ Colors should be matched in up and down pairs.

- Boundary part

$$H_{bdy} = \sum_{k=1}^{s}\left(\left|d^k\right\rangle_1\left\langle d^k\right| + \left|u^k\right\rangle_{2n}\left\langle u^k\right|\right).$$

Colored Motzkin spin model 3

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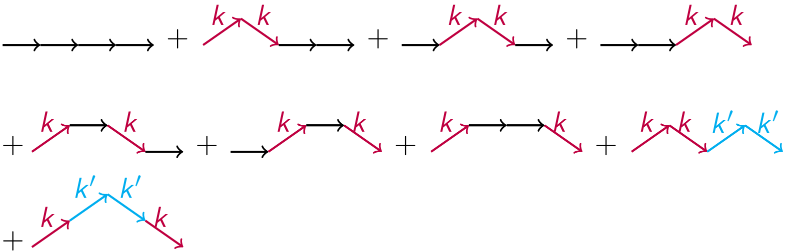


$$|P_4\rangle = \frac{1}{\sqrt{1+6s+2s^2}}\Bigg[|0000\rangle + \sum_{k=1}^{s}\left\{\left|u^k d^k 00\right\rangle + \cdots + \left|u^k 00 d^k\right\rangle\right\} + \sum_{k,k'=1}^{s}\left\{\left|u^k d^k u^{k'} d^{k'}\right\rangle + \left|u^k u^{k'} d^{k'} d^k\right\rangle\right\}\Bigg].$$

Colored Motzkin spin model 4

[Movassagh, Shor 2014]

Entanglement entropy

- Paths from $(0,0)$ to (n,h), $P_n^{(0\to h)}$, have h unmatched up steps.
  Let $\tilde{P}_n^{(0\to h)}(\{\kappa_m\})$ be paths with the colors of unmatched up steps frozen.
  (unmatched up from height $(m-1)$ to m) $\to u^{\kappa_m}$
- Similarly,

$$P_n^{(h\to 0)} \to \tilde{P}_n^{(h\to 0)}(\{\kappa_m\}),$$

  (unmatched down from height m to $(m-1)$) $\to d^{\kappa_m}$.

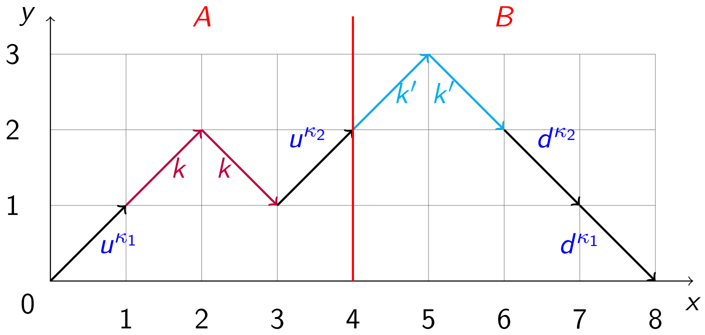
- The numbers satisfy $M_n^{(h)} = s^h\, \tilde{M}_n^{(h)}$.

Colored Motzkin spin model 5

[Movassagh, Shor 2014]

Example

$2n = 8$ case, $h = 2$



Colored Motzkin spin model 6

[Movassagh, Shor 2014]

- Schmidt decomposition

$$
\begin{aligned}
|P_{2n}\rangle \quad = \quad & \sum_{h\geq 0}\sum_{\kappa_1=1}^{s}\cdots\sum_{\kappa_h=1}^{s}\sqrt{p_{n,n}^{(h)}} \\
& \times \left|\tilde{P}_n^{(0\to h)}(\{\kappa_m\})\right\rangle \otimes \left|\tilde{P}_n^{(h\to 0)}(\{\kappa_m\})\right\rangle
\end{aligned}
$$

with

$$
p_{n,n}^{(h)} = \frac{\left(\tilde{M}_n^{(h)}\right)^2}{M_{2n}}.
$$

- Reduced density matrix

$$
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\rho_A \quad = \quad & \sum_{h\geq 0}\sum_{\kappa_1=1}^{s}\cdots\sum_{\kappa_h=1}^{s} p_{n,n}^{(h)} \\
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\end{aligned}
$$

Colored Motzkin spin model 7

[Movassagh, Shor 2014]

- For $n \to \infty$,

$$p_{n,n}^{(h)} \sim \frac{\sqrt{2}\, s^{-h}}{\sqrt{\pi}\,(\sigma\, n)^{3/2}} (h+1)^2\, e^{-\frac{(h+1)^2}{2\sigma n}} \times [1 + O(1/n)]$$

with $\sigma \equiv \frac{\sqrt{s}}{2\sqrt{s}+1}$.

