Quantization of interacting topological super particle field theory

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In collaboration with Y. Kaneko and A. D'Adda

N.K. and Watabiki: Commun. Math. Phys. 144 (1992)641;
Mod. Phys. Lett. A 7 (1992) 1137.

N.K., K. Suehiro, T.Tsukioka and H.Umetsu: Nucl. Phys. (1998)

D'Adda, N.K., Shimode, Tsukioka, Phys. Lett. (2017)

Generalized Gauge Theories in arbitrary dimensions

N.K. & Watabiki '91

gauge field	$A = T^a A^a_\mu dx_\mu$	$\mathcal{A} = \mathbf{1}\psi + \mathbf{i}\hat{\psi} + \mathbf{j}A + \mathbf{k}\hat{A}$
gauge parameter	$v = T^a v^a$	$\mathcal{V} = \mathbf{1}\hat{a} + \mathbf{i}a + \mathbf{j}\hat{\alpha} + \mathbf{k}\alpha$
derivative	$d = dx^\mu \partial_\mu$	$\mathcal{Q} = \mathbf{j}d$
curvature	$F = dA + A^2$	$\mathcal{F} = \mathcal{Q}\mathcal{A} + \mathcal{A}^2$
gauge trans.	$\delta A = dv + [A, v]$	$\delta\mathcal{A} = \mathcal{Q}\mathcal{V} + [\mathcal{A}, \mathcal{V}]$
Chern-Simons	$\int Tr(\frac{1}{2}AdA \; + \; \frac{1}{3}A^3)$	$\int Tr_{\mathbf{k}}(\frac{1}{2}\mathcal{A}\mathcal{Q}\mathcal{A} \; + \; \frac{1}{3}\mathcal{A}^3)$
Topological Yang-Mills	$\int Tr(FF)$	$\int Str_{\mathbf{1}}(\mathcal{F}\mathcal{F})$
Yang-Mills	$\int Tr(F \star F)$	$\int Tr_{\mathbf{1}}(\mathcal{F} \mathrm{v} \mathcal{F}) \star 1$

Quantization of Abelian 2-dim. BF theory

$$BF = B(d\omega + \omega^2) \quad \text{(4-dim.)}$$

$$\int_{M_2} d^2x[\underbrace{\epsilon^{\mu\nu}\phi\partial_\mu\omega_\nu}_{\phi d\omega} + b\partial^\mu\omega_\mu - i\bar{c}\partial^\mu\partial_\mu c] \qquad \delta\omega = dv$$

ϕ^A	$s\phi^A$	$s_\mu\phi^A$	$\widetilde{s}\phi^A$
ϕ	0	$-\epsilon_{\mu\nu}\partial^\nu\bar{c}$	0
ω_ν	$\partial_\nu c$	0	$-\epsilon_{\nu\rho}\partial^\rho c$
c	0	$-i\omega_\mu$	0
$\bar{c}$	$-ib$	0	$-i\phi$
b	0	$\partial_\mu\bar{c}$	0

on shell N=2 twisted SUSY invariance

$$s^2 = \{s, \tilde{s}\} = \tilde{s}^2 = \{s_\mu, s_\nu\} = 0,$$
$$\{s, s_\mu\} = -i\partial_\mu, \{\tilde{s}, s_\mu\} = i\epsilon_{\mu\nu}\partial^\nu.$$

Kato, N.K. Uchida '03

$$S_{\text{off-shell AQBF}} = \int_{M_2} d^2x[\epsilon^{\mu\nu}\phi\partial_\mu\omega_\nu + b\partial^\mu\omega_\mu - i\bar{c}\partial^\mu\partial_\mu c - i\lambda\rho]$$

$$= \int_{M_2} d^2x s\tilde{s}\frac{1}{2}\epsilon^{\mu\nu}s_\mu s_\nu(-i\bar{c}c)$$

ϕ^A	$s\phi^A$	$s_\mu\phi^A$	$\widetilde{s}\phi^A$
ϕ	$i\rho$	$-\epsilon_{\mu\nu}\partial^\nu\bar{c}$	0
ω_ν	$\partial_\nu c$	$-i\epsilon_{\mu\nu}\lambda$	$-\epsilon_{\nu\rho}\partial^\rho c$
c	0	$-i\omega_\mu$	0
$\bar{c}$	$-ib$	0	$-i\phi$
b	0	$\partial_\mu\bar{c}$	$-i\rho$
λ	$\epsilon^{\mu\nu}\partial_\mu\omega_\nu$	0	$-\partial_\mu\omega^\mu$
ρ	0	$-\partial_\mu\phi - \epsilon_{\mu\nu}\partial^\nu b$	0

off shel N=2 twisted SUSY invariance

Gauge Theory on the Random Lattice

Form | **Simplex**

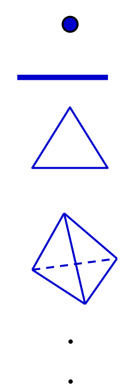
0 ϕ

1 $\omega = \omega_\mu dx^\mu$

2 $B = B_{\mu\nu} dx^\mu dx\nu$

3 $\Omega = \Omega_{\mu\nu\rho} dx^\mu dx^\nu dx\rho$

$\vdots$



$$\overset{\scriptstyle 1}{}\overset{\scriptstyle 3}{}\overset{\scriptstyle 0}{}\overset{\scriptstyle 2}{}$$

$$\mathcal{A} = \mathbf{j}(\omega + \Omega + \cdots) + \mathbf{k}(\phi + B + \cdots)$$
$$+\mathbf{1}(\chi^{(1)} + \chi^{(3)} + \cdots) + \mathbf{i}(\chi^{(0)} + \chi^{(2)} + \cdots)$$

(quaternion based formulation: origin ?)

$$S = \int Tr(\tfrac{1}{2}\mathcal{A}Q\mathcal{A} \ + \ \tfrac{1}{3}\mathcal{A}^3)$$

$$\delta\mathcal{A} = d\mathcal{V} + [\mathcal{A}, \mathcal{V}]$$

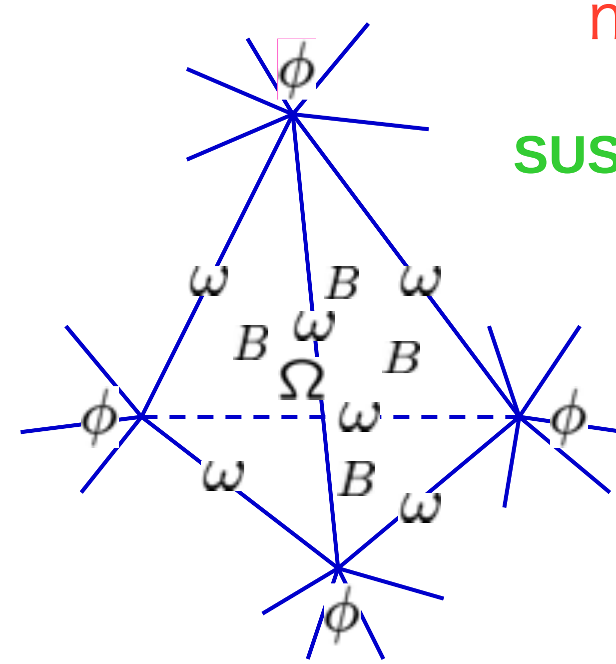
N.K.＆Watabiki '91

Gauge Theory + Gravity ？

Boson ⟷ Fermion ?

matter fermion ?

SUSY ?



$$U_{\tilde{l}} = (e^\omega)_{\tilde{l}}$$

$$(\tilde{l} = \text{dual link})$$

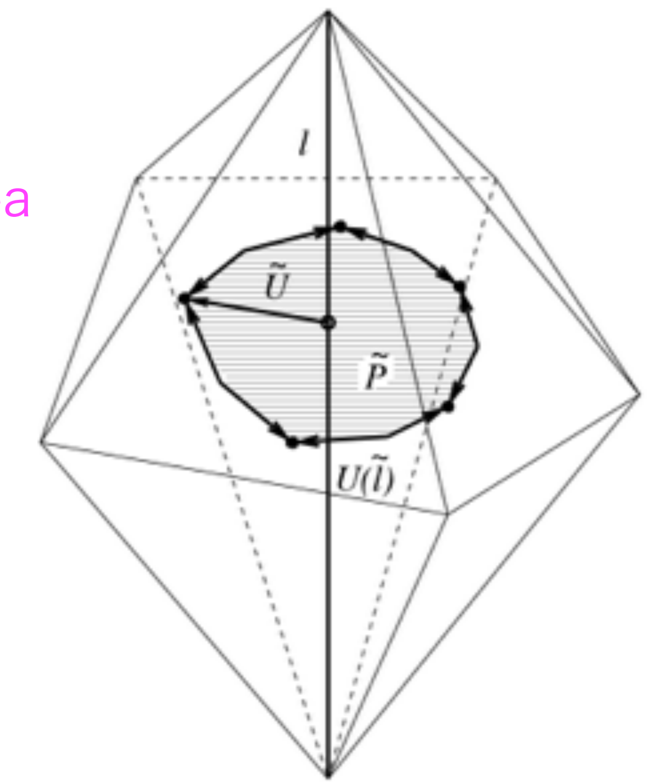
3d Chern-Simons gravity by witten (1988/89)

