Entanglement Entropy for probe branes.

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work with Han-Chih Chang and Christoph Uhlemann

talk at the International Workshop on Strings, Black Holes and Quantum Information, Sendai, Sep 7, 2015

Motivation

Entanglement and Entanglement Entropy have emerged as two of the most important players in the recent attempts to reconstruct spacetime from quantum information.

Motivation

Some of the more interesting holographic setups involve probe objects.

- quantum liquids with unusual correlations
- quantum Hall systems
- EPR pairs

Motivation:

Question: How to calculate holographic entanglement entropies for probe branes?

Probe branes

Probe Branes

(AK, Katz)

d+1 dimensional spacetime

n+1 dimensional worldvolume

$$\frac{L^{d-1}}{G_N} \gg T_0 L^{n+1} \gg 1$$

Newton's constant

Probe brane tension **N**

N^2

L=curvature radius of background geometry

Probe Branes

(AK, Katz)

Gravity classical
dominant physics at $O(N^2)$
loops contribute at $O(1)$

"the glue"

$$\frac{L^{d-1}}{G_N} \gg 1$$

Newton's constant

N^2

L=curvature radius of background geometry

Probe Branes

(AK, Katz)

Worldvolume equations classical
dominant flavor physics at O(N)
loops contribute at O(1)

“the quarks”

$$T_0 L^{n+1} \gg 1$$

Probe brane tension N

L=curvature radius of background geometry

Probe Branes

(AK, Katz)

No gravitational backreaction from probe brane
O(N^2) physics unaffected by order N physics
O(N) physics: wordvolume of flavor brane and leading order backreaction
at O(1) all hell breaks lose: 2nd order backreaction, glue loops, quark loops