Note: Effectively $h \lesssim O(\sqrt{n})$.

- Entanglement entropy

$$S_A \;=\; -\sum_{h\geq 0} s^h\, p_{n,n}^{(h)} \ln p_{n,n}^{(h)}$$

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up to terms vanishing as $n \to \infty$.

Grows as $\sqrt{n}$.

Colored Motzkin spin model 8

[Movassagh, Shor 2014]

Comments

- Matching color $\Rightarrow$ s^{-h} factor in $p_{n,n}^{(h)}$
  $\Rightarrow$ crucial to $O(\sqrt{n})$ behavior in S_A

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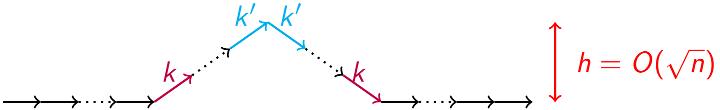
- Typical configurations:



$+$ (equivalence moves).

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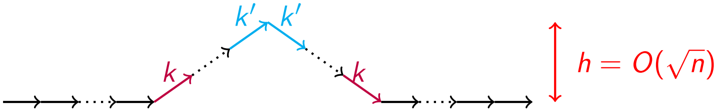

$+$ (equivalence moves).

- For spin 1/2 chain (only up and down), the model in which similar behavior exhibits in colored as well as uncolored cases has been constructed. (Fredkin model) [Salberger, Korepin 2016]

Colored Motzkin spin model 9

[Movassagh, Shor 2014]

- Correlation functions [Dell'Anna et al, 2016]

$$\langle S_{z,1} S_{z,2n} \rangle_{\text{connected}} \to -0.034... \times \frac{s^3 - s}{6} \neq 0 \qquad (n \to \infty)$$

$\Rightarrow$ Violation of cluster decomposition property for $s > 1$

Colored Motzkin spin model 9

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  due to highly entangled ground state even though the system has local interactions

- Deformation of models to achieve the volume law behavior ($S_A \propto n$)
  Weighted Motzkin/Dyck walks [Zhang et al, Salberger et al 2016]

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  Semigroup version of the symmetric group S_k
  $\mathcal{S}_p^k \ (p = 1, \cdots, k)$
- $x_{a,b} \in \mathcal{S}_1^k$ maps a to b. $(a, b \in \{1, \cdots, k\})$
  Product rule: $x_{a,b} * x_{c,d} = \delta_{b,c}\, x_{a,d}$

$$x_{1,2} * x_{2,1} = x_{1,1}, \qquad x_{2,1} * x_{1,2} = x_{2,2}$$

(partial identities)

$(x_{1,2})^{-1} = x_{2,1}$ (unique inverse)

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  $\mathcal{S}_p^k$ $(p = 1, \cdots, k)$
- $x_{a,b} \in \mathcal{S}_1^k$ maps a to b. $(a, b \in \{1, \cdots, k\})$
  Product rule: $x_{a,b} * x_{c,d} = \delta_{b,c}\, x_{a,d}$

$$x_{1,2} * x_{2,1} = x_{1,1}, \qquad x_{2,1} * x_{1,2} = x_{2,2}$$

(partial identities)

$(x_{1,2})^{-1} = x_{2,1}$ (unique inverse)

- $x_{a_1,a_2;\, b_1,b_2} \in \mathcal{S}_2^k$ etc, ...

Symmetric Inverse Semigroups (SISs)

- Inverse Semigroup ($\subset$ Semigroup):
  An unique inverse exists for every element.
  But, no unique identity (partial identities).
- SIS ($\subset$ Semigroup):
  Semigroup version of the symmetric group S_k
  $\mathcal{S}_p^k \ (p = 1, \cdots, k)$
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(unique inverse)

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SIS Motzkin model 1

[Sugino, Padmanabhan 2017]

- Change the spin d.o.f. as $|x_{a,b}\rangle$ with $a, b \in \{1, 2, \cdots, k\}$.