3-dim. Chern- Simon Gravity

$$S_{cont} = \int (\tfrac{1}{2} A dA + \tfrac{1}{3} A^3) = \int Tr(e \wedge (d\omega + \omega^2))$$ (Witten 1988/89)

$$(A = e^a P_a + \omega^{ab} J_{ab})$$

regge calculus idea

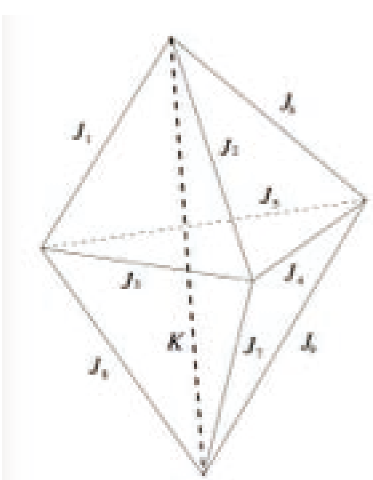


$$S_{Lat} = \sum_l \epsilon_{abc} e^a(l) [\ln \prod_{\partial \tilde{P}(l)} U]^{bc} \qquad U = e^{\omega}$$

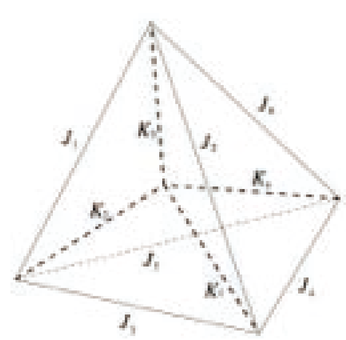
$$Z = \int dU de e^{-S} \sim \prod_{\text{edges}} (2J+1) \prod_{\text{tetrahedra}} (-1)^{\sum J_i} \begin{Bmatrix} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \end{Bmatrix}$$

$$\sim \sum e^{-S_{Regge}}$$ Ponzano-Regge gravity

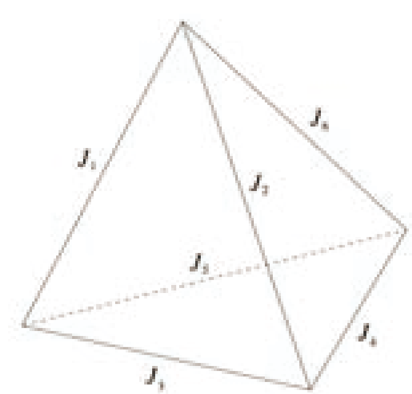
N.K., H.B.Nielsen & N.Sato (1999)



2-3 move



1-4 move



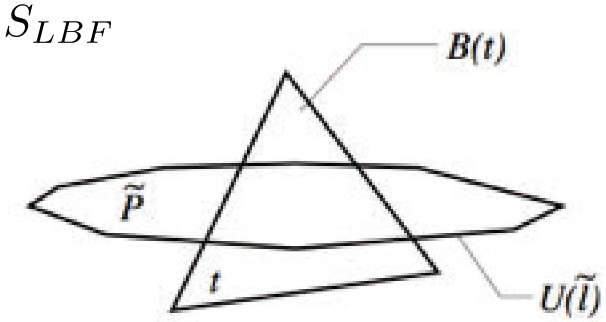
$$(-1)^{\sum_{i=1}^{6} J_i} \begin{Bmatrix} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \end{Bmatrix} \sim \frac{1}{\sqrt{12\pi V}} \cos\left(S_{\text{Regge}} + \frac{\pi}{4}\right) \quad (\text{all } J_i \gg 1)$$

4-dim. BF Gravity

(N.K., K.Sato & Uchida (2000))

$$Z = \int \mathcal{D}U\mathcal{D}B\ \delta\left(\left(\prod U(\tilde{l})\right) B \left(\prod U(\tilde{l})\right)^{\dagger} - B\right) \sum_{N} \delta(|B| - N) e^{i\, S_{LBF}}$$

$$S_{LBF} = \sum_{t} \mathrm{tr}\left(-iB(t)\left[\ln \prod_{\tilde{l}\in\partial\tilde{P}} U(\tilde{l})\right]\right) = \frac{1}{2}\sum_{t} B^{a}(t) F^{a}(t)$$



$$Z_{LBF} = \sum_{J}\prod_{\text{site}}\Lambda^{-1}\prod_{\text{link}}\Lambda\prod_{\text{triangle}}(2J+1)\prod_{\text{tetrahedron}}(2J+1)\prod_{\text{4-simplex}}\begin{Bmatrix} J_1 & J_2 & J_3 & J_4 & J_5 \\ J_6 & J_7 & J_8 & J_9 & J_{10} \\ J_{11} & J_{12} & J_{13} & J_{14} & J_{15}\end{Bmatrix}$$

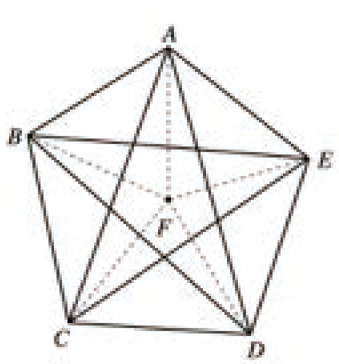
$$\left(F^{a} = e^{d\omega+\omega^2+\cdots}\right)$$

$$\begin{Bmatrix} J_1 & J_2 & J_3 & J_4 & J_5 \\ J_6 & J_7 & J_8 & J_9 & J_{10} \\ J_{11} & J_{12} & J_{13} & J_{14} & J_{15}\end{Bmatrix} = \sum_{\text{all } m_i}(-)^{\sum_{i=1}^{15}(J_i-m_i)}\begin{pmatrix} J_1 & J_7 & J_6 \\ m_1 & m_7 & m_6\end{pmatrix}\begin{pmatrix} J_3 & J_8 & J_7 \\ -m_3 & -m_8 & -m_7\end{pmatrix}$$

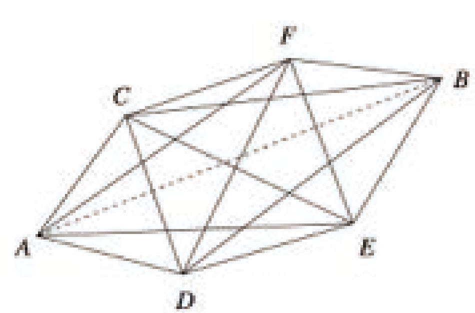
$$\times\begin{pmatrix} J_4 & J_8 & J_9 \\ m_4 & m_8 & m_9\end{pmatrix}\begin{pmatrix} J_1 & J_9 & J_{10} \\ -m_1 & -m_9 & -m_{10}\end{pmatrix}\begin{pmatrix} J_2 & J_{11} & J_{10} \\ m_2 & m_{11} & m_{10}\end{pmatrix}$$

$$\times\begin{pmatrix} J_4 & J_{11} & J_{12} \\ -m_4 & -m_{11} & -m_{12}\end{pmatrix}\begin{pmatrix} J_5 & J_{13} & J_{14} \\ m_5 & m_{13} & m_{14}\end{pmatrix}\begin{pmatrix} J_2 & J_{14} & J_{13} \\ -m_2 & -m_{14} & -m_{13}\end{pmatrix}$$

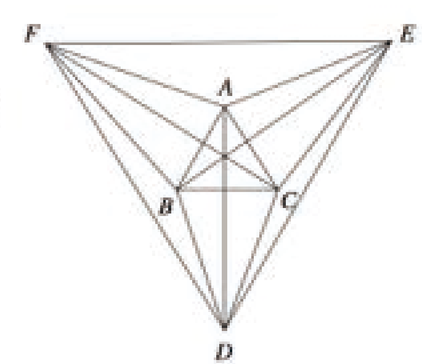
$$\times\begin{pmatrix} J_3 & J_{14} & J_{15} \\ m_3 & m_{14} & m_{15}\end{pmatrix}\begin{pmatrix} J_5 & J_{15} & J_6 \\ -m_5 & -m_{15} & -m_6\end{pmatrix}$$



1-5 move



2-4 move



3-3 move

Generalized Chern-Simons actions in arbitrary dimensions

$$S = \int Tr\left(\frac{1}{2}\mathcal{A}Q\mathcal{A} + \frac{1}{3}\mathcal{A}^3\right) = \mathbf{1}S^1 + \mathbf{i}S^i + \mathbf{j}S^j + \mathbf{k}S^k \ \in \mathbf{\Lambda}_-$$

is invariant under a generalized gauge transformation:

$$\delta\mathcal{A} = Q\mathcal{V} + [\mathcal{A}, \mathcal{V}] = \mathbf{1}\delta\psi_1 + \mathbf{i}\delta\hat{\psi}_0 + \mathbf{j}\delta A_0 + \mathbf{k}\delta\hat{A}_1$$