$$\frac{L^{d-1}}{G_N} \gg T_0 L^{n+1} \gg 1$$

Newton's constant

Probe brane tension

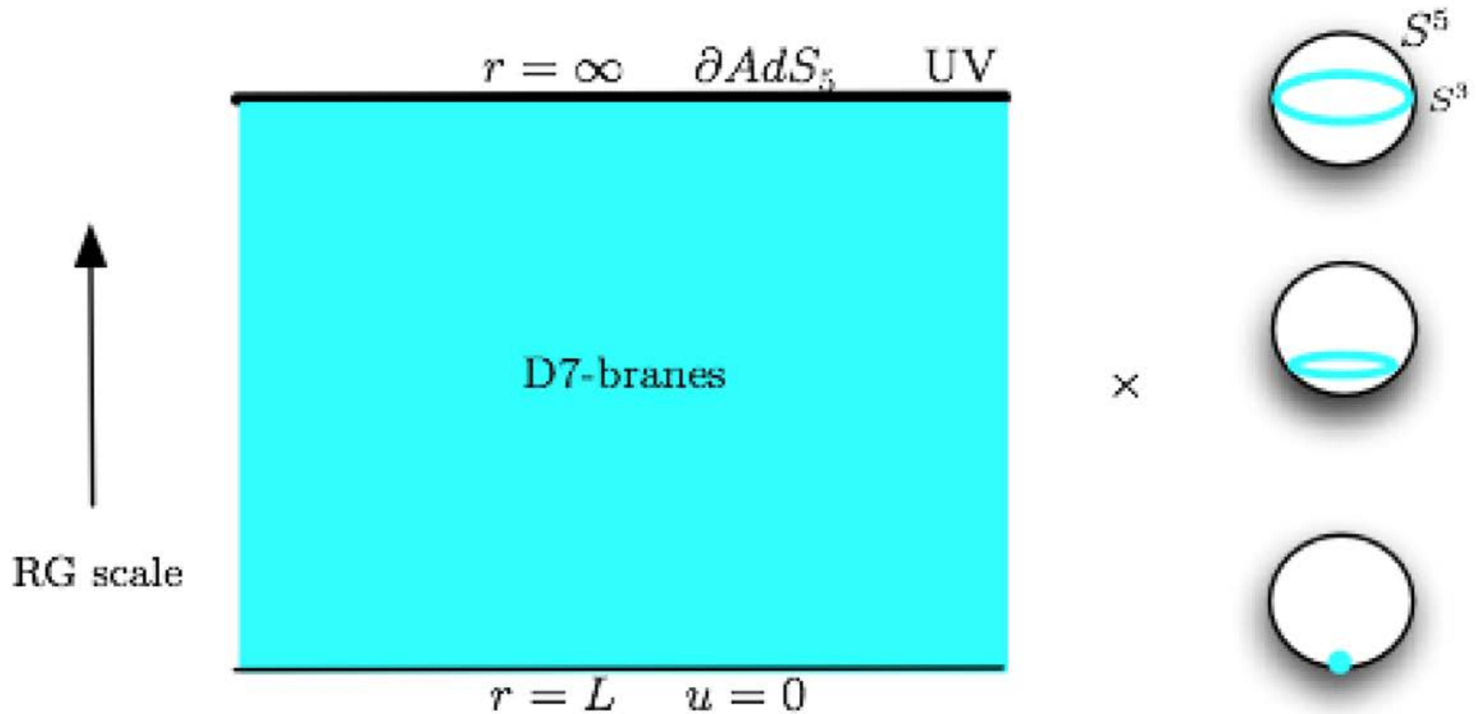
N

N^2

L=curvature radius of background geometry

Example: Flavor Branes

(AK, Katz)

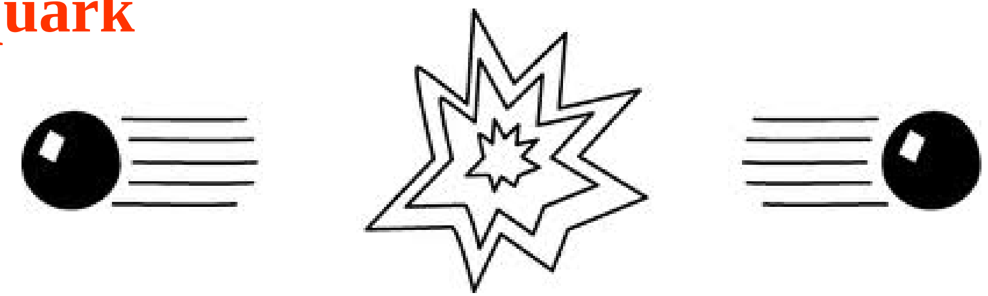


(picture from CLMRW-review, 2011)