- $a < b$ case: 'up' $\Leftrightarrow$ ${}^{a}\nearrow^{b}$

  $a > b$ case: 'down' $\Leftrightarrow$ ${}^{a}\searrow^{b}$

  $a = b$ case: 'flat' $\Leftrightarrow$ $\overset{a\quad b}{\longrightarrow}$

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  $a > b$ case: ‘down’ $\Leftrightarrow$ ${}^{a}\!\searrow_{b}$

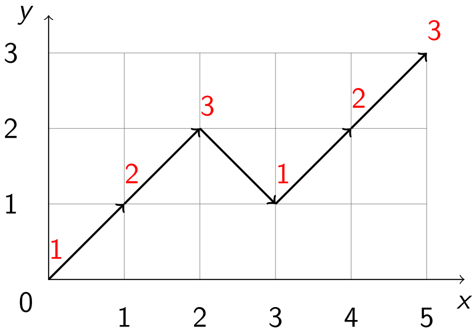
  $a = b$ case: ‘flat’ $\Leftrightarrow$ $\xrightarrow{a\quad b}$
- We regard the configuration of adjacent sites $|(x_{a,b})_j\rangle\,|(x_{c,d})_{j+1}\rangle$ as a connected path for $b = c$.
  c.f.) Analogous to the product rule of Symmetric Inverse Semigroup ($\mathcal{S}_1^k$): $x_{a,b} * x_{c,d} = \delta_{b,c}\, x_{a,d}$

  a, b: semigroup indices
- Inner product: $\langle x_{a,b}|x_{c,d}\rangle = \delta_{a,c}\delta_{b,d}$
- Let us consider the $k = 3$ case.

SIS Motzkin model 2

[Sugino, Padmanabhan 2017]

- Maximum height is lower than the original Motzkin case.

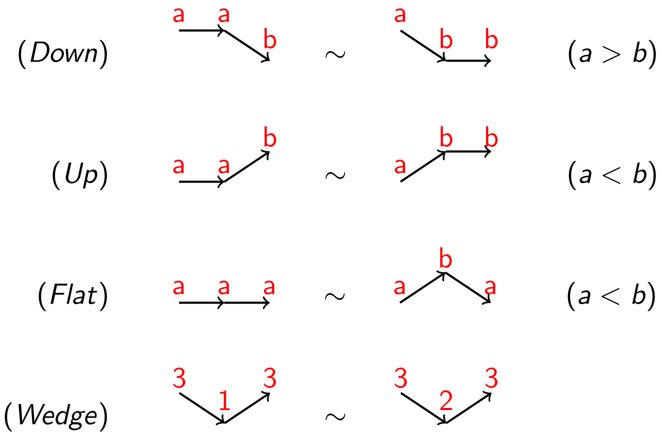


SIS Motzkin model 3

[Sugino, Padmanabhan 2017]

Hamiltonian $H_{S31Motzkin} = H_{bulk} + H_{bulk,disc} + H_{bdy}$

- H_{bulk}: local interactions corresponding to the following moves:



SIS Motzkin model 4

[Sugino, Padmanabhan 2017]

- $H_{bulk,disc}$ lifts disconnected paths to excited states.

$\Pi^{|\psi\rangle}$: projector to $|\psi\rangle$

$$H_{bulk,disc} = \sum_{j=1}^{2n-1} \sum_{a,b,c,d=1;b\neq c}^{3} \Pi^{\left|(x_{a,b})_j,(x_{c,d})_{j+1}\right\rangle}$$

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-

$$\begin{aligned} H_{bdy} &= \sum_{a>b} \Pi^{\left|(x_{a,b})_1\right\rangle} + \sum_{a<b} \Pi^{\left|(x_{a,b})_{2n}\right\rangle} \\ &\quad + \Pi^{|(x_{1,3})_1,(x_{3,2})_2,(x_{2,1})_3\rangle} + \Pi^{|(x_{1,2})_{2n-2},(x_{2,3})_{2n-1},(x_{3,1})_{2n}\rangle} \end{aligned}$$

The last 2 terms have no analog to the original Motzkin model.