To prove generalized gauge invariance:

1. $Q^2 = 0$

2. $\{\overrightarrow{Q}, \lambda_-\} = Q\lambda_-, \quad [\overrightarrow{Q}, \lambda_+] = Q\lambda_+$

3. $Tr(\lambda_+\lambda'_+) = Tr(\lambda'_+\lambda_+), \quad Tr(\lambda_-\lambda_+) = Tr(\lambda_+\lambda_-), \quad Tr(\lambda_-\lambda'_-) = -Tr(\lambda'_-\lambda_-)$

quaternion algebra:

$$S_{GCS} = \int Tr\left(\frac{1}{2}\mathcal{A}Q\mathcal{A} + \frac{1}{3}\mathcal{A}^3\right)$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1},$$
$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

$$\delta\mathcal{A} = Q\mathcal{V} + [\mathcal{A}, \mathcal{V}] \quad (Q = \mathbf{j}d)$$

Let us introduce two-type of fields $a + b + c = \mathrm{odd\ or\ even}$

(Abelian) $[\mathbf{q}(a,b,c), \Phi_{(a',b',c')}] = 0$

$$\begin{aligned}
\mathcal{A} &= \mathbf{q}(1,1,1)\Phi_{(1,1,1)} + \mathbf{q}(1,0,0)\Phi_{(1,0,0)} + \mathbf{q}(0,1,0)\Phi_{(0,1,0)} + \mathbf{q}(0,0,1)\Phi_{(0,0,1)} \\
&= \mathbf{1}\Phi_{(1,1,1)} + \mathbf{i}\Phi_{(1,0,0)} + \mathbf{j}\Phi_{(0,1,0)} + \mathbf{k}\Phi_{(0,0,1)} \qquad \in \mathbf{\Lambda}_- \\
\mathcal{V} &= \mathbf{q}(0,0,0)\Phi_{(0,0,0)} + \mathbf{q}(0,1,1)\Phi_{(0,1,1)} + \mathbf{q}(1,0,1)\Phi_{(1,0,1)} + \mathbf{q}(1,1,0)\Phi_{(1,1,0)} \\
&= \mathbf{1}\Phi_{(0,0,0)} + \mathbf{i}\Phi_{(0,1,1)} + \mathbf{j}\Phi_{(1,0,1)} + \mathbf{k}\Phi_{(1,1,0)} \qquad \in \mathbf{\Lambda}_+
\end{aligned}$$

Z(2) grading structure

$$\mathcal{A}\mathcal{A}' = -\mathcal{A}'\mathcal{A} \in \mathbf{\Lambda}_+ \quad \mathcal{A}\mathcal{V} = \mathcal{V}\mathcal{A} \in \mathbf{\Lambda}_- \quad \mathcal{V}\mathcal{V}' = \mathcal{V}'\mathcal{V} \in \mathbf{\Lambda}_+$$

different grading commutes with quaternion	=	total 3-gradings (anti-commuting) without quaternion

Higher form gauge fields and non-Abelian extension

$$\Phi_{(a,b,c)}(x)$$

boson-fermion

even- and odd-form

$c = 0, 1$

even-form

odd-from

boson

$$\Phi_{(0,0,c)}(x) \equiv \hat{A}_c(x) = \text{direct sum of bosonic even forms},$$

$$\Phi_{(0,1,c)}(x) \equiv A_c(x) = \text{direct sum of bosonic odd forms},$$

fermion

$$\Phi_{(1,0,c)}(x) \equiv \hat{\psi}_c(x) = \text{direct sum of fermionic even forms},$$

$$\Phi_{(1,1,c)}(x) \equiv \psi_c(x) = \text{direct sum of fermionic odd forms}$$

Derivative operator: $d = dx^\mu \partial_\mu$ as $\mathcal{A}$-type operator:

$$Q = \mathbf{q}(0,1,0)d = \mathbf{j}d$$

Non Abelian extension

$$\mathcal{A} = \mathcal{A}^B T^B = (\mathbf{1}\psi_1^B + \mathbf{i}\hat{\psi}_0^B + \mathbf{j}A_0^B + \mathbf{k}\hat{A}_1^B)T^B$$

$$\mathcal{V} = \mathcal{V}^B T^B = (\mathbf{1}\hat{a}_0^B + \mathbf{i}a_1^B + \mathbf{j}\hat{\alpha}_1^B + \mathbf{k}\alpha_0^B)T^B$$

Generalized Chern-Simons actions

fermionic-odd

$$S^1 = \int Tr\left[-\psi_1(dA_0 + A_0^2 + \hat{A}_1^2 + \hat{\psi}_0^2) + \hat{A}_1(d\hat{\psi}_0 + [A_0, \hat{\psi}_0]) + \frac{1}{3}\psi_1^3\right],$$

fermionic-even

$$S^i = \int Tr\left[-\hat{\psi}_0(dA_0 + A_0^2 + \hat{A}_1^2 - \psi_1^2) - \hat{A}_1(d\psi_1 + \{A_0, \psi_1\}) - \frac{1}{3}\hat{\psi}_0^3\right],$$

bosonic-odd

$$S^j = \int Tr\left[\underline{-\frac{1}{2}A_0 dA_0 - \frac{1}{3}A_0^3} + \frac{1}{2}\hat{A}_1(d\hat{A}_1 + [A_0, \hat{A}_1]) + \frac{1}{2}\hat{\psi}_0(d\hat{\psi}_0 + [A_0, \hat{\psi}_0]) + \frac{1}{2}\psi_1(d\psi_1 + \{A_0, \psi_1\}) - \hat{\psi}_0\{\psi_1, \hat{A}_1\}\right]$$

bosonic-even

$$S^k = \int Tr\left[\underline{-\hat{A}_1(dA_0 + A_0^2} + \hat{\psi}_0^2 - \psi_1^2) - \frac{1}{3}\hat{A}_1^3 - \psi_1(d\hat{\psi}_0 + [A_0, \hat{\psi}_0])\right]$$

are generalized gauge invariant:

$$\underline{\delta A_0 = d\hat{a}_0 + [A_0, \hat{a}_0]} + \{\hat{A}_1, a_1\} + [\psi_1, \hat{\alpha}_1] - \{\hat{\psi}_0, \alpha_0\},$$

$$\underline{\delta \hat{A}_1 = -da_1 - \{A_0, a_1\}} + [\hat{A}_1, \hat{a}_0] + [\psi_1, \alpha_0] + \{\hat{\psi}_0, \hat{\alpha}_1\},$$

$$\delta\psi_1 = -d\hat{\alpha}_1 - [A_0, \hat{\alpha}_1] - [\hat{A}_1, \alpha_0] + [\psi_1, \hat{a}_0] - [\hat{\psi}_0, a_1],$$

$$\delta\hat{\psi}_0 = d\alpha_0 + \{A_0, \alpha_0\} - \{\hat{A}_1, \hat{\alpha}_1\} + [\psi_1, a_1] + [\hat{\psi}_0, \hat{a}_0]$$

All anti-commutators turn into commutators for Lie algebra !

(hidden grading c)

$$\{A_0, \alpha_0\} = A_0^B \alpha_0^C [T^B, T^C], \quad \{\hat{\psi}_0, \hat{\alpha}_1\} = \hat{\psi}_0^B \hat{\alpha}_1^C [T^B, T^C], \quad \{\hat{A}_1, a_1\} = \hat{A}_1^B a_1^C [T^B, T^C]$$

D-dimensional generalized Chern-Simons actions

generalized gauge fields

$\psi_c^{(i)}$ $\quad i$: i-form

$\phi_1^{(0)}, \omega_0^{(1)}, B_1^{(2)}, \Omega_0^{(3)}, H_1^{(4)}, ..$

$$\begin{aligned}\mathcal{A} &= \mathbf{1}\psi_1 + \mathbf{i}\hat{\psi}_0 + \mathbf{j}A_0 + \mathbf{k}\hat{A}_1 \\ &= \mathbf{1}(\psi_1^{(1)} + \psi_1^{(3)} + \cdots) + \mathbf{i}(\psi_0^{(0)} + \psi_0^{(2)} + \psi_0^{(4)} + \cdots) \\ &+ \mathbf{j}(\omega_0^{(1)} + \Omega_0^{(3)} + \cdots) + \mathbf{k}(\phi_1^{(0)} + B_1^{(2)} + H_1^{(4)} + \cdots),\end{aligned}$$

generalized gauge parameters

$\alpha_c^{(i)}$ $\quad i$: i-form

$v_0^{(0)}, u_1^{(1)}, b_0^{(2)}, U_1^{(3)}, h_0^{(4)}, ..$

$$\begin{aligned}\mathcal{V} &= \mathbf{1}\hat{a}_0 + \mathbf{i}a_1 + \mathbf{j}\hat{\alpha}_1 + \mathbf{k}\alpha_0 \\ &= \mathbf{1}(v_0^{(0)} + b_0^{(2)} + h_0^{(4)} + \cdots) + \mathbf{i}(u_1^{(1)} + U_1^{(3)} + \cdots) \\ &+ \mathbf{j}(\alpha_1^{(0)} + \alpha_1^{(2)} + \alpha_1^{(4)} \cdots) + \mathbf{k}(\alpha_0^{(1)} + \alpha_0^{(3)} + \cdots)\end{aligned}$$

