Janossy densities

for chiral random matrices

and multi-flavor 2-color QCD

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SMN   PoS LATTICE2015, 057 = 1606.00276 [hep-lat]

H. Fuji, I. Kanamori, SMN   to be submitted to JHEP

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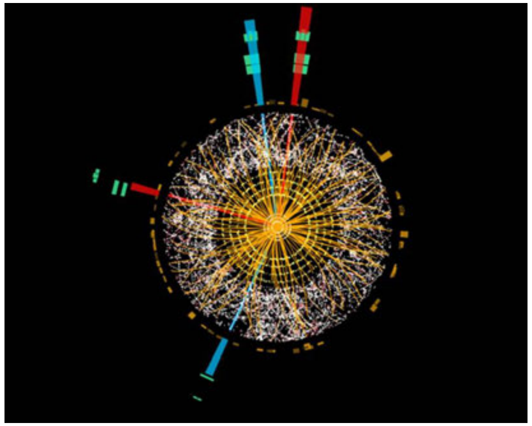
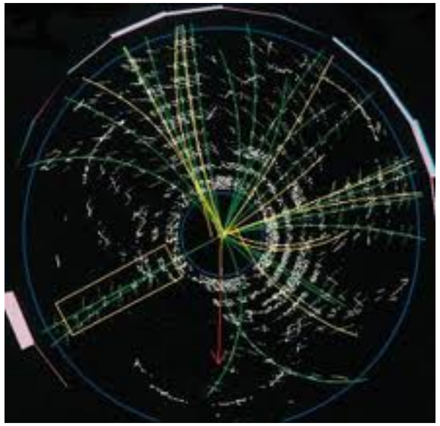
Hierarchy problem

Higgs boson M=125GeV



Top quark m=173GeV

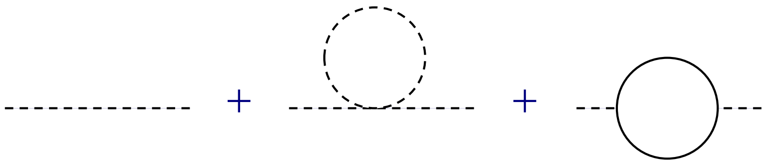


radiative correction of masses

$$M^2 = M_0^2 + \# \Lambda^2 + \# m_0^2 \log \frac{\Lambda}{m_0} + \cdots$$

$$m = m_0 + \# m_0 \log \frac{\Lambda}{m_0} + \cdots$$

Weisskopf '34



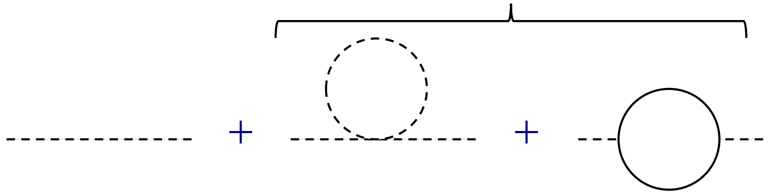
extremely-fine tuning needed to account for M_H=125GeV

Hierarchy problem

Solution 1: SUSY $\quad (H\ ,\ \psi_H)$ degenerate m_0

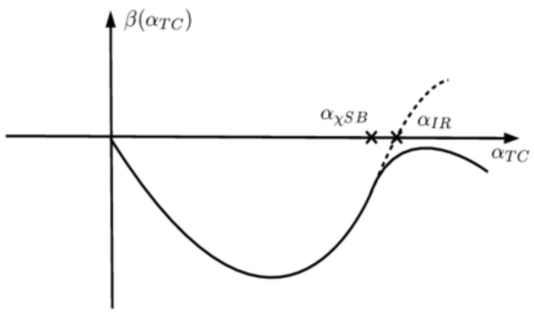
$$m^2 = m_0^2 + \underset{\uparrow}{0} + \# m_0^2 \log\frac{\Lambda}{m_0} + \cdots \qquad\qquad m = m_0 + \# m_0 \log\frac{\Lambda}{m_0} + \cdots$$

$\mathrm{O}(\Lambda^2)$ cancelled

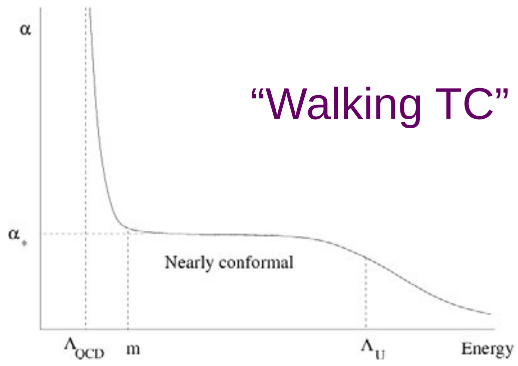


Solution 2: Higgs is ~~fundamental~~ ⇨ composite in (new) gauge theory w/o scalar

$$H \sim \bar{t}\, t \ \ \text{(top condensation)}\ \ \text{or}\ \ \bar{Q}\, Q \ \ \text{(technicolor)}$$



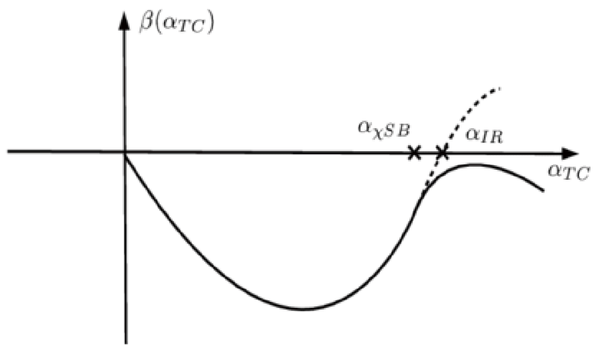
⇨



"Walking TC"

Hierarchy problem

Solution 2: Higgs is ~~fundamental~~ ⇨ composite in (new) gauge theory w/o scalar



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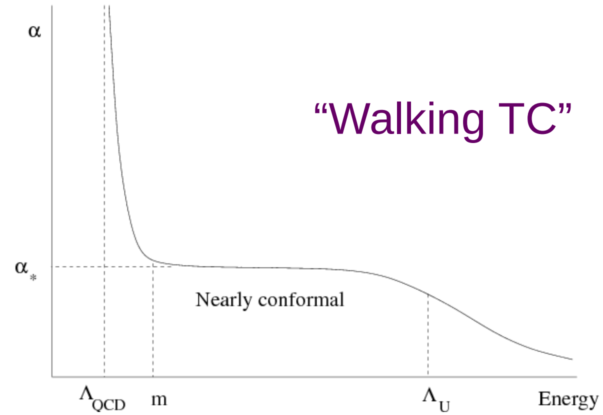

"Walking TC"

$$\beta(\alpha) = -\underbrace{\left(\frac{11}{3}N - \frac{2}{3}n_F - \frac{4}{3}N\, n_{Ad}\right)}_{\text{small but >0}}\alpha^2 - \underbrace{\left(\frac{34}{3}N^2 - \left(\frac{13}{3}N - \frac{1}{N}\right)n_F - \frac{32}{3}N^2 n_{Ad}\right)}_{<0}\alpha^3 - \cdots$$

- $N = 3\,,\; n_F = 12$ KMI group; Kuti; Hasenfratz
- $N = 2\,,\; n_{Ad} = 2$ "Minimal WTC" Catterall-Sannino
- $N = 2\,,\; n_F = 8$ NCTU group, Helsinki group...

χSB or (near-)Conformality tested for 2C QCD on Lattices

Chiral Random Matrices

[Shuryak-Verbaarschot '93]

$$\langle \cdots \rangle_{\text{CHIRAL GAUSSIAN } H} = \frac{1}{Z(m)} \int dH \; \mathrm{e}^{-\mathrm{tr}\, H^2} \prod_f \det\left(H + i m_f\right) \cdots \quad , \quad H = \begin{array}{cc} & \overbrace{}^{N+\nu} \\ \left[\begin{array}{c|c} 0 & M \\ \hline M^+ & 0 \end{array}\right] & \end{array} \Big\} N$$

HS transf.