Example: holographic EPR

anti-quark

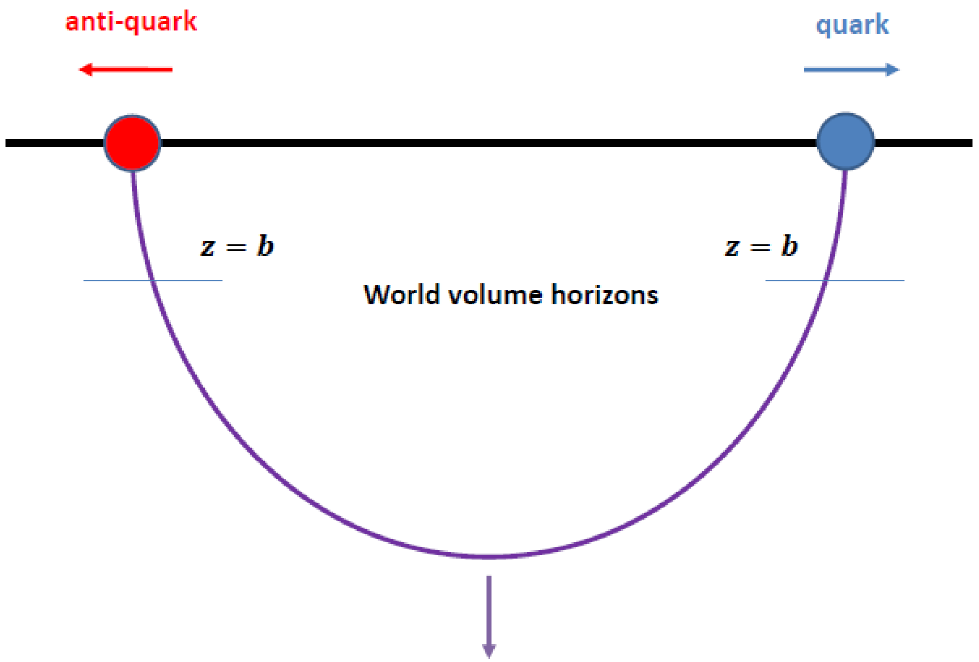
quark



pair created in background electric field

$$|\Psi> = \frac{1}{\sqrt{2}}(|\uparrow\downarrow> - |\downarrow\uparrow>)$$

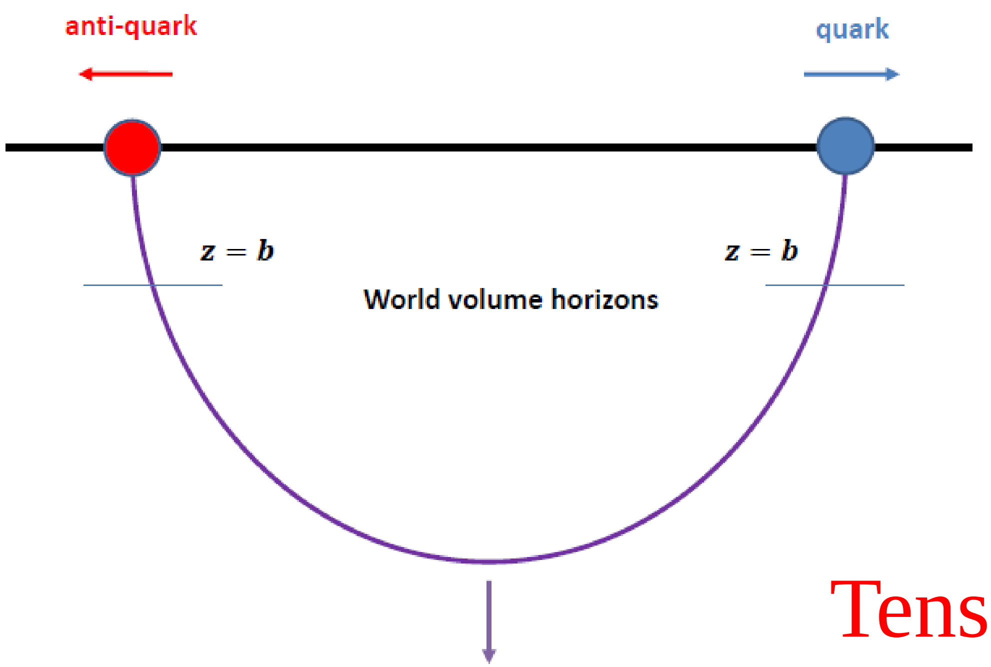
Example: Holographic EPR (Jensen, AK)



$$x^2 = t^2 + b^2 - z^2 \, .$$

Worldsheet = ER bridge (finite distance, no causal connection)

Holographic EPR



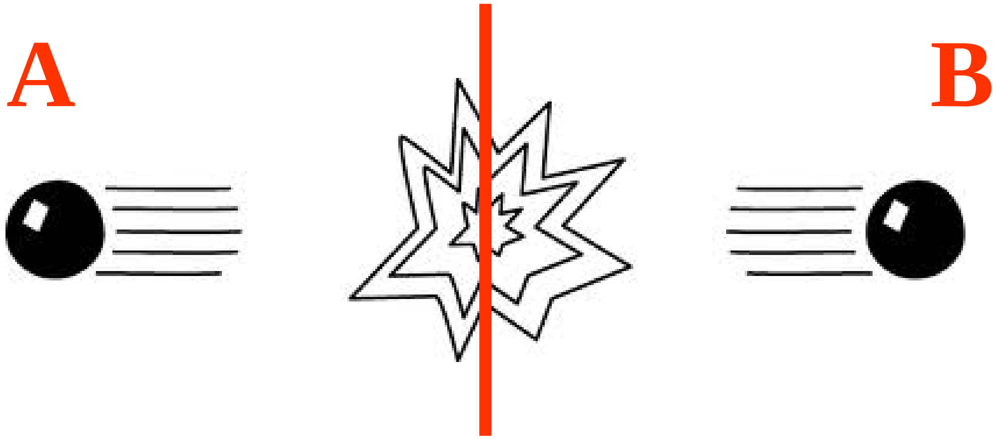
$$x^2 = t^2 + b^2 - z^2\,.$$

(Xiao)

Tension $\sim\sqrt{\lambda}$

$N^2 \gg \sqrt{\lambda} \gg 1$

S_{EE} for the holographic EPR pair



$$S_{EE} = \sqrt{\lambda}/_{3}$$

quark not just a single parton, $\sqrt{\lambda}$ gluons part of quasi-particle

Holographic EPR pair:

Susskind, Maldacena: **ER=EPR**

what does "=" mean?

- Our construction makes clear that this is holographic duality
- "=" means: has mathematically equivalent description in terms of
- can be generalized to include dynamical gravity (RS)
- can be generalized to entangled Hawking pairs.