2-grading system

(exclude fermionic gauge fields and parameters)

$$\psi = 0, \quad \alpha = 0$$

$$S_2^k(\psi=0) = \int Tr\left[-\phi_1(d\omega_0 + \omega_0^2) - \phi_1^2 B_1\right]$$
$$S_3^j(\psi=0) = \int Tr\Big[-\frac{1}{2}\omega_0 d\omega_0 - \frac{1}{3}\omega_0^3 + \phi_1(dB_1 + [\omega_0, B_1]) - \Omega_0\phi_1^2\Big],$$
$$S_4^k(\psi=0) = \int Tr\Big[-B_1(d\omega_0 + \omega_0^2) - \phi_1(d\Omega_0 + \{\omega_0, \Omega_0\} + B_1^2 - H_1\phi_1^2\Big]$$

(even-form gauge field) 1

$$\begin{aligned}
\delta\phi_1 &= [\phi_1, v_0], \\
\delta\omega_0 &= dv_0 + [\omega_0, v_0] + \{\phi_1, u_1\}, \\
\delta B_1 &= -du_1 - \{\omega_0, u_1\} + [\phi_1, b_0] + [B_1, v_0], \\
\delta\Omega_0 &= db_0 + [\omega_0, b_0] + \{\phi_1, U_1\} + \{B_1, u_1\} + [\Omega_0, v_0] \\
\delta H_1 &= -dU_1 - \{\omega_0, U_1\} + [\phi_1, h_0] + [B_1, b_0] - \{\Omega_0, u_1\} + [H_1, v_0]
\end{aligned}$$

(odd-form gauge parameter) 1

0-,⋯,4-dimensional generalized Chern-Simons actions

(differential form degree: sector-wise equivalence for G-C-S)

leading as BF type

$$S_0^k = \int Tr\left[-\phi_1(\hat{\psi}_0^{(0)})^2 - \frac{1}{3}\phi_1^3\right]$$

$$S_1^j = \int Tr\left[\underline{\frac{1}{2}\phi_1(d\phi_1 + [\omega_0, \phi_1])} - \hat{\psi}_0^{(0)}\{\psi_1^{(1)}, \phi_1\} + \frac{1}{2}\hat{\psi}_0^{(0)}(d\hat{\psi}_0^{(0)} + [\omega_0, \hat{\psi}_0^{(0)}])\right]$$

$$S_2^k = \int Tr\left[\underline{-\phi_1(d\omega_0 + \omega_0^2} + \{\hat{\psi}_0^{(0)}, \hat{\psi}_0^{(2)}\} - (\psi_1^{(1)})^2) - \psi_1^{(1)}(d\hat{\psi}_0^{(0)} + [\omega_0, \hat{\psi}_0^{(0)}]) - \phi_1^2 B_1\right]$$

$$S_3^j = \int Tr\Big[\underline{-\frac{1}{2}\omega_0 d\omega_0 - \frac{1}{3}\omega_0^3} + \hat{\psi}_0^{(0)}(d\hat{\psi}_0^{(2)} + [\omega_0, \hat{\psi}_0^{(2)}] - \{\psi_1^{(1)}, B_1\} - \{\psi_1^{(3)}, \phi_1\})$$
$$+ \phi_1(dB_1 + [\omega_0, B_1]) - \Omega_0(\phi_1^2 + (\hat{\psi}_0^{(0)})^2)\Big]$$

$$S_4^k = \int Tr\Big[\underline{-B_1(d\omega_0 + \omega_0^2} + \{\hat{\psi}_0^{(0)}, \hat{\psi}_0^{(2)}\} - (\psi_1^{(1)})^2) - \phi_1(d\Omega_0 + \{\omega_0, \Omega_0\} + B_1^2 + \{\hat{\psi}_0^{(0)}, \hat{\psi}_0^{(4)}\}$$
$$- \{\psi_1^{(1)}, \psi_1^{(3)}\}) - H_1((\hat{\psi}_0^{(0)})^2 + \phi_1^2) - \psi_1^{(1)}(d\hat{\psi}_0^{(2)} + [\omega_0, \hat{\psi}_0^{(2)}] + [\Omega_0, \hat{\psi}_0^{(0)}]) - \psi_1^{(3)}[\omega_0, \hat{\psi}_0^{(0)}]\Big]$$

bosonic generalized gauge transformations

$$
\begin{aligned}
\delta\phi_1 &= [\phi_1, v_0] + \{\hat{\psi}_0^{(0)}, \hat{\alpha}_1^{(0)}\}, \\
\underline{\delta\omega_0 = dv_0 + [\omega_0, v_0]} &+ \{\phi_1, u_1\} + [\psi_1^{(1)}, \hat{\alpha}_1^{(0)}] - \{\hat{\psi}_0^{(0)}, \alpha_0^{(1)}\}, \\
\underline{\delta B_1 = -du_1 - \{\omega_0, u_1\}} &+ [\phi_1, b_0] + [B_1, v_0] + \{\hat{\psi}_0^{(0)}, \hat{\alpha}_1^{(2)}\} + [\psi_1^{(1)}, \alpha_0^{(1)}] + \{\hat{\psi}_0^{(2)}, \hat{\alpha}_1^{(0)}\}, \\
\delta\Omega_0 &= db_0 + [\omega_0, b_0] + \{\phi_1, U_1\} + \{B_1, u_1\} + [\Omega_0, v_0] \\
&\quad - \{\hat{\psi}_0^{(0)}, \alpha_0^{(3)}\} + [\psi_1^{(1)}, \hat{\alpha}_1^{(2)}] - \{\hat{\psi}_0^{(2)}, \alpha_0^{(1)}\} + [\psi_1^{(3)}, \hat{\alpha}_1^{(0)}], \\
\delta H_1 &= -dU_1 - \{\omega_0, U_1\} + [\phi_1, h_0] + [B_1, b_0] - \{\Omega_0, u_1\} + [H_1, v_0] \\
&\quad + \{\hat{\psi}_0^{(0)}, \alpha_1^{(4)}\} + [\psi_1^{(1)}, \alpha_0^{(3)}] + \{\hat{\psi}_0^{(2)}, \hat{\alpha}_1^{(2)}\} + [\psi_1^{(3)}, \alpha_0^{(1)}] + \{\hat{\psi}_0^{(4)}, \hat{\alpha}_1^{(0)}\}
\end{aligned}
$$

Kalb-Ramond $\longrightarrow$ BF $\longrightarrow$ this formulation

fermionic generalized gauge transformations

$$
\begin{aligned}
\delta\hat{\psi}_0^{(0)} &= [\hat{\psi}_0^{(0)}, v_0] - \{\phi_1, \hat{\alpha}_1^{(0)}\}, \\
\delta\psi_1^{(1)} &= -d\hat{\alpha}_1^{(0)} - [\omega_0, \hat{\alpha}_1^{(0)}] - [\phi_1, \alpha_0^{(1)}] - [\hat{\psi}_0^{(0)}, u_1] + [\psi_1^{(1)}, v_0], \\
\delta\hat{\psi}_0^{(2)} &= d\alpha_0^{(1)} + \{\omega_0, \alpha_0^{(1)}\} - \{\phi_1, \hat{\alpha}_1^{(2)}\} - \{B_1, \hat{\alpha}_1^{(0)}\} + [\hat{\psi}_0^{(0)}, b_0] + [\psi_1^{(1)}, u_1] + [\hat{\psi}_0^{(2)}, v_0], \\
\delta\psi_1^{(3)} &= -d\hat{\alpha}_1^{(2)} - [\omega_0, \hat{\alpha}_1^{(2)}] - [\phi_1, \alpha_0^{(3)}] - [B_1, \alpha_0^{(1)}] - [\Omega_0, \hat{\alpha}_1^{(0)}] \\
&\quad - [\hat{\psi}_0^{(0)}, U_1] + [\psi_1^{(1)}, b_0] - [\hat{\psi}_0^{(2)}, u_1] + [\psi_1^{(3)}, v_0], \\
\delta\hat{\psi}_0^{(4)} &= d\alpha_0^{(3)} + \{\omega_0, \alpha_0^{(3)}\} - \{\phi_1, \alpha_1^{(4)}\} - \{B_1, \hat{\alpha}_1^{(2)}\} + \{\Omega_0, \alpha_0^{(1)}\} - \{H_1, \hat{\alpha}_1^{(0)}\} \\
&\quad + [\hat{\psi}_0^{(0)}, h_0] + [\psi_1^{(1)}, U_1] + [\hat{\psi}_0^{(2)}, b_0] + [\psi_1^{(3)}, u_1] + [\hat{\psi}_0^{(4)}, v_0]
\end{aligned}
$$