$N \to \infty \;,\; m_f \to 0$

$\mu_f = N m_f \;$ fixed

captures all
Global & Discrete
symmetries of QCD

$$M_{ab} \in \begin{cases} \mathbf{R} \text{ (chGOE)} & \beta=1 \\ \mathbf{C} \text{ (chGOE)} & \beta=2 \\ \mathbf{H} \text{ (chGSE)} & \beta=4 \end{cases}$$

0D reduction
of chPT

$$\int_{U(N_F)} dU \; \left(\det U\right)^{\nu} \exp\left\{\mathrm{Re}\,\mathrm{tr}\,\mathrm{diag}(\mu_f) U\right\} = \mathrm{cst.} \frac{\det\left[\mu_i^{j-1} I_{\nu+j-i}(\mu_i)\right]_{i,j=1}^{N_F}}{\Delta(\mu_f^2)}$$

for β=2 [Brower-Rossi-Tan '81]

similar forms with Pf ~ qdet for β=1,4 [Smilga-Verbaarschot '95, Nagao-SMN '00]

analytically solvable, symmetry-based model of QCD in **χSB phase**

Chiral Random Matrices

[Shuryak-Verbaarschot '93]

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HS transf.

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$$M_{ab} \in \begin{cases} \mathbf{R}\ (\text{chGOE}) & \beta=1 \\ \mathbf{C}\ (\text{chGOE}) & \beta=2 \\ \mathbf{H}\ (\text{chGSE}) & \beta=4 \end{cases}$$

$$\int_{U(N_F)} dU \; \left(\det U\right)^{\nu} \exp\left\{\mathrm{Re}\,\mathrm{tr}\,\mathrm{diag}(\mu_f)U\right\} = \mathrm{cst.} \frac{\det\left[\mu_i^{j-1} I_{\nu+j-i}(\mu_i)\right]_{i,j=1}^{N_F}}{\Delta(\mu_f^2)}$$

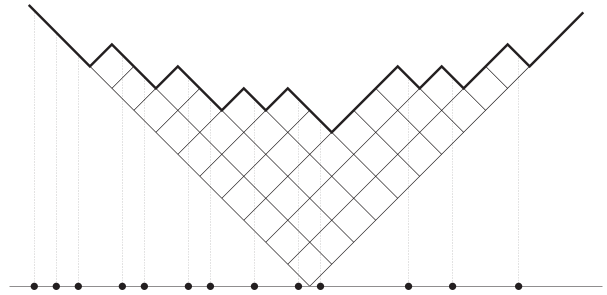
$\updownarrow$

chPT: $V_4 \Sigma\, \mathrm{Re}\,\mathrm{tr}\,\mathrm{diag}(m_f)\, U$

if QCD is in **χSB phase**,

- Dirac EVDs on various V_4 collapse onto chRM result
- can determine Σ by fitting

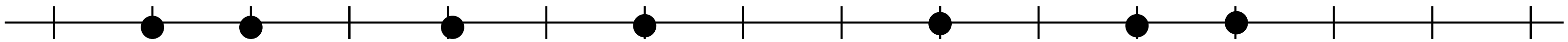
Determinantal point process



- Plancherel measure on {YT}
- directed percolation
- continuous ⇨ invariant RMEs

$$\mathrm{Prob}_N(n_1,\ldots,n_N)=\frac{1}{N!}\det\left[K(n_i,n_j)\right]_{i,j=1}^{N} \qquad n_i\in\mathbf{Z}$$

$$\mathbf{K}=\left[K(n,m)\right]_{n,m\in\mathbf{Z}} \ : \text{projective } \mathbf{K}\cdot\mathbf{K}=\mathbf{K}\ ,\ \ \mathrm{tr}\,\mathbf{K}=N$$



then,

$$R_{N-1}(n_1,\ldots,n_{N-1})=N\sum_{m\in\mathbf{Z}}\frac{1}{N!}\det\begin{bmatrix}\left[K(n_i,n_j)\right]_{i,j=1}^{N-1} & \left[K(m,n_j)\right]_{j=1}^{N-1}\\ \left[K(n_i,m)\right]_{i=1}^{N-1} & K(m,m)\end{bmatrix}=\frac{N-(N-1)}{(N-1)!}\det\left[K(n_i,n_j)\right]_{i,j=1}^{N-1}$$

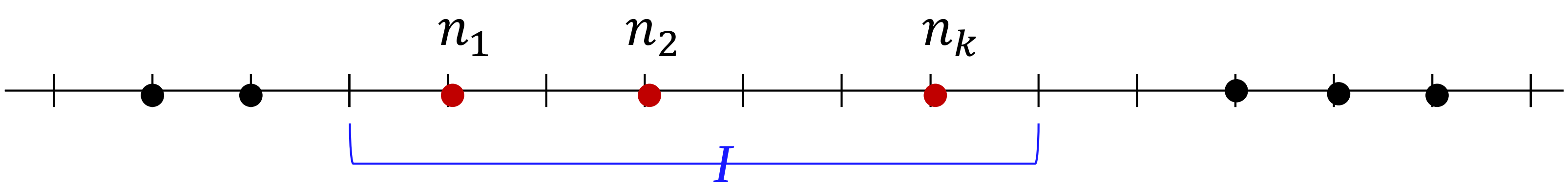
$\Downarrow$ repeat

$$R_k(n_1,\ldots,n_k)=\det\left[K(n_i,n_j)\right]_{i,j=1}^{k} \qquad = \text{repeat} \Rightarrow \qquad R_1(n)=K(n,n)$$

Janossy density

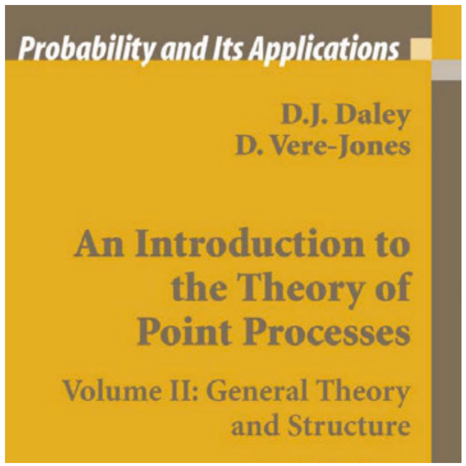
for Det point process $R_k(x_1,\ldots,x_k) = \det\left[K(x_i,x_j)\right]_{i,j=1}^{k}$

$J_{k,I}(n_1,\ldots,n_k) = \text{Prob of}$



no pts in I except for k designated pts

[Textbook '88]

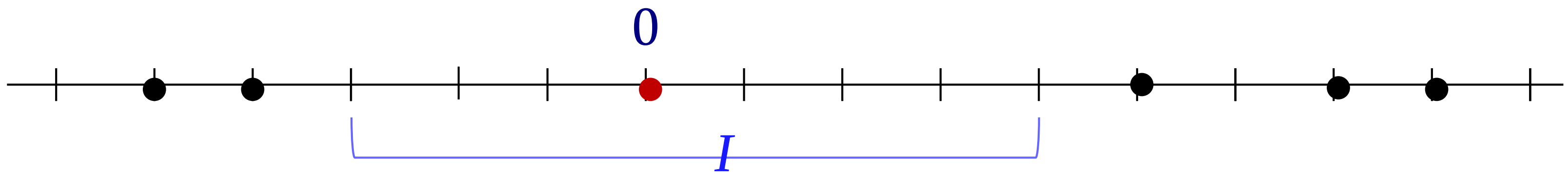


$$\mathbf{K}_I := \left[K(n,m)\right]_{n,m\in I}$$

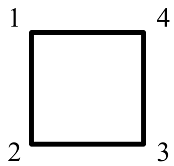
$$J_{k,I}(n_1,\ldots,n_k) = \det\big(1-\overbrace{\mathbf{K}_I}^{\text{all pts in } I}\big)\cdot \det\Big[\overbrace{\langle n_i|\mathbf{K}_I(1-\mathbf{K}_I)^{-1}|n_j\rangle}^{\text{designated pts }\bullet}\Big]_{i,j=1}^{k}$$