(last two: Jensen, AK, Robinson)

Backreaction?

Backreaction of the quark sector

How far can I get without ever having to solve Einstein's equations?

There are some new quantities in the flavor sector (order N physics) that are manifestly only due to probe fields (e.g. chiral condensate); no backreaction.

Does backreaction ever matter at order N?

Backreaction?

(see e.g. AK, O'Bannon, Thompson)

Generically, at order N backreaction matters.

$$S = N^2 S_{grav} + N\, S_{probe}$$

glue and quark sector

backreaction suppressed….

$$\delta h \propto G_N\, T_{\mu\nu} \propto {}^{1}\!/_{N}\;\; h$$

$$\boxed{\varepsilon = \; N^2 \varepsilon_{grav} + N\,(\varepsilon_{probe} + \frac{\delta\varepsilon_{grav}}{\delta g}\,\mathrm{h})}$$

$\boldsymbol{\varepsilon = {}^{\delta S}\!/_{\delta g_{00}}}$ = energy density

… but enters at order N

Backreaction

(see e.g. AK, O'Bannon, Thompson)

Exception: the free energy = on-shell action.

$$S = N^2 S_{grav} + N\, S_{probe}$$

glue and quark sector

backreaction suppressed

$$\delta h \propto G_N\, T_{\mu\nu} \propto {}^{1}/_{N}\;\; h$$

$$\boxed{\omega = \; N^2 S_{grav} + N\,(S_{probe} + \cancel{\frac{\delta S_{grav}}{\delta g}}^{=0}\,\mathrm{h})}$$

contribution due to backreaction vanishes by equations of motion!

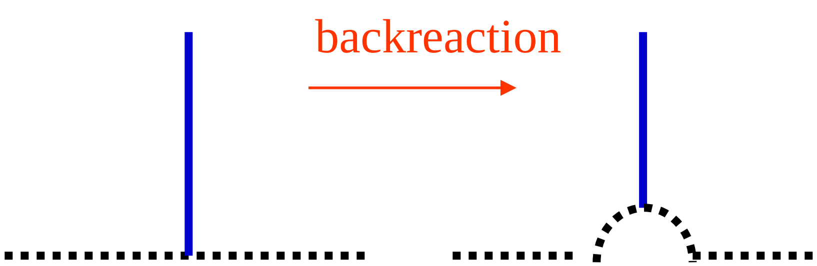
Backreaction: Entropy

(see e.g. AK, O'Bannon, Thompson)

$$\boxed{\omega = N^2 S_{grav} + N\, S_{probe}}$$

good news! all equilibrium properties can be calculated without ever having to deal with backreaction!!!!

Including thermal entropy:



honest calculation:
calculate change in horizon area

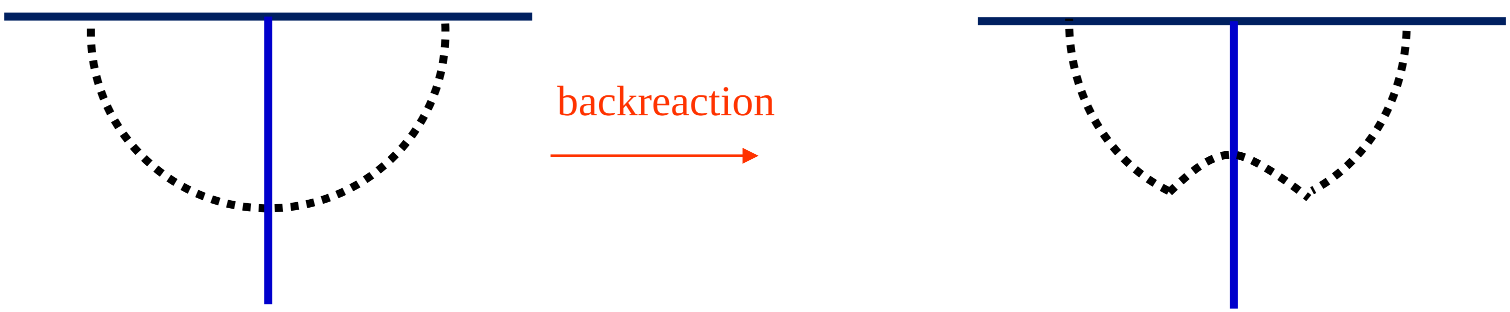
or

$$s = -\frac{\partial \omega}{\partial T}$$

but "cheat" works!