Quantization by Batalin-Vilkovisky formalism

BRST transformation for gauge field and ghost:

$$[Q_B, A^a_\mu] = D_\mu c^a, \quad [Q_B, c^a] = \frac{1}{2} f^a_{bc} c^b c^c$$

nilpotency of: Q_B

$$[Q_B, D_\mu c^a] = 0, \quad [Q_B, \frac{1}{2} f^a_{bc} c^b c^c] = 0$$

Consider the action with external fields:

$$S[J, K] = \int d^d x \left(\mathcal{L} + J^\mu_a A^a_\mu + J_a c^a + K^\mu_a D_\mu c^a - \frac{1}{2} K_a f^a_{bc} c^b c^c \right)$$

Ward-Takahashi identity:

$$0 = \langle [Q_B, e^{iS[J,K]}] \rangle = i \int d^d x \langle \left(J_a^\mu D_\mu c^a - \frac{1}{2} J_a f^a_{bc} c^b c^c \right) e^{iS[J,K]} \rangle$$

$$e^{i\Gamma[J,K]} = \langle e^{iS[J,K]} \rangle$$

$$\Phi^a_\mu = \frac{\partial}{\partial J^\mu_a} \Gamma[J,K] = \langle A^a_\mu e^{iS} \rangle, \quad \Phi^a = \frac{\partial}{\partial J_a} \Gamma[J,K] = \langle c^a e^{iS} \rangle$$

Effective action;

$$W[\Phi, K] = \Gamma[J,K] + J \cdot \Phi \qquad J^\mu_a = \frac{\partial W}{\partial \Phi^a_\mu}, \quad J_a = \frac{\partial W}{\partial \Phi^a}$$

WT-id = Slavnov-Taylor identity:

$$0 = \left(J^\mu_a \cdot \frac{\partial}{\partial K^a_\mu} + J_a \cdot \frac{\partial}{\partial K^a} \right) \Gamma[J,K] = \frac{\partial W}{\partial \Phi^a_\mu} \frac{\partial W}{\partial K^\mu_a} + \frac{\partial W}{\partial \Phi^a} \frac{\partial W}{\partial K_a} = (W, W)$$

$$\delta_{BRST} \Phi = (\Phi, W)$$

fermionic bosonic

Batalin-Vilkovisky quantization is the generalization of this.

Batalin-Vilkovisky quantization

Define Poisson bracket: $\{F,G\} = \frac{F\overleftarrow{\partial}}{\partial\Phi^A}\frac{\partial G}{\partial\Phi^*_A} - \frac{F\overleftarrow{\partial}}{\partial\Phi^*_A}\frac{\partial G}{\partial\Phi^A}$ graded Jacobi id.

degree of anti-field $\Phi^*_A: \quad |\Phi^*_A| = -|\Phi_A| - 1$ $\qquad \frac{F\overleftarrow{\partial}}{\partial\Phi^A} = (-1)^{|\Phi_A|(|F|+|\Phi_A|)}\frac{\partial F}{\partial\Phi_A}$

$$W[\Phi,\Phi^*] = \int d^dx(\mathcal{L} + J^\mu_a A^a_\mu + J_a c^a + A^{\mu*}_a D_\mu c^a - c^*_a\frac{1}{2}f^a_{bc}c^b c^c + \bar{c}^*_a b^a)$$

non-minimal sector (gauge fixing)

Quantized gauge action satisfies **master equation:**

$$\{W,W\} = 2\frac{\partial W}{\partial A^a_\mu}\frac{\partial W}{\partial A^{\mu*}_a} - 2\frac{\partial W}{\partial c^a}\frac{\partial W}{\partial c^*_a} - 2\underbrace{\frac{\partial W}{\partial \bar{c}^a}\frac{\partial W}{\partial \bar{c}^*_a}}_{=0}$$

$$= 2\langle J^\mu_a D_\mu c^a - J_a\frac{1}{2}f^a_{bc}c^b c^c\rangle = 0$$

$$\{W,W\} = 0$$

BRST transformation: $\delta_B\Phi^A = \{W,\Phi^A\}$

anti-field Φ^*_A can be replaced by derivative of gauge fermion Ψ

$$\Phi^*_A = \frac{\Psi \overleftarrow{\partial}}{\partial \Phi^A} \qquad (\mathrm{deg}.\Psi = -1)$$

$$\int D\Phi^A e^{\frac{i}{\hbar} W(\Phi, \frac{(\Psi+\Delta\Psi)\overleftarrow{\partial}}{\partial\Phi^A})} - \int D\Phi^A e^{\frac{i}{\hbar} W(\Phi, \frac{\Psi\overleftarrow{\partial}}{\partial\Phi^A})}$$
$$\propto \int D\Phi^A \Delta\Psi \left(i\hbar \Delta_{BV} W - \frac{1}{2}\{W, W\}\right) e^{\frac{i}{\hbar}W}$$

Yang-Mills

$$\Psi = \overline{c}^a \partial_\mu A^\mu_a$$
$$A^{a*}_\mu = \frac{\partial \Psi}{\partial A^\mu_a} = \overline{c}^a \partial_\mu$$
$$c^*_a = -\frac{\partial\Psi}{\partial c^a} = 0$$
$$\overline{c}^*_a = -\frac{\partial\Psi}{\partial \overline{c}^a} = \partial_\mu A^\mu_a$$

W is independent of the choice of gauge fixing fermion Ψ

if the quantum master equation is satisfied:

$$i\hbar \Delta_{BV} W - \frac{1}{2}\{W, W\} = 0 \qquad W = \int d^4x(\mathcal{L} + \overline{c}_a \partial_\mu D^\mu c^a + b^a \partial_\mu A^\mu_a)$$

$$\hbar \to 0 : \text{calssical master equation}, \quad \{W, W\} = 0$$

Symplectic structure in classical mechanics

Hamiltonian; $H(q,p) = p_i \dot{q}^i - L(q,\dot{q}), \qquad p_i = \frac{\partial L}{\partial \dot{q}^i}$

$$\frac{\partial H}{\partial p_i} = \dot{q}^i, \qquad \frac{\partial H}{\partial q^i} = -\frac{\partial L}{\partial q^i} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = -\dot{p}_i$$

(BRST tr.) $\dot{F}(q,p) = \{F, H\} = \frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i} = \frac{\partial F}{\partial q^i}\dot{q}^i + \frac{\partial F}{\partial p_i}\dot{p}_i$

Poisson bracket

(Master eq.) $\frac{\partial H}{\partial t} = \{H, H\} = 0$ Energy conservation

Lagrangian: $L(q,\dot{q}) = p_i \dot{q}^i - H(q,p)$

Action: $S = \int dt (p\dot{q} - H)$

AKSZ construction of graded symplectic manifold

(Alexandrov, Kontsevich, Schwartz and Zaboronsky)

QP manifold: $(M, \{\cdot,\cdot\}, \Theta), \quad \{\Theta, \Theta\} = 0$

graded Mfd.

Hamiltonian

graded Poisson bracket

grade 1 vector field (BRST charge)

$$Q\cdot \equiv \{\Theta, \cdot\}$$

$$\left(Q = Q^i \frac{\partial}{\partial x^i} = \frac{\partial \Theta}{\partial q^i}\frac{\partial}{\partial p_i} - \frac{\partial \Theta}{\partial p_i}\frac{\partial}{\partial q^i}, \ (x^i) = (q^i, p_i) \right)$$

$$Q^2 \cdot = Q\{\Theta, \cdot\} = \{\Theta, \{\Theta, \cdot\}\} = \frac{1}{2}\{\underbrace{\{\Theta, \Theta\}}_{=0}, \cdot\} = 0$$

graded Jacobi identity

Every solution to the classical master equation determines
QP manifold = quantized physical phase space

closed symplectic two form: $d\omega = 0,\ \omega = \omega_{ab} dx^a \wedge dx^b$

Poisson bracket: $\{F, G\} = (\omega^{-1})^{ab} \dfrac{\partial F}{\partial x^a} \dfrac{\partial G}{\partial x^b}$

Symplectic: $\mathcal{L}_Q \omega = (i_Q d + d i_Q)\omega = d i_Q \omega = 0$ $\quad \left(i_Q = Q^a \dfrac{\partial}{\partial (dx^a)} \right)$

existence of Hamiltonian Θ : $i_Q \omega = d\Theta \longrightarrow \{\Theta, \Theta\} = 0$

AKSZ construction of quantized action:

$$S = \int d^d x d^d \theta (p \mathbf{d} q + \Theta) \quad (\mathbf{d} = \theta^\mu \partial_\mu,\ \theta_\mu \text{ as odd coordinate})$$

QP manifold can be geometrically understood as follows:

Consider the differential equations: $v^i(t) = \frac{dx^i(t)}{dt} = X^i(x(t))$ (*)