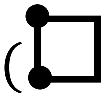
2 Janossy density in DPP

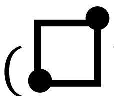
Proof



$$
\begin{aligned}
& J_{1,I}(0) \\
= \;& R_1(0) - \sum_{n \in I} R_2(0,n) + \frac{1}{2!} \sum_{n,n' \in I} R_3(0,n,n') - \cdots \\
= \;& K(0,0) - \sum_{n \in I} \begin{vmatrix} K(0,0) & K(0,n) \\ K(n,0) & K(n,n) \end{vmatrix} + \frac{1}{2!} \sum_{n,n' \in I} \begin{vmatrix} K(0,0) & K(0,n) & K(0,n') \\ K(n,0) & K(n,n) & K(n,n') \\ K(n',0) & K(n',n) & K(n',n') \end{vmatrix} \cdots \\
= \;& \langle 0|\mathbf{K}_I|0\rangle - \left\{ \langle 0|\mathbf{K}_I|0\rangle \ \mathrm{tr}\mathbf{K}_I - \left\langle 0|\mathbf{K}_I^2|0\right\rangle \right\} \\
& + \frac{1}{2!} \left\{ \langle 0|\mathbf{K}_I|0\rangle \left(\mathrm{tr}\mathbf{K}_I\right)^2 - \langle 0|\mathbf{K}_I|0\rangle \ \mathrm{tr}\mathbf{K}_I^2 - 2\left\langle 0|\mathbf{K}_I^2|0\right\rangle \mathrm{tr}\mathbf{K}_I + 2\left\langle 0|\mathbf{K}_I^3|0\right\rangle \right\} - \cdots \\
= \;& \langle 0|\mathbf{K}_I|0\rangle \left\{ 1 - \mathrm{tr}\mathbf{K}_I + \frac{1}{2!}\left(\mathrm{tr}\mathbf{K}_I\right)^2 - \cdots \right\} \left\{ 1 - \frac{1}{2}\mathrm{tr}\mathbf{K}_I^2 + \cdots \right\} \cdots \\
+ \;& \left\langle 0|\mathbf{K}_I^2|0\right\rangle \left\{ 1 - \mathrm{tr}\mathbf{K}_I + \cdots \right\} \cdots \\
+ \;& \left\langle 0|\mathbf{K}_I^3|0\right\rangle \left\{ 1 - \cdots \right\} \cdots + \cdots \\
= \;& \left\langle 0|\mathbf{K}_I + \mathbf{K}_I^2 + \mathbf{K}_I^3 + \cdots |0\right\rangle \det(\mathbf{1} - \mathbf{K}_I) = \left\langle 0|\mathbf{K}_I(\mathbf{1} - \mathbf{K}_I)^{-1}|0\right\rangle \det(\mathbf{1} - \mathbf{K}_I)
\end{aligned}
$$

Example: 2 Fermions on 

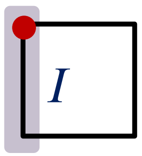
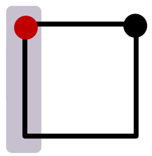
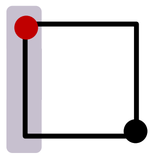
Prob() = 1/8

Prob() = 1/4

$\Longleftrightarrow$ $$\mathrm{Prob}(n,m) = \frac{1}{2!}\begin{vmatrix} K(n,n) & K(n,m) \\ K(m,n) & K(m,m) \end{vmatrix}$$

$$\mathbf{K} = \frac{1}{4}\begin{pmatrix} 2 & 1-i & 0 & 1+i \\ 1+i & 2 & 1-i & 0 \\ 0 & 1+i & 2 & 1-i \\ 1-i & 0 & 1+i & 2 \end{pmatrix} = \mathbf{K}\cdot\mathbf{K}$$

tr $\mathbf{K}$= 2

$J_{1,I}(1)$ = Prob() =  +  $= \frac{1}{8} + \frac{1}{4}$

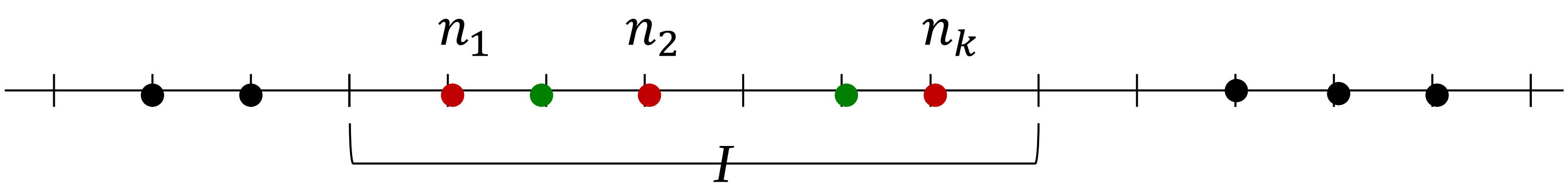
$$= \det(\mathbf{1} - \mathbf{K}_I)\cdot\langle 1|\mathbf{K}_I(\mathbf{1}-\mathbf{K}_I)^{-1}|1\rangle$$

$$= \left|\mathbf{1} - \tfrac{1}{4}\begin{pmatrix} 2 & 1-i \\ 1+i & 2 \end{pmatrix}\right| \cdot \begin{pmatrix} 3 & 2-2i \\ 2+2i & 3 \end{pmatrix}_{1,1} = \frac{1}{8}\cdot 3$$

Janossy density

Det. point process: $$R_k(x_1,\ldots,x_k) = \det\left[K(x_i,x_j)\right]_{i,j=1}^{k}$$

$J_{p,k,I}(n_1,\ldots,n_k)$ = Prob of



exactly p pts ● in I except for k designated pts ●

$$\mathbf{K}_I := \left[K(n,m)\right]_{n,m\in I}$$

$$J_{p,k,I}(n_1,\ldots,n_k) = \frac{1}{p!}\left(-\partial_\xi\right)^p \det\left(1-\xi\mathbf{K}_I\right)\cdot \det\left[\left\langle n_i \middle| \mathbf{K}_I\left(1-\xi\mathbf{K}_I\right)^{-1} \middle| n_j\right\rangle\right]_{i,j=1}^{k}\Bigg|_{\xi=1}$$

2 Janossy density in DPP

- In case $(\mathbf{1}-\mathbf{K}_I)$ may be singular : use $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D|\,|A - CD^{-1}B|$

$$J_{k,I}(n_1,\dots,n_k) = \det(\mathbf{1}-\mathbf{K}_I)\cdot\det\left[\langle n_i|\mathbf{K}_I(\mathbf{1}-\mathbf{K}_I)^{-1}|n_j\rangle\right]_{i,j=1}^{k}$$

$$= (-)^k \ \det \left| \begin{array}{cc} \overbrace{-\left[\langle n_i|\mathbf{K}_I|n_j\rangle\right]_{i,j=1}^{k}}^{\text{designated pts } \bullet} & \overbrace{-\left[\langle m|\mathbf{K}_I|n_j\rangle\right]_{j=1,\dots,k}^{m\in I}}^{\text{all pts in } I} \\ -\left[\langle n_i|\mathbf{K}_I|m\rangle\right]_{m\in I}^{j=1,\dots,k} & \mathbf{1}-\mathbf{K}_I \end{array} \right| \begin{array}{l} \left.\right\}\ \bullet \\ \left.\right\}\ I \end{array}$$

- Continuous distributions : Fredholm Det from Quadrature approx. of I

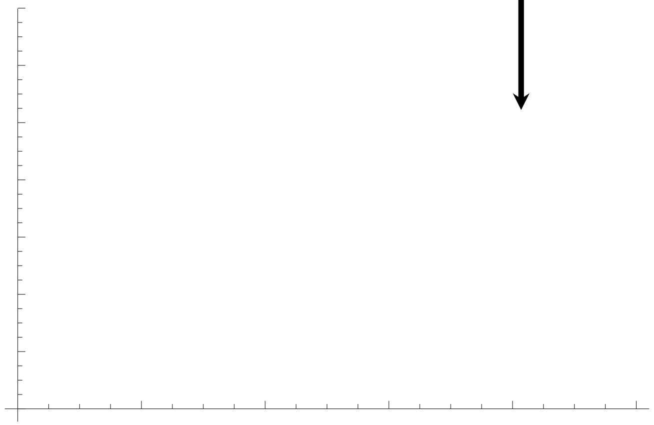
$$K(n,m) \to \sqrt{\Delta y_a}\,K(y_a,y_b)\sqrt{\Delta y_b}$$

$$J_{k,I}(x_1,\dots,x_k) = \lim_{\Delta y_a\to 0}$$

$$(-)^k \det \left| \begin{array}{cc} -\left[K(x_i,x_j)\right]_{i,j=1}^{k} & -\left[\sqrt{\Delta y_a}\,K(y_a,x_i)\right]_{y_a\in I}^{i=1,\dots,k} \\ -\left[K(x_i,y_b)\sqrt{\Delta y_b}\right]_{y_b\in I}^{j=1,\dots,k} & \mathbf{1}-\left[\sqrt{\Delta y_a}\,K(y_a,y_b)\sqrt{\Delta y_b}\right]_{y_a,y_b\in I} \end{array} \right|$$

new? not explicit in [Borodin-Soshnikov ’03][Forrester-Witte ’07][Forrester-Witte-Bornemann ’12] exercise 0 : 1st ∼8th EV distributions of chGUE , chGSE