SIS Motzkin model 5

[Sugino, Padmanabhan 2017]

- Ground states correspond to connected paths starting at $(0, 0)$, ending at $(2n, 0)$ and not entering $y < 0$. $\mathcal{S}_1^3$ MWs

SIS Motzkin model 5

[Sugino, Padmanabhan 2017]

- Ground states correspond to connected paths starting at $(0,0)$, ending at $(2n,0)$ and not entering $y<0$. $\mathcal{S}_1^3$ MWs
- The ground states have 5 fold degeneracy according to the initial and finial semigroup indices:
  $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$ and $(3,3)$ sectors
  The $(3,3)$ sector is trivial, consisting of only one path:

$$x_{3,3}x_{3,3}\cdots x_{3,3}.$$

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  $x_{3,3}x_{3,3}\cdots x_{3,3}$.
- The number of paths can be obtained by recursion relations. For length-n paths from the semigroup index a to b ($P_{n,a\to b}$),

$$\begin{aligned} P_{n,1\to 1} &= x_{1,1}P_{n-1,1\to 1} + x_{1,2}\sum_{i=1}^{n-2} P_{i,2\to 2}\, x_{2,1}P_{n-2-i,1\to 1} \\ &\quad + x_{1,3}\sum_{i=1}^{n-2} P_{i,3\to 3}\, x_{3,1}P_{n-2-i,1\to 1} \\ &\quad + x_{1,3}\sum_{i=1}^{n-2} P_{i,3\to 3}\, x_{3,2}P_{n-2-i,2\to 1}, \qquad \text{etc.} \end{aligned}$$

SIS Motzkin model 6

[Sugino, Padmanabhan 2017]

Result

- The entanglement entropies $S_{A,1\to1}$, $S_{A,1\to2}$, $S_{A,2\to1}$ and $S_{A,2\to2}$ take the same form as in the case of the Motzkin model.

  Logarithmic violation of the area law

  - The form of $p_n^{(h)} \sim \frac{(h+1)^2}{n^{3/2}}\, e^{-(\text{const.})\frac{(h+1)^2}{n}}$ is universal.

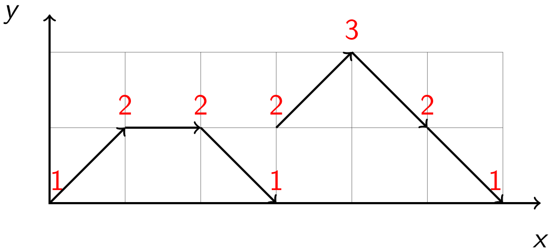
- $S_{A,3\to3} = 0$.

SIS Motzkin model 7

Localization

[Padmanabhan, F.S., Korepin 2018]

- There are excited states corresponding to disconnected paths.
  Example) One such path in $2n = 6$ case,



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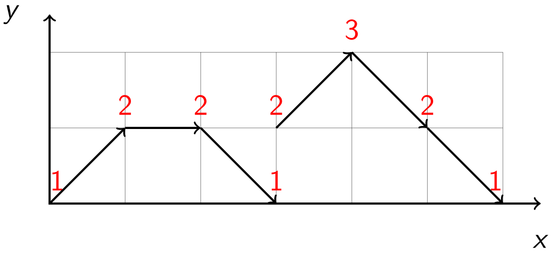

Corresponding excited state: $\left|P_{3,\,1\to1}\right\rangle \otimes \left|P^{(1\to0)}_{3,\,2\to1}\right\rangle$

Each connected component has no entanglement with other components.

"2nd quantization" of paths

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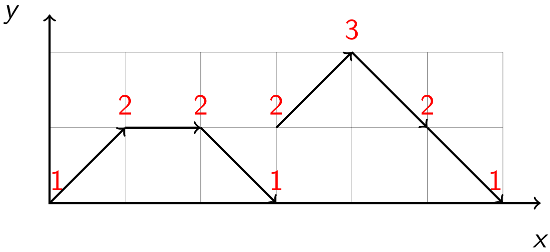

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Each connected component has no entanglement with other components.

"2nd quantization" of paths

$\Rightarrow$ 2pt connected correlation functions of local operators belonging to separate connected components vanish.