Assume grade 1 vector field: $X = X^i \frac{\partial}{\partial x^i}$

$$x^i(t) = x^i + tv^i \qquad \deg\{t\} = -1, \quad t^2 = 0$$

$$X^i(x^i(t)) = X^i(x + vt) = X^i(x) + \underbrace{tv^j \frac{\partial X^i}{\partial x_j}}_{\displaystyle \begin{gathered} \| \\ tX^j \frac{\partial X^i}{\partial x_j} \end{gathered}}$$

$$\frac{dX^i(x(t))}{dt} = X^j \frac{\partial X^i}{\partial x^j} = 0$$ integrability of (*)

stationally flow (time independent)

$$X^i = Q^i \qquad Q^2|_i = Q^j \frac{\partial Q^i}{\partial x^j} = 0 \leftrightarrow \{\Theta, \Theta\} = 0$$

Explicit examples QP mfd

graded coordinate: (q^a, p_a) grading $(1, n-1)$: $(x^i \sim (q^a, p_a))$

Poisson bracket:

$$\{F, G\} = \frac{F\overleftarrow{\partial}}{\partial q^a}\frac{\partial G}{\partial p_a} + (-1)^n \frac{F\overleftarrow{\partial}}{\partial p_a}\frac{\partial G}{\partial q^a} \qquad \left(\frac{F\overleftarrow{\partial}}{\partial x^a} = (-1)^{|x^a|(|F|-|x^a|)}\frac{\partial F}{\partial x^a}\right)$$

Hamiltonian, ex.1 : $\Theta(q,p) = \frac{1}{2} f_{bc}^a p_a q^b q^c \qquad \{\Theta, \Theta\} = 0$

QP manifold: $(M, \{\cdot,\cdot\}, \Theta) = 0$ (find Θ to get QP-mfd)

nilpotent degree 1 vector field =BRST charge:

$$Q\cdot = \{\Theta, \cdot\} = \frac{\Theta\overleftarrow{\partial}}{\partial q^a}\frac{\partial}{\partial p_a} + \frac{\Theta\overleftarrow{\partial}}{\partial p_a}\frac{\partial}{\partial q^a} = (-1)^n\left(f_{ab}^c q^b p_c \frac{\partial}{\partial p_a} + \frac{1}{2} f_{bc}^a q^b q^c \frac{\partial}{\partial q^a}\right)$$

$$Q^2 = 0$$

$$\Sigma \to \mathcal{M} : (x, \theta) \mapsto (q^a, p^a) = (\Phi^a(x,\theta), \Psi^a(x,\theta))$$

field theory projection

$$\Psi^a(x,\theta) = c^a(x) + \theta^\mu A^a_\mu + \frac{1}{2}\theta^\mu\theta^\nu \phi^{*a}_{\mu\nu}(x),$$

$$\Phi_a(x,\theta) = \phi_a(x) + \theta^\mu A^*_{a,\mu}(x) + \frac{1}{2}\theta^\mu\theta^\nu c^*_{a,\mu\nu}(x)$$

$(\varphi^* = \text{anti-field of } \varphi)$

$(|\varphi^*| = -|\varphi| - 1)$

$$\mathbf{d} = \theta^\mu \partial_\mu$$

$$\left(\int d^2x d^2\theta \ \theta^\mu\theta^\nu \Phi_{\mu\nu} = \int \Phi_{\mu\nu} dx^\mu \wedge dx^\nu\right)$$

AKSZ action

$$W = \int d^2x \int d^2\theta (\Phi_a \mathbf{d}\Psi^a + \Theta(\Psi, \Phi))$$

$$= \int d^2x \int d^2\theta \ \Phi_a \left(\mathbf{d}\Psi^a + \frac{1}{2} f^a_{bc}\Psi^b\Psi^c\right)$$

$$= \int d^2x \left(\frac{1}{2}\phi_a \epsilon^{\mu\nu} F^a_{\mu\nu} - \epsilon^{\mu\nu} A^*_{a,\nu}(\partial_\mu c^a + f^a_{bc} A^b_\mu c^c) + \frac{1}{4}\epsilon^{\mu\nu} c^*_{a,\mu\nu} f^a_{bc} c^b c^c + \frac{1}{2}\epsilon^{\mu\nu}\phi^{*b}_{\mu\nu} f^a_{bc}\phi_a c^c\right)$$

quantized 2-d. minimal BF action by formalism

fermions and 2-form boson can be identified as anti-fields

AKSZ construction coincides with 2-dim. G-C-S action

Graded manifold $\mathcal{M}$: coordinates: (q^a, p_{1a})

grading $(1, 0)$

p_{1a} has independent hidden grading 1

Poisson bracket: $$\{F, G\} = \frac{F\overleftarrow{\partial}}{\partial q^a}\frac{\partial G}{\partial p_{1a}} - \frac{F\overleftarrow{\partial}}{\partial p_{1a}}\frac{\partial G}{\partial q^a}$$

Hamiltonian, ex.2: $$\Theta(q^a, p_{1a}) = \Theta_1(q^a, p_{1a}) + \Theta_3(q^a, p_{1a})$$

$$\Theta(q^a, p_{1a}) = \Theta_1(q^a, p_{1a}) + \Theta_3(q^a, p_{1a})$$

$$\Theta_1(q^a, p_{1a}) = \frac{1}{2} f_{bc}^a p_{1a} q^b q^c \quad \longleftrightarrow \quad \Theta(q, p) = \frac{1}{2} f_{bc}^a p_a q^b q^c$$

$$\Theta_3(q^a, p_{1a}) = \frac{1}{3!} f_{abc} p_{1a} p_{1b} p_{1c}$$

$$\{\Theta, \Theta\} = \{\Theta_1 + \Theta_3, \Theta_1 + \Theta_3\} = 0$$

Map: $\Sigma \to \mathcal{M} : (x^\mu, \theta^\mu) \mapsto (q^a, p_{1a}) = (\Psi(x,\theta), \Phi(x,\theta))$

component fields:

$$\Phi^a(x,\theta) = \phi_1^a(x) + \theta^\mu \psi_{1,\mu}^a(x) + \frac{1}{2}\theta^\mu\theta^\nu B_{1,\mu\nu}^a(x)$$

$$\Psi^a(x,\theta) = \hat{\psi}_0^a(x) + \theta^\mu \omega_{0,\mu}^a(x) + \frac{1}{2}\theta^\mu\theta^\nu \hat{\psi}_{0,\mu\nu}^a(x)$$

$$W_{AKSZ} = \int d^2x d^2\theta \left(\Phi^a \mathbf{d}\Psi_a + \Theta_1(\Phi,\Psi) + \Theta_3(\Phi,\Psi)\right)$$

$$= \int d^2x d^2\theta \left(\Phi^a \mathbf{d}\Psi_a + \frac{1}{2} f_{bc}^a \Phi_a \Psi^b \Psi^c + \frac{1}{3!} f_{abc} \Phi^a \Phi^b \Phi^c\right)$$

$$= \int Tr\{\phi_1(d\omega_0 + \omega_0^2 + \{\hat{\psi}_0^{(0)}, \hat{\psi}_0^{(2)}\} + \psi_1^2)$$

$$+ \psi_1(d\hat{\psi}_0^{(0)} + [\omega_0, \hat{\psi}_0^{(0)}]) + B_1(\phi_1^2 + (\hat{\psi}_0^{(0)})^2)\}$$

ghost #=2

Coincides with 2-dim. generalized C-S with fermions

(different gradings anti-commute here)

AKSZ construction and BV formalism

(ghost #, hidden grading)

(0,1) $$\Phi^a(x,\theta) = \phi_1^a(x) + \theta^\mu \psi_{1,\mu}^a(x) + \frac{1}{2}\theta^\mu\theta^\nu B_{1,\mu\nu}^a(x)$$

(1,0) $$\Psi^a(x,\theta) = \hat{\psi}_0^a(x) + \theta^\mu \omega_{0,\mu}^a(x) + \frac{1}{2}\theta^\mu\theta^\nu \hat{\psi}_{0,\mu\nu}^a(x)$$

$\Downarrow$ anti-fields identification

$$\Phi^a(x,\theta) = \phi_1^a(x) - \theta^\mu \epsilon_{\mu\nu}\omega_{1a}^{*\nu}(x) + \theta^1\theta^2 c_1^{*a}(x)$$
$$\Psi^a(x,\theta) = c_0^a(x) + \theta^\mu \omega_{0\mu}^a(x) + \theta^1\theta^2 \phi_0^{*a}(x)$$

$$(|\varphi^*| = -|\varphi| - 1)$$

physical field (ghost #=0)