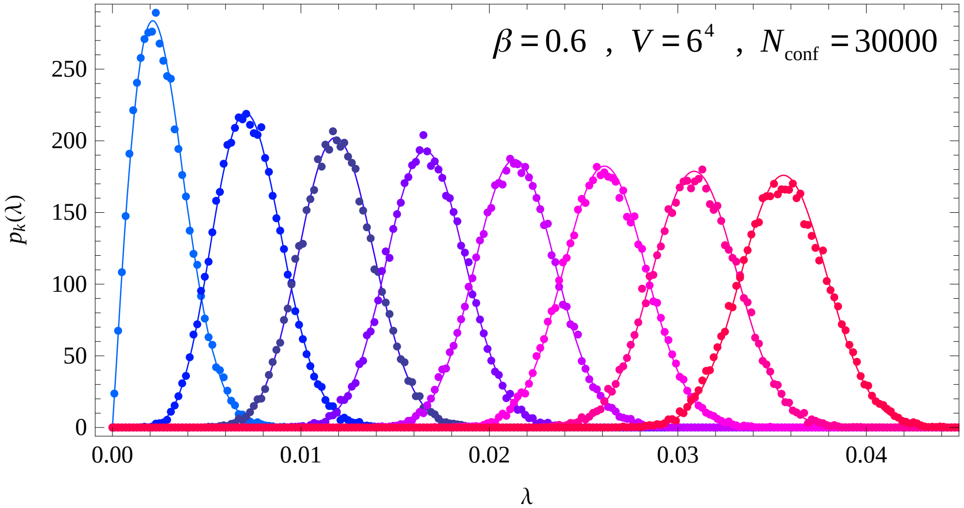
flattening oscillation $\Rightarrow$ larger error for fitting

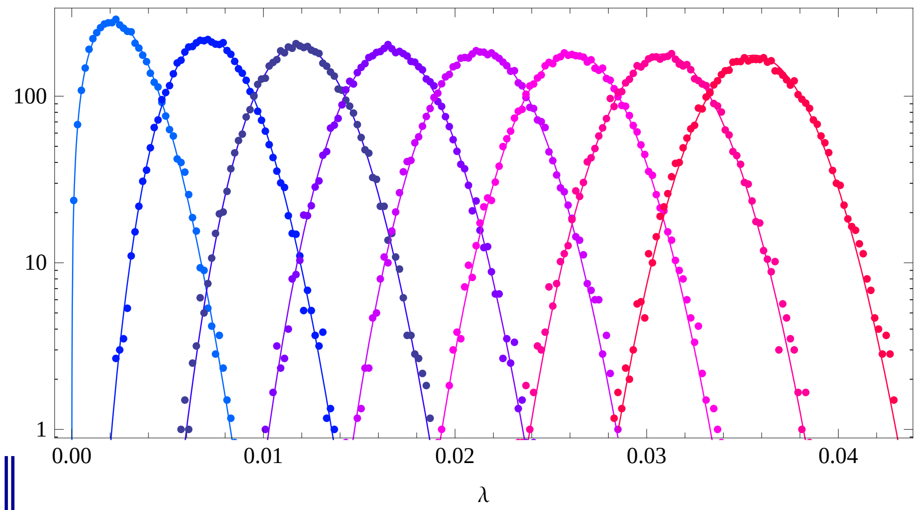


quasi-Gaussian $\Rightarrow$ precise fitting possible

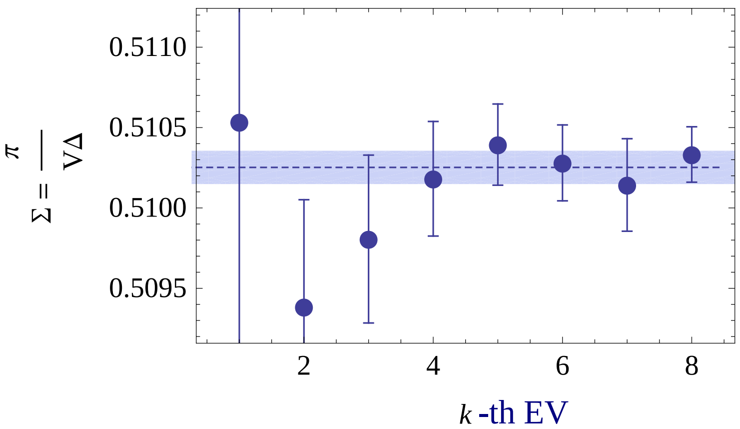
Chiral condensate from Individual EV distributions

exercise 1 : quenched U(1) Dirac spectrum vs chGUE





chiral condensate

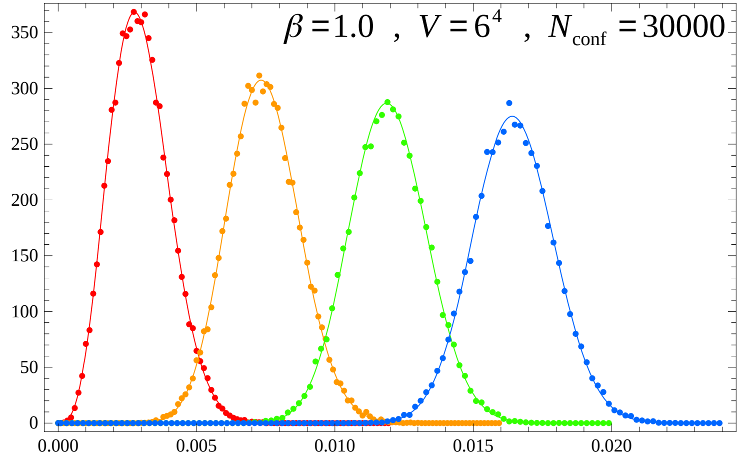


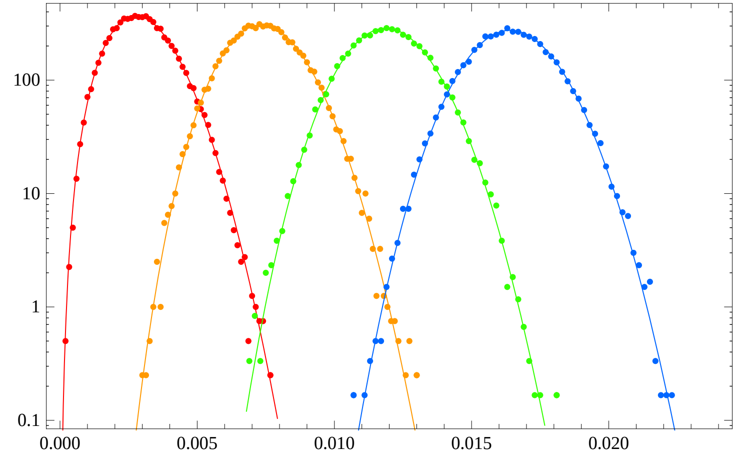
$$\Sigma a^3 = 0.51025(10)$$

[SMN '15]

Chiral condensate from Individual EV distributions

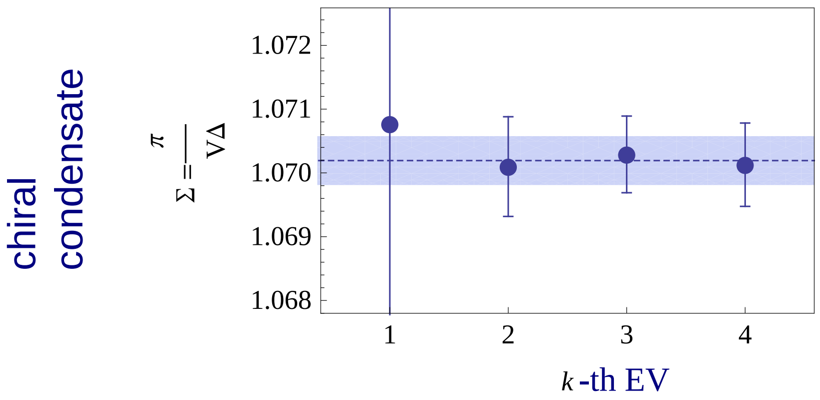
exercise 2 : quenched SU(2) Dirac spectrum vs chGSE