Backreaction: EE

But for the EE we are stuck with needing leading order backreaction to even get order N contribution.



$$S^{EE} = \underbrace{S^{EE}_{grav}}_{O(N^2)} + \underbrace{\frac{\delta Area}{4G}}_{O(N)}$$

O(1/N)

O(1/N^2)

Exceptions: no backreaction

Casini, Huerta, Myers:

- spherical entangling surface
- conformal field theory

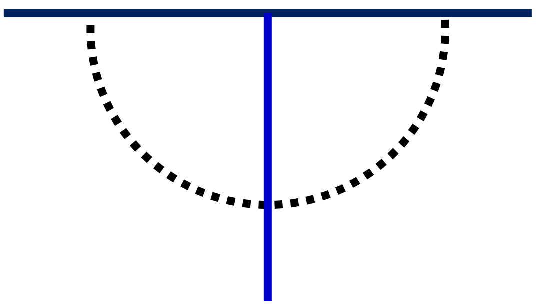
$$\text{EE} \xrightarrow{\text{conformal map}} \text{thermal entropy on hyperboloid}$$

Jensen, O'Bannon:

still applies for conformal flavors (massless quarks!) on conformal defect

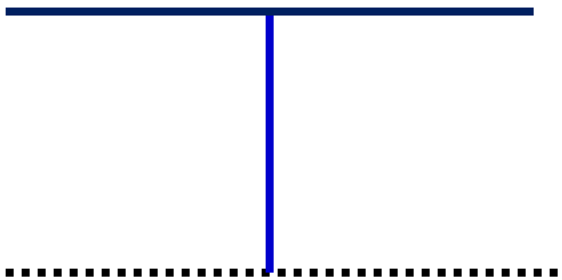
In this case cheat still applies.

CHM:



(CHM)

=



$$ds^2 = \frac{-dt^2 + dx^2 + dz^2}{z^2}$$

$$ds^2 = -h(r)dt^2 + dH^2 + {dr^2}/{h(r)}$$

h(r)= $r^2 - 1$

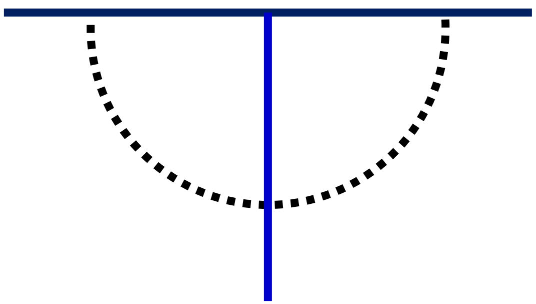
T=0

T=1/2π

Casimir stress tensor on H appears thermal

Entangling surface = boundary of H

Jensen-O'Bannon Calculation



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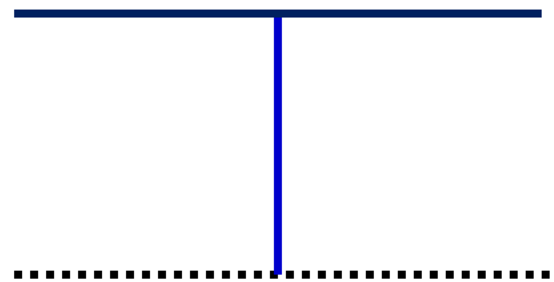


O(N²): (CHM)

constant

$$S = \frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{g}\left(R + \frac{d(d-1)}{L^2}\right)$$

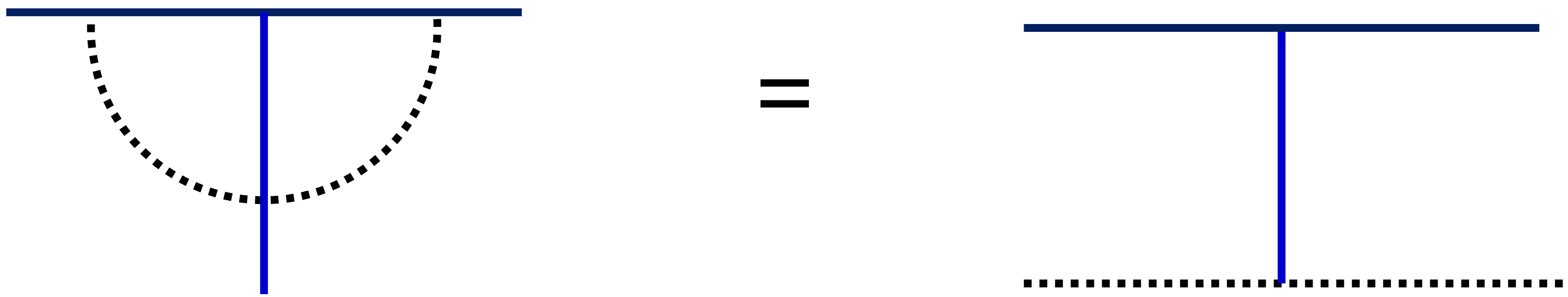
$$S = \frac{V^H_{d-1}}{4G_N} \quad \xleftarrow{\ d/dT\ } $$