We define BV bracket:

$$(F,G) = \int d^2x \left(\frac{F\overleftarrow{\partial}}{\partial\phi_{1a}}\frac{\partial G}{\partial\phi_0^{*a}} - \frac{F\overleftarrow{\partial}}{\partial\phi_0^{*a}}\frac{\partial G}{\partial\phi_{1a}} + \frac{F\overleftarrow{\partial}}{\partial\omega_{0\mu}^a}\frac{\partial G}{\partial\omega_{1a}^{*\mu}} - \frac{F\overleftarrow{\partial}}{\partial\omega_{1a}^{*\mu}}\frac{\partial G}{\partial\omega_{0\mu}^a} + \frac{F\overleftarrow{\partial}}{\partial c_0^a}\frac{\partial G}{\partial c_{1a}^*} - \frac{F\overleftarrow{\partial}}{\partial c_{1a}^*}\frac{\partial G}{\partial c_0^a} \right)$$

$$\left(\frac{F\overleftarrow{\partial}}{\partial\Phi} = (-1)^{|\Phi|(|\Phi|+|F|)}(-1)^{[\Phi]([\Phi]+[F])}\frac{\partial F}{\partial\Phi} \right)$$

hidden grading

$$W_{AKSZ} = W_{BF} + W'$$

$$W_{AKSZ} = W_{BF} + W' \qquad\qquad (W_{AKSZ}, W_{AKSZ}) = 0$$

Quantized minimal BF action + modification

$$W_{BF} = \int d^2x \, \frac{1}{2}\phi_{1a}\epsilon^{\mu\nu}F^a_{\mu\nu} - \int d^2x \left(\omega^{*\mu}_{1a}(\partial_\mu c^a_0 + f^a_{bc}\omega^b_{0\mu}c^c_0) + \frac{1}{2}f^a_{bc}c^*_{1a}c^b_0 c^c_0 - \phi^{*a}_0 f^c_{ab}c^b_0\phi_{1c} \right)$$

$$W' = \frac{1}{2}\int d^2x \left(f^a_{bc}\phi_{1a}\epsilon_{\mu\nu}\omega^{*b\mu}_1\omega^{*c\nu}_1 + f_{abc}\phi^a_1\phi^b_1 c^{*c}_1 \right)$$

Puzzle: W_{AKSZ} is equivalent to the 2-dim. generalized Chern-Simons action with fermions and at the same time quantized BF action with extra term W'

ghost number of $B^a_1 = c^{*a}_1 = -2$

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \theta^1\theta^2 c^{*a}_1(x) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \\ \Phi^a(x,\theta) = \phi^a_1(x) + \theta^\mu\psi^a_{1,\mu}(x) + \dfrac{1}{2}\theta^\mu\theta^\nu B^a_{1,\mu\nu}(x) \\ \Psi^a(x,\theta) = \hat{\psi}^a_0(x) + \theta^\mu\omega^a_{0,\mu}(x) + \dfrac{1}{2}\theta^\mu\theta^\nu\hat{\psi}^a_{0,\mu\nu}(x) \\ \qquad\qquad\quad\;\; \| \\ \qquad\qquad\quad c^a_0(x) \end{array}$$

On the other hand we can consider 2-dim. generalized C-S action as starting classical action :

simply two form gauge field with ghost number 0

(bosons only) $$S_2^k(\psi=0) = \int Tr\left[-\phi_1(d\omega_0+\omega_0^2) - \phi_1^2 B_1\right]$$

gauge trans.
$$\begin{aligned}
\delta\phi_1 &= [\phi_1, v_0], \\
\delta\omega_0 &= dv_0 + [\omega_0, v_0] + \{\phi_1, u_1\}, \\
\delta B_1 &= -du_1 - \{\omega_0, u_1\} + [\phi_1, b_0] + [B_1, v_0],
\end{aligned}$$

We need to introduce the following fields:

$$X^a,\ (0,1),\ Y^a,\ (1,0),\ Z^a,\ (2,1), \qquad \text{(ghost number, hidden grading)}$$

$$\begin{aligned}
X^a &= \phi_1^a + \cdots \\
Y^a &= \cdots + \theta^\mu \omega_{0\mu} + \cdots \\
Z^a &= \cdots + \frac{1}{2}\theta^\mu\theta^\nu B_{1\mu\nu}
\end{aligned}$$

finite number of fields is not enough to obtain nilpotent Q with physical ϕ_1, ω_0, B_1 ghost # =0

$\longleftrightarrow$ infinite reducibility

We consider a supermanifold which has the following coordinates:

$$(X^a_{(2m)}, Y^a_{(2m+1)}), \quad (m = -\infty, \cdots, \infty)$$

where the suffix denotes ghost number. X has hidden deg. 1 and Y has 0.

We can find deg. 1 nilpotent vector field (BRST operator):

$$\mathcal{Q} = \sum_m \mathcal{Q}_m, \qquad \mathcal{Q}^2 = 0$$

$$\mathcal{Q}_{2m+1} = \frac{A}{2} f^a_{bc} \left(\sum_k Y^b_{(2k+1)} Y^c_{(-2k+2m+1)} \right) \frac{\partial}{\partial Y^a_{(2m+1)}} + \frac{B}{2} f^a_{bc} \left(\sum_k X^b_{(2k)} X^c_{(-2k+2m+2)} \right) \frac{\partial}{\partial Y^a_{(2m+1)}}$$

$$\mathcal{Q}_{2m} = A f^a_{bc} \left(\sum_k Y^b_{(2k+1)} X^c_{(-2k+2m)} \right) \frac{\partial}{\partial X^a_{(2m)}} \qquad (A, B = \text{const.})$$

We can find hamiltonian Θ satisfying:

$$\mathcal{Q}^2 \cdot = \{\Theta, \{\Theta, \cdot\}\} = -\frac{1}{2}\{\{\Theta, \Theta\}, \cdot\} = 0$$

Hamiltonian:

$$\Theta = \frac{A}{2}\sum_k \sum_m f_{abc} Y^b_{(2k+1)} Y^c_{(-2k+2m+1)} X^a_{(-2m)} + \frac{B}{3!}\sum_k \sum_m f_{abc} X^a_{(2k)} X^b_{(-2k+2m+2)} X^c_{(-2m)}$$

AKSZ action: $$S = \int d^2x d^2\theta \left(\sum_m X_{(-2m)a} \mathbf{d} Y^a_{(2m+1)} + \Theta \right)$$

We already found in our BV minimal action for 2-dim. G-C-S
with infinite reducibility. (N.K., Suehiro, Tsukioka & Umetsu '98)

$$\tilde{S} = -\int d^2x \mathrm{tr} \sum_{n=-\infty}^{\infty} \{ C^B_n (\epsilon^{\mu\nu} \partial_\mu C^B_{n\nu} + \sum_{m=\infty}^{\infty} (\epsilon^{\mu\nu} C^B_{m\mu} C^B_{-(m+n)\nu} + \{C^F_m, \tilde{C}^F_{-(m+n)}\} - \epsilon^{\mu\nu} C^F_{m\mu} C^F_{-(m+n)\nu}))$$
$$+ \tilde{C}^B_n \sum_{m=-\infty}^{\infty} (C^F_m C^F_{-(m+n)} + C^B_m C^B_{-(m+n)})$$
$$- C^F_n \epsilon^{\mu\nu} (\partial_\mu C^F_{-n\mu} + \sum_{m=-\infty}^{\infty} [C^B_{m\mu}, C^F_{-(m+n)\nu}])\}$$

infinite reducibility:

gauge tr. $\mathcal{V}_2 = \delta_1 \mathcal{A} = [Q + \mathcal{A}, \mathcal{V}_1]$ (eq. of motion) $\mathcal{F} = 0$

$$\delta_2 \delta_1 \mathcal{A} = [Q + \mathcal{A}, \mathcal{V}_2] = \{Q + \mathcal{A}, [Q + \mathcal{A}, \mathcal{V}]\} = \frac{1}{2}[\{Q + \mathcal{A}, Q + \mathcal{A}\}, \mathcal{V}] = \frac{1}{2}[\mathcal{F}, \mathcal{V}] = 0$$

In the current AKSZ procedure we can identify:

$$Y^a_{(2m+1)} = C^{F,a}_{2m+1} + \theta^\mu C^{B,a}_{2m\mu} + \frac{1}{2} C^{F,a}_{2m-1\mu\nu}$$

$$X^a_{(2m)} = C^{B,a}_{2m} + \theta^\mu C^{F,a}_{2m-1\mu} + \frac{1}{2} C^{B,a}_{2m-2\mu\nu}$$

renaming

$$Y^a_{(2m+1)} = \psi^a_{(2m+1)} + \theta^\mu \omega^a_{(2m)\mu} + \frac{1}{2}\theta^\mu\theta^\nu \hat{\psi}^a_{(2m-1)\mu\nu}$$

$$X^a_{(2m)} = \phi^a_{(2m)} + \theta^\mu \psi^a_{(2m-1)\mu} + \frac{1}{2}\theta^\mu\theta^\nu B^a_{(2m-2)\mu\nu}$$

$$\Psi^a = \sum_m Y^a_{(2m+1)} = \sum_m \psi^a_{(2m+1)} + \theta^\mu \sum_m \omega^a_{(2m)\mu} + \frac{1}{2}\theta^\mu\theta^\nu \sum_m \hat{\psi}^a_{(2m-1)\mu\nu}$$
$$= \boxed{\psi^a + \theta^\mu \omega^a_\mu + \theta^\mu\theta^\nu \hat{\psi}^a_{\mu\nu}}$$

$$\Phi^a = \sum_m X^a_{(2m)} = \sum_m \phi^a_{(2m)} + \theta^\mu \sum_m \psi^a_{(2m-1)\mu} + \frac{1}{2}\theta^\mu\theta^\nu \sum_m B^a_{(2m-2)\mu\nu}$$
$$= \boxed{\phi^a + \theta^\mu \psi^a_\mu + \theta^\mu\theta^\nu B^a_{\mu\nu}}$$

In this way classical fields $\phi^a, \omega^a_\mu, B^a_{\mu\nu}$ can be introduced as physical fields having ghost number zero for bosons. At this stage fermions still carry ghost number. We can extend to introduce fermionic classical fields (ghost #=0). We need to introduce infinite ghost fields.