⇓

[SMN ’15]

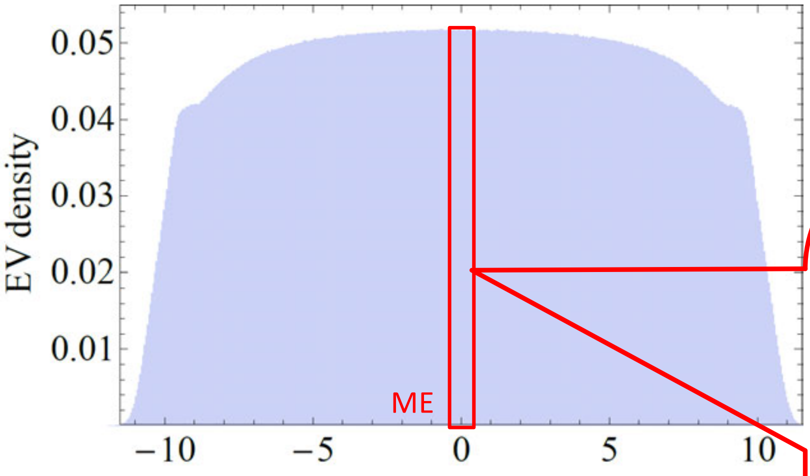


$\Sigma a^3 = 1.07019(38)$

Σ within $O(10^{-4})$ error!

Risan ’17

q^{-1}-Hermite ensemble vs Critical statistics: Anderson H

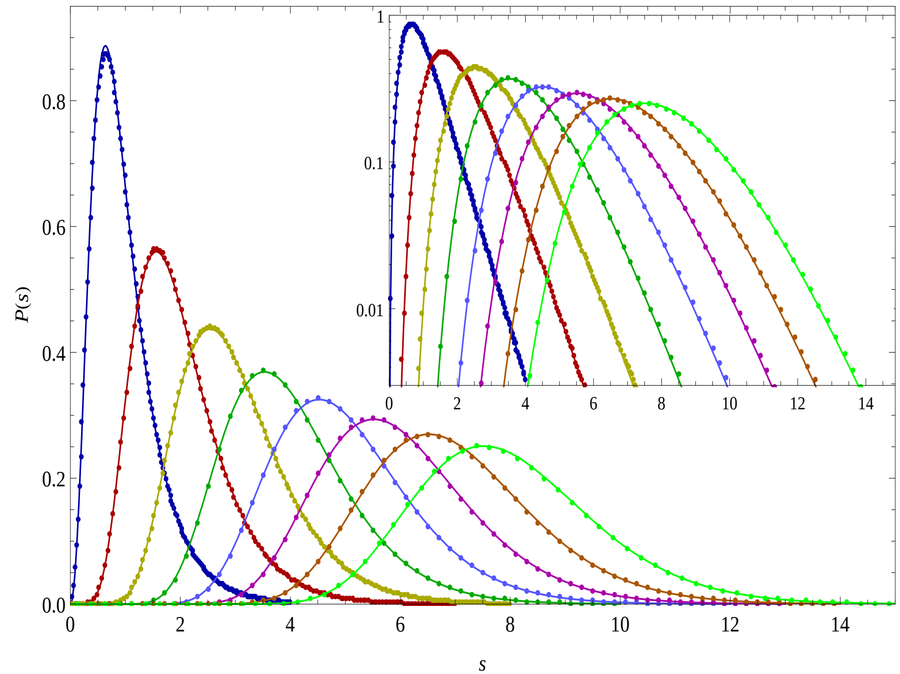


AH on $12^3,16^3,20^3$, $N_{\rm conf}=10^4$

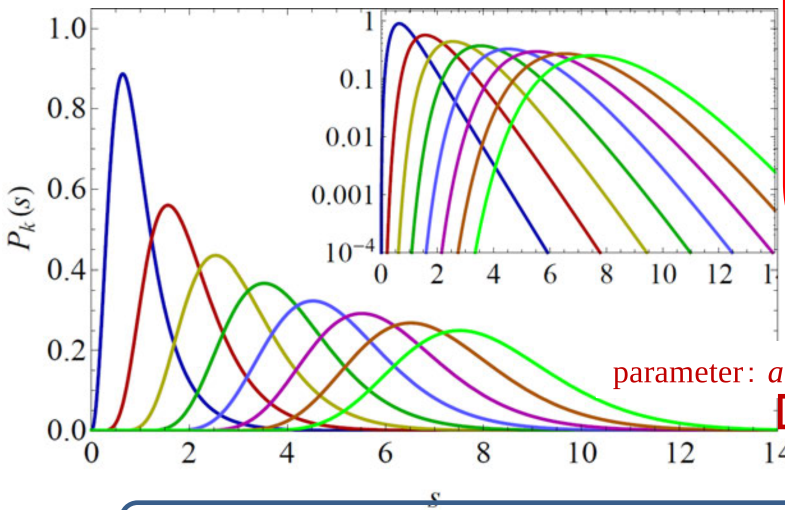
randomness $W=18.1$

mag. flux $\Phi=0.4\pi$

[SMN ’15]



MLSDs: q^{-1}-HE



parameter: $a = 3.55$

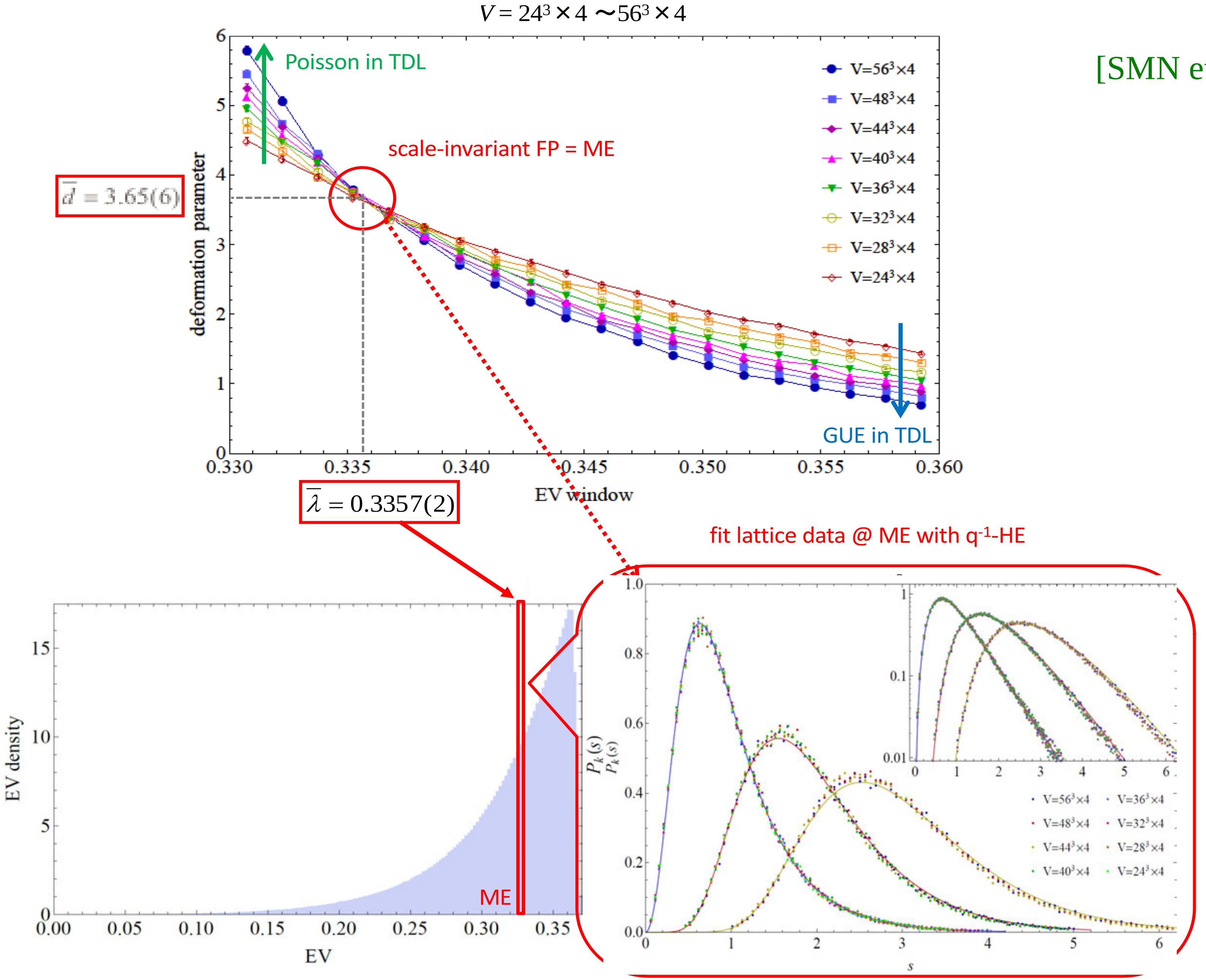
fit MLSDs of AH at ME with MLSDs of q^{-1}-HE