$\Rightarrow$ Localization!

Colored SIS Motzkin model 1

[Sugino, Padmanabhan 2017]

The SIS $\mathcal{S}_2^3$

- 18 elements $x_{ab,cd}$ with $ab \in \{12, 23, 31\}$ and $cd \in \{12, 23, 31, 21, 32, 13\}$ satisfying

$$x_{ab,cd} * x_{ef,gh} = \delta_{c,e}\delta_{d,f}\, x_{ab,gh} + \delta_{c,f}\delta_{d,e}\, x_{ab,hg}\,.$$

- can be regarded as 2 sets of $\mathcal{S}_1^3$. $\Rightarrow$ color d.o.f.

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- can be regarded as 2 sets of $\mathcal{S}_1^3$. $\Rightarrow$ color d.o.f.
- Spin variables: $x^s_{a,b}$ $(s = 1, 2)$ $(a, b = 1, 2, 3)$
- The new moves (C moves) introduced to the Hamiltonian.

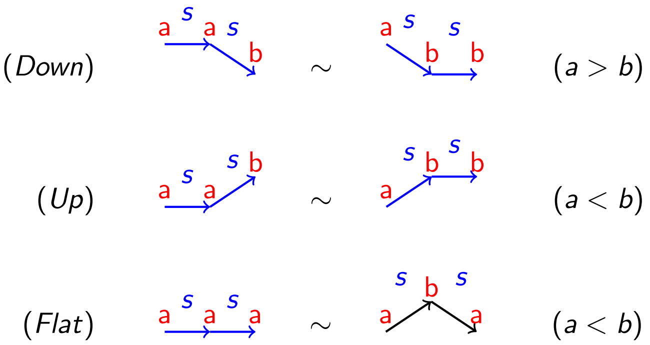
$$\mathrm{a} \xrightarrow{1} \mathrm{a} \;\sim\; \mathrm{a} \xrightarrow{2} \mathrm{a}$$

Colored SIS Motzkin model 2

[Sugino, Padmanabhan 2017]

Hamiltonian: $H_{cS31Motzkin} = H_{bulk} + H_{bulk,disc} + H_{bdy}$

- In H_{bulk}, (Down), (Up) and (Flat) are essentially the same as before.



Colored SIS Motzkin model 3

[Sugino, Padmanabhan 2017]

- Wedge move:

$(Wedge)$

- $$(Cross)_{j,j+1} = \sum_{b>a,c} \left[\Pi^{\left|(x^1_{a,b})_j,(x^2_{b,c})_{j+1}\right\rangle} + \Pi^{\left|(x^2_{a,b})_j,(x^1_{b,c})_{j+1}\right\rangle} \right]$$

  forbids unmatched up and down steps in ground states.

$$\Downarrow$$

$$\begin{aligned} H_{bulk} &= \mu \sum_{j=1}^{2n} C_j + \sum_{j=1}^{2n-1} \left[(Down)_{j,j+1} + (Up)_{j,j+1} \right. \\ &\quad \left. + (Flat)_{j,j+1} + (Wedge)_{j,j+1} + (Cross)_{j,j+1} \right] \end{aligned}$$

Colored SIS Motzkin model 4

[Sugino, Padmanabhan 2017]

- $$H_{bulk,disc} = \sum_{j=1}^{2n-1} \sum_{a,b,c,d=1;b\neq c}^{3} \sum_{s,t=1}^{2} \Pi^{\left|(x^s_{a,b})_j,(x^t_{c,d})_{j+1}\right\rangle}$$
- $$\begin{aligned} H_{bdy} &= \sum_{a>b} \sum_{s=1}^{2} \Pi^{\left|(x^s_{a,b})_1\right\rangle} + \sum_{a<b} \sum_{s=1}^{2} \Pi^{\left|(x^s_{a,b})_{2n}\right\rangle} \\ &+ \sum_{s,t=1}^{2} \Pi^{\left|(x^s_{1,3})_1,(x^s_{3,2})_2,(x^t_{2,1})_3\right\rangle} \\ &+ \sum_{s,t=1}^{2} \Pi^{\left|(x^s_{1,2})_{2n-2},(x^t_{2,3})_{2n-1},(x^t_{3,1})_{2n}\right\rangle} \end{aligned}$$