What is the physical meaning of the result ?

→ Quantization of topological Super Point Particle Field Theory

We introduce particle coordinates: $(x^\mu(\tau), p_\mu(\tau)) \quad (\theta^\mu(\tau), \eta_\mu(\tau))$

nilpotent BRST transformation

$$\begin{aligned}
\delta_B x^\mu(\tau) &= \theta^\mu(\tau) \\
\delta_B \theta^\mu(\tau) &= 0 \\
\delta_B p_\mu(\tau) &= 0 \\
\delta_B \eta_\mu(\tau) &= -p_\mu(\tau)
\end{aligned}$$

Consider the following BRST exact action
(gauge fixing of topological point particle):

$$\begin{aligned}
S = -\int d\tau \delta_B(\eta_\mu(\dot{x}^\mu - p^\mu)) &= -\int d\tau(\delta_B\eta_\mu(\dot{x}^\mu - p^\mu) - \eta_\mu\delta_B\dot{x}^\mu) \\
&= \int d\tau(p_\mu\dot{x}^\mu - p_\mu p^\mu + \eta_\mu\dot{\theta}^\mu)
\end{aligned}$$

equation of motion:

$$2p_\mu = \dot{x}^\mu$$

$$S_{\text{on-shell}} = -\int d\tau \left(\frac{1}{4}\dot{x}^\mu\dot{x}_\mu + \eta_\mu\dot{\theta}^\mu\right)$$

conjugate momentum:

$$\pi_{x,\mu} = \frac{\mathcal{L}\overleftarrow{\partial}}{\partial \dot{x}^\mu} = p_\mu$$

$$\pi_{\theta,\mu} = \frac{\mathcal{L}\overleftarrow{\partial}}{\partial \dot{\theta}^\mu} = \eta_\mu$$

commutation relation:

$$[x^\mu, p_\nu] = i\delta^\mu_\nu$$

$$\{\theta^\mu, \eta_\nu\} = i\delta^\mu_\nu$$

first quantized BRST charge: $Q_B = i\theta^\mu p_\mu = \theta^\mu \partial_\mu = \mathbf{d}$

super coordinate eigenstate: $|x, \theta>$

field of super point particle:

bosonic: $\Phi(x,\theta) = <x,\theta|\Phi> = \phi(x) + \theta^\mu \psi_\mu(x) + \frac{1}{2}\theta^\mu\theta^\nu B_{\mu\nu}$

fermionic: $\Psi(x,\theta) = <x,\theta|\Psi> = \hat{\psi}(x) + \theta^\mu \omega_\mu(x) + \frac{1}{2}\theta^\mu\theta^\nu \hat{\psi}_{\mu\nu}$

Point particle field theory

physical state condition:

$$< x,\theta|Q_B|\Phi >= \theta_\mu \partial_\mu \Phi(x,\theta) = 0$$
$$< x,\theta|Q_B|\Psi >= \theta_\mu \partial_\mu \Psi(x,\theta) = 0$$

Since BRST symmetry is nilpotent, there is gauge symmetry:

$$\delta|\Phi >= Q_B|\Lambda_B >, \quad \delta|\Psi >= Q_B|\Lambda_F >$$

Action satisfying physical state condition:

$$S =< \Psi|Q_B|\Phi >= \int d^2x d^2\theta \ \Psi(x,\theta) Q_B \Phi(x,\theta)$$

We can introduce Yukawa interaction and 3-point interaction in BRST invariant way:

$$S = \int d^2x d^2\theta \left(\Phi^a \mathbf{d} \Psi_a + \frac{1}{2} f^a_{bc} \Phi_a \Psi^b \Psi^c + \frac{1}{3!} f_{abc} \Phi^a \Phi^b \Phi^c \right)$$

Quantized interacting topological super particle field theory action

This coincides with even dimensional generalized Chern-Simons action !

Comparison with AKSZ construction

Introduce grading: $\quad \mathrm{deg.\ of}\ (q^a, q_a) = (1,0)$

$$\{F,G\} = \frac{F\overleftarrow{\partial}}{\partial q^a}\frac{\partial G}{\partial q_a} - \frac{F\overleftarrow{\partial}}{\partial q_a}\frac{\partial G}{\partial q^a} \qquad \Theta = \frac{1}{3!} f_{abc} q^a q^b q^c, \quad \{\Theta,\Theta\} = 0$$

$$S = \int \left(\frac{1}{2} q_a \mathbf{d} q^a + \frac{1}{3!} f_{abc} q^a q^b q^c \right)$$

We claimed this structure can be extended more general framework with 3 gradings:

$$q^a \to \mathcal{A}^a = \mathbf{1}\psi_1^a + \mathbf{i}\hat{\psi}_0^a + \mathbf{j}A_0^a + \mathbf{k}\hat{A}_1^a \quad Q = jd$$

$$\mathcal{V}^a = \mathbf{1}\hat{a}_0^a + \mathbf{i}a_1^a + \mathbf{j}\hat{\alpha}_1^a + \mathbf{k}\alpha_0^a$$

$$S_{GCS} = \int Tr\left(\frac{1}{2}\mathcal{A}Q\mathcal{A} + \frac{1}{3}\mathcal{A}^3\right) \qquad (\mathcal{A} = \mathcal{A}^a T^a, \quad \mathcal{V} = \mathcal{V}^a T^a)$$

$$\delta\mathcal{A} = Q\mathcal{V} + [\mathcal{A}, \mathcal{V}]$$

$$
\begin{aligned}
\mathcal{A}^a &= \mathbf{1}\psi_1^a + \mathbf{i}\hat{\psi}_0^a + \mathbf{j}A_0^a + \mathbf{k}\hat{A}_1^a \\
&= \mathbf{1}\psi_1^a + \mathbf{k}\hat{A}_1^a + \mathbf{i}(\hat{\psi}_0^a - \mathbf{k}A_0^a) \\
&\to (\psi_1^a + \hat{A}_1^a) + (\hat{\psi}_0^a - A_0^a) \\
&\equiv \Phi_1^a \quad + \quad \Psi_0^a = q^a
\end{aligned}
$$

quaternion + commuting different gradings

||

no quaternion + anti-commuting different gradings

no-quaternion + 3 total gradings:

$$
\begin{aligned}
S &= \int \left(\frac{1}{2} q_a \mathbf{d} q^a + \frac{1}{3!} f_{abc} q^a q^b q^c \right) \\
&= \int \left[\frac{1}{2} (\Phi_{1a} + \Psi_{0a}) \mathbf{d} (\Phi_1^a + \Psi_0^a) + \frac{1}{3!} f_{abc} (\Phi_1^a + \Psi_0^a)(\Phi_1^b + \Psi_0^b)(\Phi_1^c + \Psi_0^c) \right] \\
&= S_E + S_O
\end{aligned}
$$

target space dim. Quantized topological super particle field theory actions

even dim. $$S_E = \int \left(\Phi_{1a} \mathbf{d} \Psi_0^a + \frac{1}{2} f_{abc} \Phi_1^a \Psi_0^b \Psi_0^c + \frac{1}{3!} f_{abc} \Phi_1^a \Phi_1^b \Phi_1^c \right)$$

odd dim. $$S_O = \int \left(\frac{1}{2} (\Phi_{1a} \mathbf{d} \Phi_1^a + \Psi_{0a} \mathbf{d} \Psi_0^a) + \frac{1}{2} f_{abc} \Psi_0^a \Phi_1^b \Phi_1^c + \frac{1}{3!} f_{abc} \Psi_0^a \Psi_0^b \Psi_0^c \right)$$

hidden grading is needed for S_O

Conclusion

- Quantization of G-C-S actions is reformulated by AKSZ formalism.
- AKSZ formulation of G-C-S is quantized topological super point particle field theory for arbitrary even- and odd-dimensions.
- Odd dimensional formulation needs 3rd grading.
- To keep component fields of G-C-S as physical classical fields infinite components are needed.