✓ MLSDs of q^{-1}-HE perfectly fit to Scale-inv Critical Statistics of AH at ME

Risan '17

q^{-1}-Hermite ensemble vs Critical statistics: QCD $\not{D}$

[SMN et al '18]



Technical problems

[Damgaard-SMN '01]

"Shifting" method for chRMT

$$\lambda_i \geq 0 \quad \in \ \mathrm{Spec}(H^2)$$

$$dH \ \mathrm{e}^{-\mathrm{tr}\,H^2} \prod_f \det\left(H + i m_f\right) \ \propto \ \prod_{i=1}^{N} \left(d\lambda_i \ \lambda_i^{\beta(\nu+1)/2-1} \ \mathrm{e}^{-\lambda_i} \prod_f \left(\lambda_i + m_f^2\right)\right) \prod_{i>j}^{N} \left|\lambda_i - \lambda_j\right|^{\beta}$$

… JPD of EVs

$$P_k(\lambda_1,\ldots,\lambda_k) = \boxed{\int_{\lambda_k}^{\infty} \cdots \int_{\lambda_k}^{\infty} d\lambda_{k+1} \cdots d\lambda_N \left(JPD_{N_F}(\lambda_1,\ldots,\lambda_N;\{m\};\nu)\right)}$$

… JPD of first k EVs

$$\tilde{\lambda}_i \equiv \lambda_i - \lambda_k \quad \Downarrow$$

$$= C(\{m\}) \int_0^{\infty} \cdots \int_0^{\infty} d\tilde{\lambda}_{k+1} \cdots d\tilde{\lambda}_N \left(JPD_{\tilde{N}_F}(\tilde{\lambda}_1,\ldots,\tilde{\lambda}_N;\{\tilde{m}\};\tilde{\nu})\right) \xrightarrow{N\to\infty} \text{ratio of Bessel det's}$$

$$\Downarrow$$

$$p_k(\lambda_k) = \int_0^{\lambda_k} \cdots \int_0^{\lambda_k} d\lambda_1 \cdots d\lambda_{k-1} P_k(\lambda_1,\ldots,\lambda_k) \xrightarrow{N\to\infty} \boxed{(k\text{-}1)\text{-fold integral of ratio of Bessel det's}}$$

for this trick to work, the exponent

$$\beta\frac{\nu+1}{2} - 1 \in \begin{cases} \mathbf{N} & (\beta = 1,2) \\ 2\mathbf{N} & (\beta = 4) \end{cases} \Rightarrow \begin{array}{l} \times\ \text{chGOE}, \ \ \nu = 0,2,4\ldots \\ \times\ \text{chGSE}, \ \ N_F = 0,2,4\ldots \end{array}$$

practically unfeasible for $k \geq 5$

Nystrom-type approx to Fredholm Det

Gauss-Legendre Quadrature : $\{x_1,\ldots,x_M\}\in I,\quad \{\Delta x_1,\ldots,\Delta x_M\}>0$

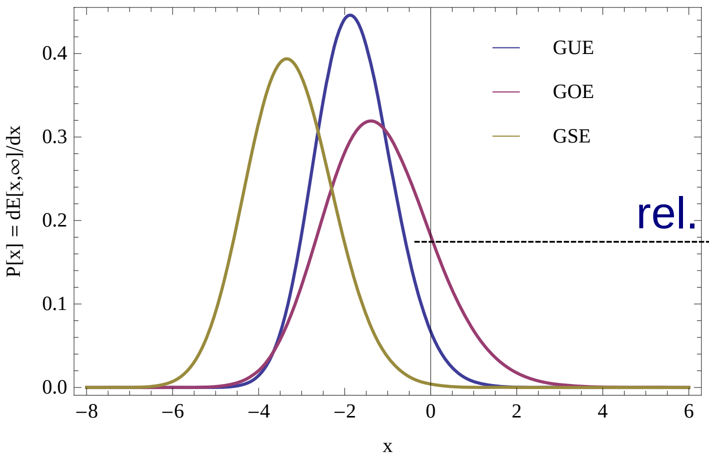
$$\int_I f(x)\,dx \cong \sum_{i=1}^{M} f(x_i)\Delta x_i \text{ , exact for } f(x)=x^M+\text{lower}$$

$$\mathrm{Det}\left(1-K_I\right) \cong \det\left[\delta_{ij}-K(x_i,x_j)\sqrt{\Delta x_i \Delta x_j}\right]_{i,j=1}^{M} + \text{relative error } O(e^{-\text{const.}M})$$

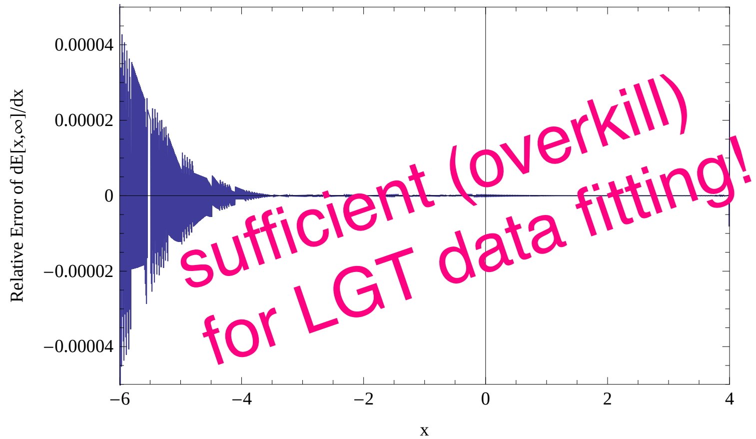
[Bornemann '10]

ex. Largest EV distribution

Nystrom approx (M=30) for K_{Airy} vs Tracy-Widom's analytic formula



rel. error



sufficient (overkill) for LGT data fitting!

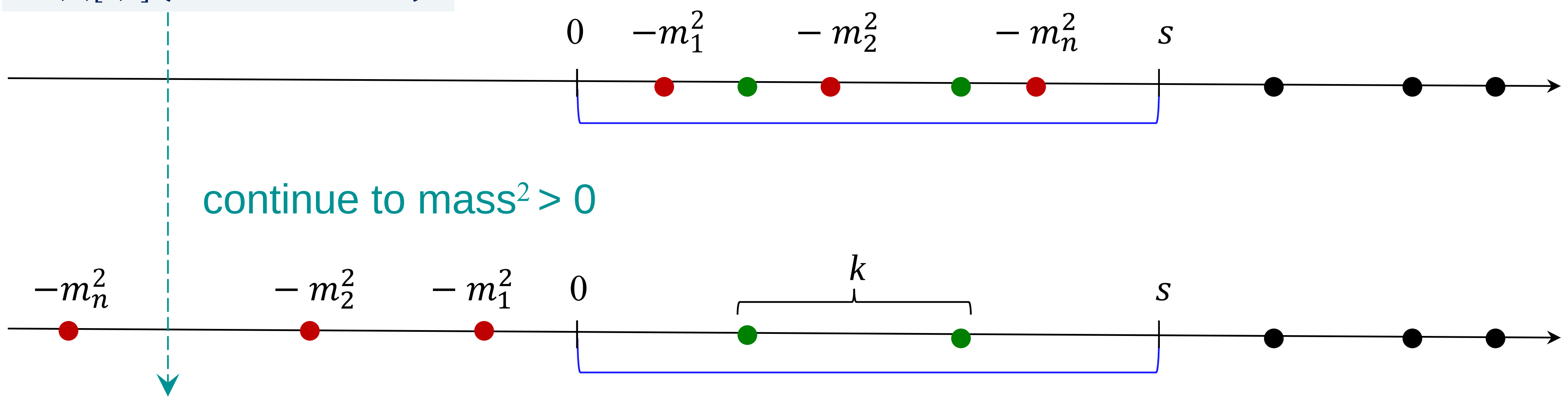
$n_F = \beta\, n$ fermions as Janossy density