Colored SIS Motzkin model 5

[Sugino, Padmanabhan 2017]

- 5 ground states of $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, $(3,3)$ sectors
- Quantum phase transition between $\mu > 0$ and $\mu = 0$ in the 4 sectors except $(3,3)$.
  - For $\mu > 0$,

$$S_A = (2\ln 2)\sqrt{\frac{2\sigma n}{\pi}} + \frac{1}{2}\ln n + \frac{1}{2}\ln(2\pi\sigma) + \gamma - \frac{1}{2} + \ln\frac{3}{2^{1/3}}$$

    with $\sigma \equiv \frac{\sqrt{2}-1}{9\sqrt{2}}$.
  - For $\mu = 0$, colors 1 and 2 decouple.

$$S_A \propto \ln n.$$

Summary and discussion 1

Summary

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- We have extended the models by introducing additional d.o.f. based on Symmetric Inverse Semigroups.
  - Quantum phase transitions
    In colored case ($\mathcal{S}_2^3$), $\sqrt{n}$ v.s. $\ln n$ for S_A
    (In uncolored case ($\mathcal{S}_1^3$), log. violation v.s. area law $O(1)$ for S_A)

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- Semigroup extension of the Fredkin model

  [Padmanabhan, F.S., Korepin 2018]
- As a feature of the extended models,
  Anderson-like localization occurs in excited states corresponding to disconnected paths.
  - "2nd quantized paths".

Summary and discussion 2

Future directions

- Continuum limit? Field theory interpretation?

  (In particular, for colored case)

  [Chen, Fradkin, Witczak-Krempa 2017]

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Thank you very much for your attention!

App. Motzkin spin model

[Bravyi et al 2012]

In terms of $S = 1$ spin matrices

$$S_z = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}, \quad S_\pm \equiv \frac{1}{\sqrt{2}}(S_x \pm i S_y) = \begin{pmatrix} & 1 & \\ & & 1 \\ & & \end{pmatrix}, \begin{pmatrix} & & \\ 1 & & \\ & 1 & \end{pmatrix},$$

$$H_{bulk} = \frac{1}{2}\sum_{j=1}^{2n-1}\Big[1_j 1_{j+1} - \frac{1}{4}S_{z\,j}S_{z\,j+1} - \frac{1}{4}S_{z\,j}^2 S_{z\,j+1} + \frac{1}{4}S_{z\,j}S_{z\,j+1}^2$$

$$-\frac{3}{4}S_{z\,j}^2 S_{z\,j+1}^2 + S_{+\,j}\left(S_z S_-\right)_{j+1} + S_{-\,j}\left(S_+ S_z\right)_{j+1} - \left(S_- S_z\right)_j S_{+\,j+1}$$

$$-\left(S_z S_+\right)_j S_{-\,j+1} - \left(S_- S_z\right)_j\left(S_+ S_z\right)_{j+1} - \left(S_z S_+\right)_j\left(S_z S_-\right)_{j+1}\Big],$$

$$H_{bdy} = \frac{1}{2}\left(S_z^2 - S_z\right)_1 + \frac{1}{2}\left(S_z^2 + S_z\right)_{2n}$$

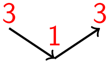
Quartic spin interactions

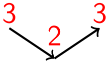
App. SIS Motzkin model

[Sugino, Padmanabhan 2017]

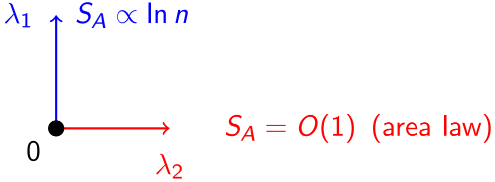
- By adding the balancing term to the Hamiltonian

$$\lambda_2 \sum_{j=1}^{2n-1} \left[\Pi^{\left|(x_{1,3})_j,(x_{3,2})_{j+1}\right\rangle} + \Pi^{\left|(x_{2,3})_j,(x_{3,1})_{j+1}\right\rangle} \right]$$

with λ_1 put to the term 

$\sim$ 

, quantum phase transition takes place in the 4 sectors except $(3,3)$:



$\lambda_1, \lambda_2 > 0$ is not frustration free (here, we do not consider).