$$r_H = 2\pi T/(d-1)$$

Free energy = C_0 volume of spacetime
= -(C_0/d) r_h^d vol(H)

$$C_0 = d/(8\pi G_N)$$

Jensen-O'Bannon Calculation



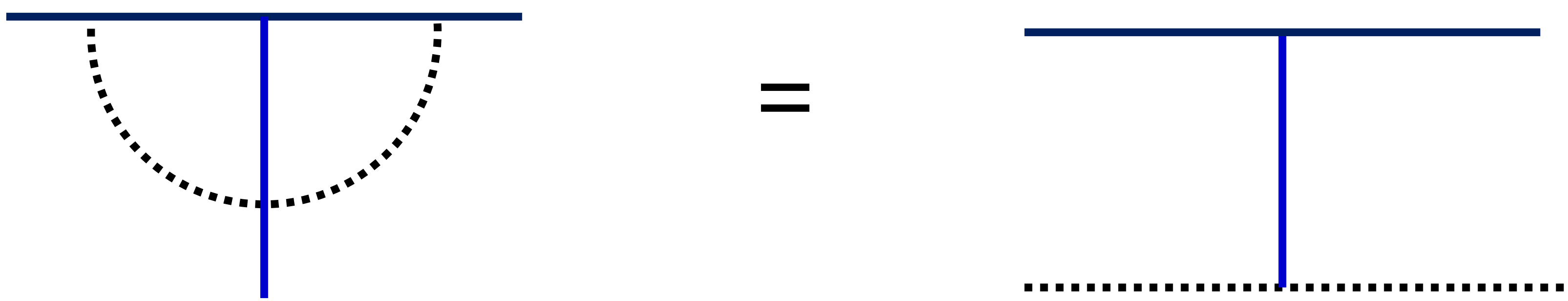
O(N²): (CHM)

$$S = \frac{V^H_{d-1}}{4G_N}$$

$$\begin{aligned} V^H_m &= V^S_{m-1}\int_\epsilon^1 dy\, \frac{(1-y^2)^{(m-2)/2}}{y^m} \\ &= p_1\left(\frac{1}{\epsilon}\right)^{m-1} + p_3\left(\frac{1}{\epsilon}\right)^{m-3} + \ldots \\ &\ldots + \begin{cases} p_{m-1}\left(\frac{1}{\epsilon}\right) + p_m + \mathcal{O}(\epsilon) & m \text{ even} \\ p_{m-2}\left(\frac{1}{\epsilon}\right) + q\log(\epsilon) + \mathcal{O}(1) & m \text{ odd.} \end{cases} \end{aligned}$$

area law

Jensen-O'Bannon Calculation



O(N): (JO)

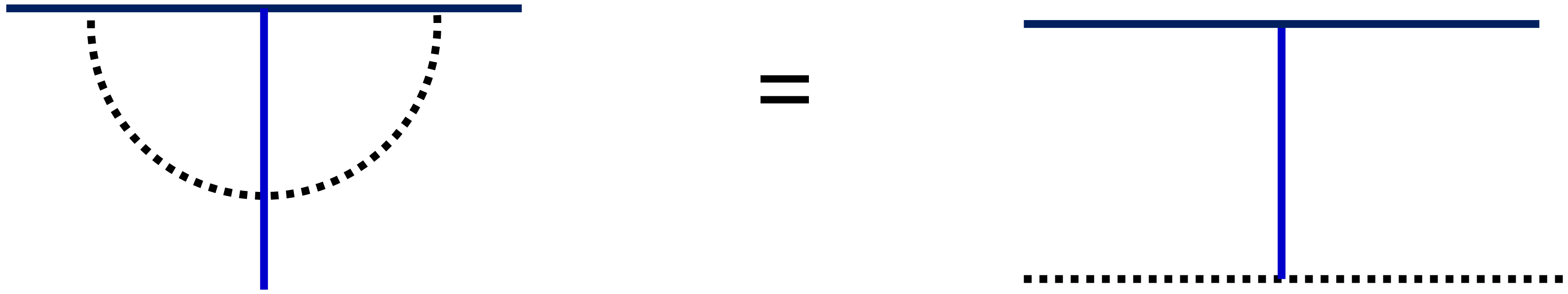
$$S_{probe} = -T_0 \int d^{n+1}z \sqrt{g_I}$$