$$\int dH\, \mathrm{e}^{-\mathrm{tr}\, H^2} \prod_{f=1}^{n} (H + i m_f)^{\beta} = \int_0^{\infty} \prod_{i=1}^{N} \left(d\lambda_i\, \lambda_i^{\frac{\beta(\nu+1)}{2}-1} \mathrm{e}^{-\lambda_i} \prod_{f=1}^{n} |\, \lambda_i + m_f^2 |^{\beta} \right) \prod_{i>j}^{N} |\lambda_i - \lambda_j|^{\beta}$$

$$\propto \int_0^{\infty} \prod_{i=1}^{N+n} \left(d\lambda_i\, \lambda_i^{\frac{\beta(\nu+1)}{2}-1} \mathrm{e}^{-\lambda_i} \right) \prod_{i>j}^{N+n} |\lambda_i - \lambda_j|^{\beta} \cdot \prod_{k=N+1}^{N+n} \delta(\lambda_k - (-m_f^2))$$

designated pts ● = −mass² > 0

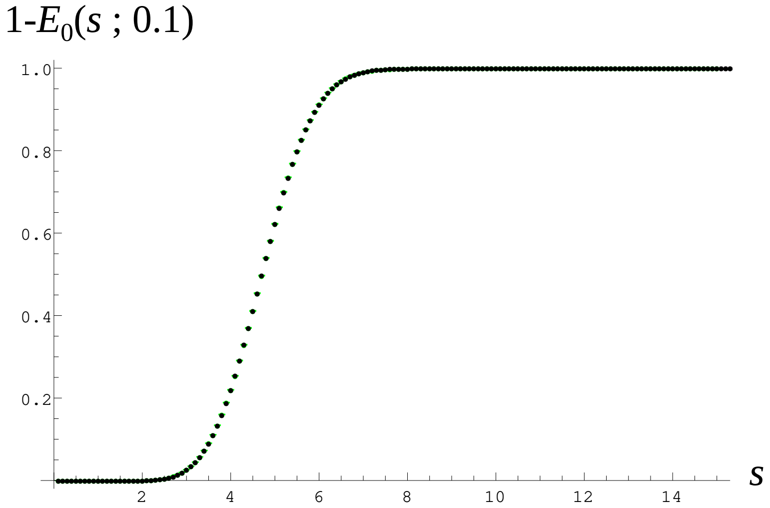
$J_{k,n,[0,s]}(-m_1^2, \dots, -m_n^2)$

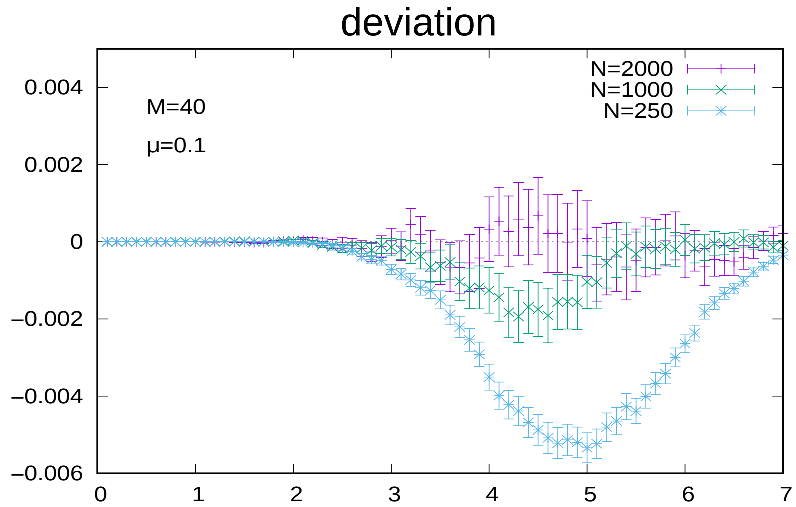


$E_k(s; m_1^2, \dots, m_n^2)$

$\Longrightarrow$ evaluate Fredholm Det by quadrature approx.

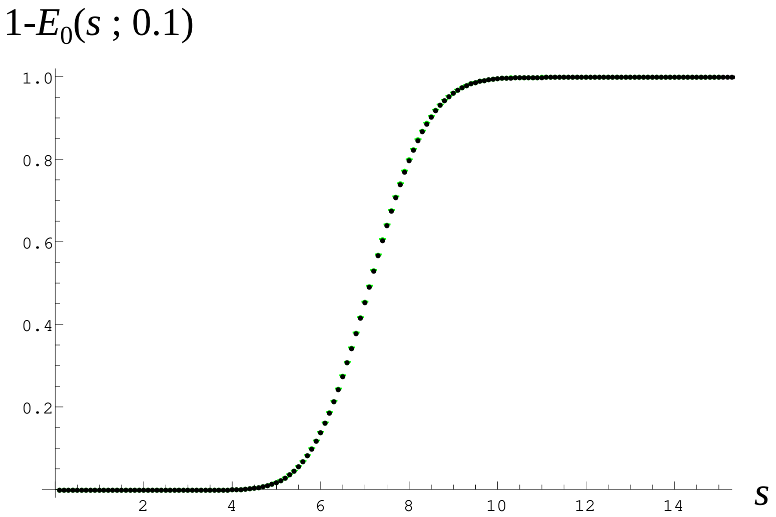
$\beta = 4$ (chGSE), $n_F = 4$, $k = 0$, $\mu = 0.1$ vs chGSE ($N = 250 \sim 2000$) by HMC

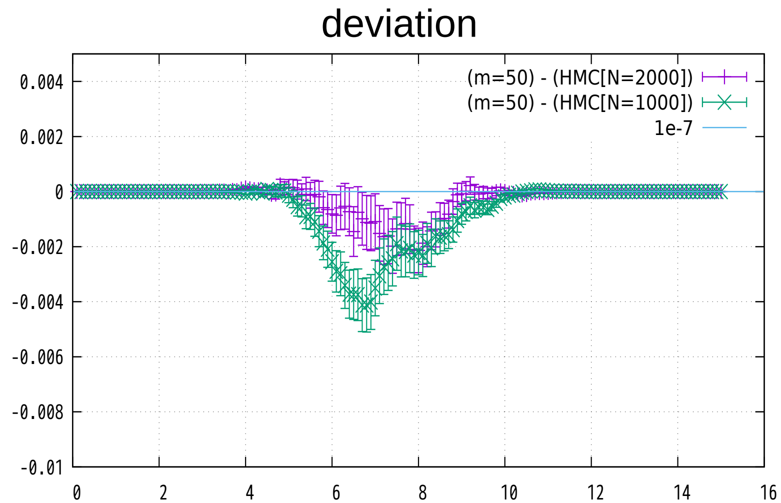




✅ Quadrature Approx of Det : systematic error too small for bare eyes!