Free energy ~ volume of spacetime

$$S = \frac{2\pi T_0}{d-1} V^H_{n-1} \xleftarrow{\;d/dT\;} S_{probe}$$

(for n<d, 1/(d-1) replaced by 1/d for n=d)

Jensen-O'Bannon Calculation



O(N): (JO)

$$S = \frac{2\pi T_0}{d-1} V^H_{n-1}$$

"flavor central charge"
number of DOFs on defect

Has exactly the functional form of a spherical entangling surface in an n+1 dimensional CFT.

Limitations of Jensen – O'Bannon

- Massless flavors only

  no topological phases

  not applicable to probe branes realizing quantum Hall effect, topological insulators, ….

- No worldvolume gauge fields

  no novel compressible quantum liquids

  no holographic quantum liquid

Beyond conformal defects?

A) Systematically include backreaction

(Chang, AK)

B) Generalize CHM to LM

Agree! (modulo RG ambiguities)

(Uhlemann, AK)

C) Find full backreaction.

has been done for smeared D3/D7 by Kontoudi/Policastro
and for some more general probes by Jones/Taylor

EE from backreaction

(Chang, AK)

EE double integral

Note that to get correction to EE, we don’t need to calculate the full backreaction.

- only need leading 1/N correction
- only need to know how it affects minimal area.

$\rightarrow$ dramatic simplification.

EE from backreaction

$$S_A = (\pi T_0) \int (d^{d-1}w\sqrt{\gamma})\,(d^{n+1}z\sqrt{g_I})\left(T^{\mu\nu}_{min} G_{\mu\nu\rho\sigma} T^{\rho\sigma}_{probe}\right)$$

tension
O(N) constant

minimal
area

probe
worldvolume

double integral

gravitational
Green's function

EE for probe branes.

= gravitational interaction between two energy distributions

$$S_A = (\pi T_0)\int \left(d^{d-1}w\sqrt{\gamma}\right)\left(d^{n+1}z\sqrt{g_I}\right)\left(T^{\mu\nu}_{min}G_{\mu\nu\rho\sigma}T^{\rho\sigma}_{probe}\right)$$

$$S_{min} = \frac{1}{4G_N}\int d^{d-1}w\,\sqrt{\gamma} \equiv \frac{1}{4G_N}\int d^{d-1}w\,\mathcal{L}_{min}$$

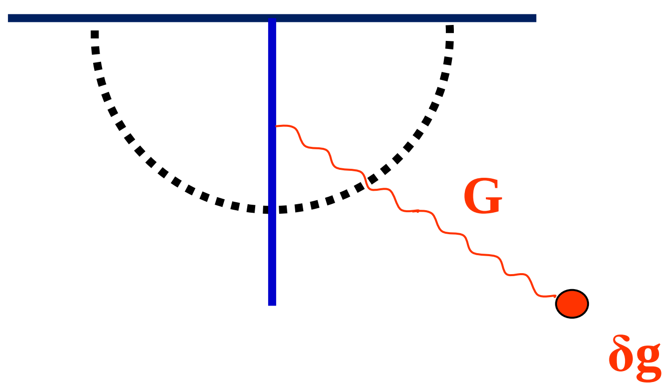
$$T^{\mu\nu}_{min} = \frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_{min}\right)}{\delta g_{\mu\nu}}$$

“stress tensor” of probe brane

“stress tensor” of minimal area

EE for probe branes - derivation

$$S_A = (\pi T_0) \int \left(d^{d-1}w\sqrt{\gamma}\right)\left(d^{n+1}z\sqrt{g_I}\right)\left(T^{\mu\nu}_{min}\underbrace{G_{\mu\nu\rho\sigma}T^{\rho\sigma}_{probe}}\right)$$