$\beta = 4$ (chGSE), $n_F = 8$, $k = 0$, $\mu = 0.1$ vs chGSE ($N = 1000 \sim 2000$) by HMC

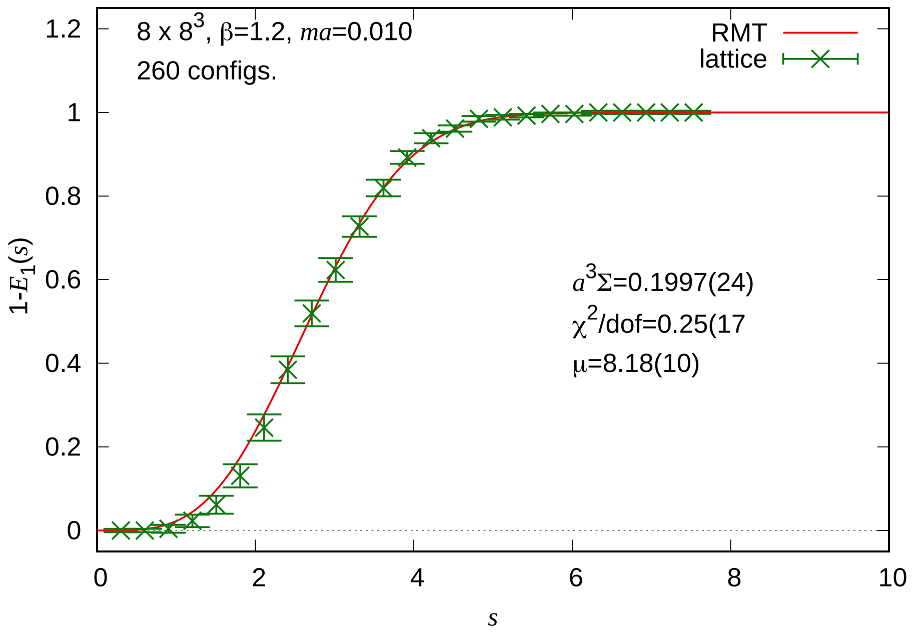




Chiral condensate from Individual EV distribution

$\beta = 4$ (chGSE), $n_F = 4$, $k = 0$ vs 2C QCD, $n_F = 8$(stag) $=2+(2+2+2)$

$\mu = 8.18$ $ma = 0.010$



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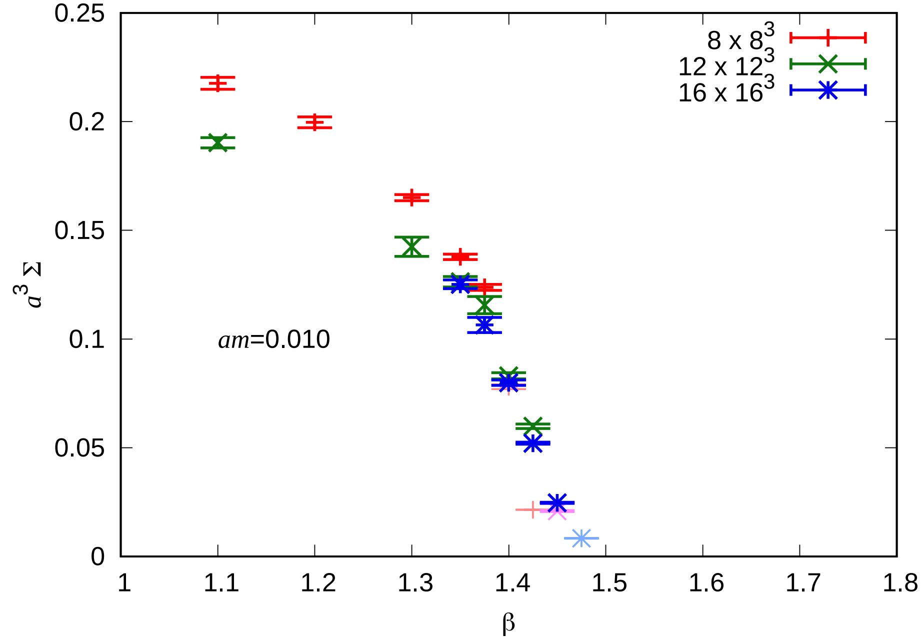
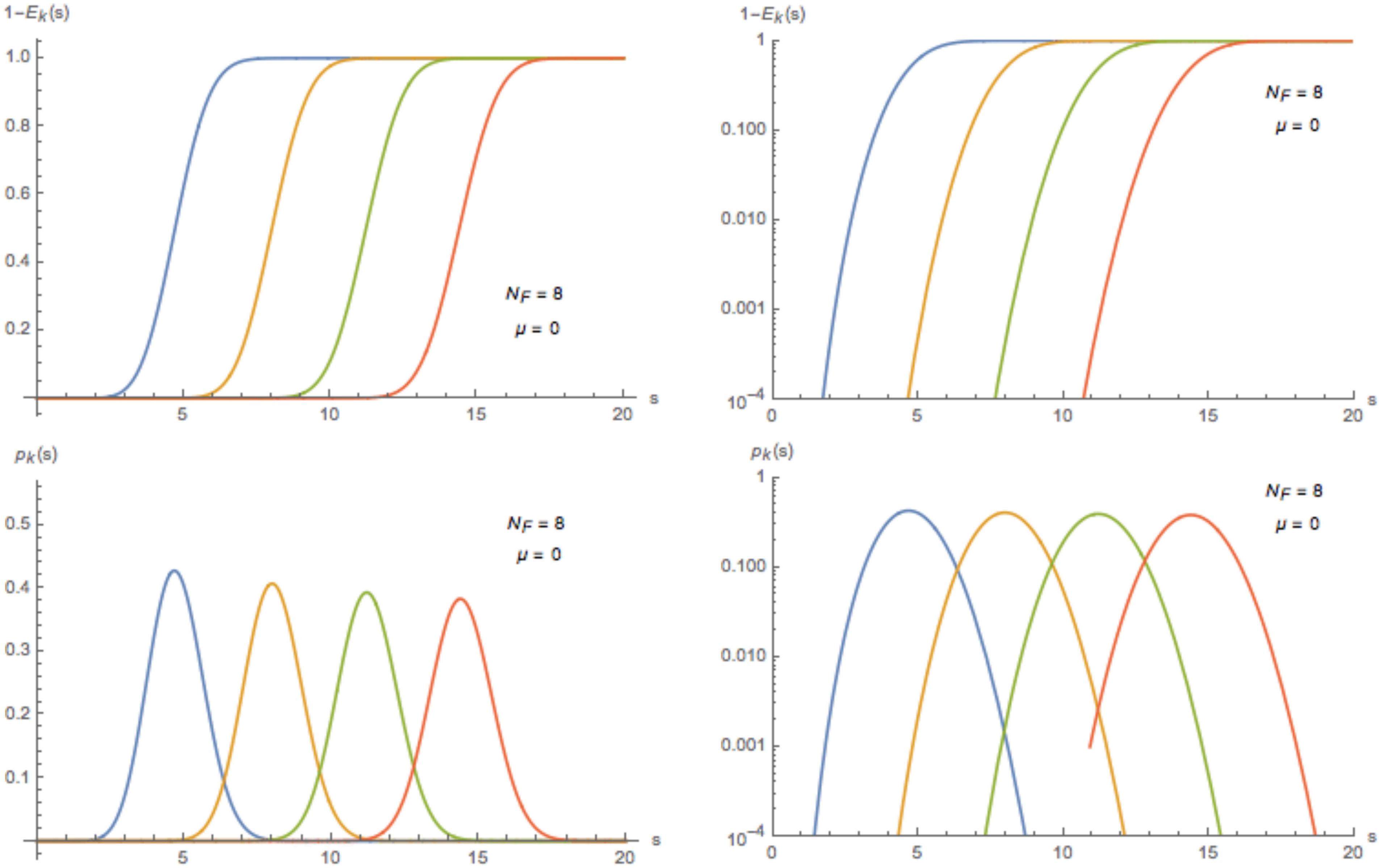


N=2, $n_F = 8$, $\beta \leq 1.4\ldots$: χS broken

4 Multi-flavor 2C QCD

$\beta = 4$ (chGSE), $n_F = 8$, $k = 0,1,2,3$, $\mu = 0,1,2,\ldots,\infty$

to be fitted with
2C QCD, $n_F = 8$ stag.



Summary

- 2 technical difficulties in evaluating Individual EVDs of massive chRME are overcome by Janossy Density formula + Quadrature method

- Individual Dirac EVDs of 2C QCD with $n_F = 4n$ staggered quarks, if the theory in χSB phase, are predicted from massive chGSE

- Chiral cond Σ of 2C QCD with $n_F = 2+(2+2+2)$ is determined by fitting $\mathrm{Spec}(\not{D})$

- feasible plan : determine whether the WTC candidates in sympl. class 2C QCD with $n_F = 8$ (stag.) , $n_{Ad} = 2$ (overlap) is χSB or conformal