$\delta g_{\mu\nu}$

linearized backreaction

EE for probe branes - derivation

$$S_A = (\pi T_0) \int (d^{d-1}w\sqrt{\gamma})\,(d^{n+1}z\sqrt{g_I})\left(T^{\mu\nu}_{min}\underbrace{G_{\mu\nu\rho\sigma}T^{\rho\sigma}_{probe}}_{\delta g_{\mu\nu}}\right)$$

corresponding change in minimal area:

$$\delta S_{min} = \frac{1}{4G_N}\int d^{d-1}w\,\sqrt{\gamma}\left(\frac{T^{\mu\nu}_{min}}{2}(\delta g)_{\mu\nu} + \cancel{\frac{\delta\mathcal{L}_{min}}{\delta x^{\mu}_{M}}}\,\delta x^{\mu}_{M}\right)$$

=0 by eom.

change of embedding

Comments

- Works in time-dependent backgrounds

Hubeny, Rangamani, Takayanagi:

Minimal area $\rightarrow$ Extremal area

same action!

use retarded Green's function for G

Comments

- Works with higher curvature corrections
  - **minimal area requirement gets replaced with more general surface**
  - **still follows from an action principle, action defining the surface is area + curvature corrections**
  - **still allows definition of $T^{min}_{\mu\nu}$**

Properties

No secondary backreaction!

D7 probe

$T_0 \sim N$

$\delta g \sim 1/N$

axion $\sim 1/N$

dilaton $\sim 1/N$

backreact on metric?

$G \sim 1/N^2$

Secondary Backreaction

axion $\sim 1/N$ $G \sim 1/N^2$

IIB-action $= 1/G\ (\text{axion})^2$

stress tensor $= N^2\ (\text{axion})^2 \sim O(1)$

$\delta g \sim G\ (\text{stress tensor}) \sim 1/N^2$

secondary backreaction is subleading

Exception:

turned on in background

axion ~ 1/N   Axion ~ 1 $G \sim 1/N^2$

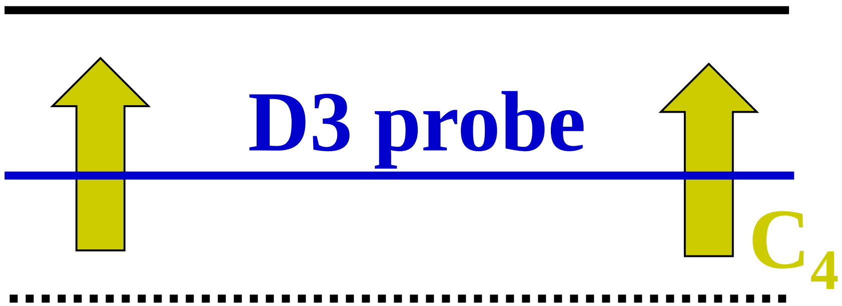
IIB-action = 1/G (Axion + axion)^2

stress tensor = … + N^2 (axion Axion) ~ O(N)

δg ~ G (stress tensor) ~ 1/N

secondary backreaction is important if brane sources field that has non-trivial background

Exception: Examples

- 

- D6 flavor brane in ABJM

(ABJM has RR 2-form background)

But: **D5 with F_{rt} is fine. Sources C_4, but orthogonal to background.**

Properties

Backreaction in internal space irrelevant.

trace reversed Einstein:

$$R_{\mu\nu} = \tilde{T}_{\mu\nu} \qquad \tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{D-2} g_{\mu\nu} T^{\rho}_{\rho}$$

Entangling surface is:

- **codimension 2 minimal surface**
- **wrapping internal manifold**

makes co-dimension 2 special

and hence does not source internal tensor modes

Properties

Backreaction in internal space irrelevant.

Instead of integrating over D-dim product space, can do integral in (d+1) dimensional spacetime with "effective" probe stress tensor.

$$T_{\mu\nu}^{probe,eff} = \int_{x_I} \sqrt{g_I}\, T_{\mu\nu}^{probe} .$$

internal space

No internal O(N) backreaction:

EE for D3/D7 (AdS_5 x S^3) =
EE for spacetime filling probe in AdS_5

EE for D3/D5 (AdS_4 x S^2) =
EE for co-dim 1 probe in AdS_5=
Randall-Sundrum brane

very simple to get fully backreacted metric!

Simplest (non-trivial) example:

Massive flavors via D3/D7:

$$\mathcal{S}^{(1)}_{\rm EE} = -\frac{t_0 L^3 V_{\rm S2}}{32G}\left(\frac{\ell^2}{6\epsilon^2} + \frac{4\mu^2+3}{6}\log\frac{\epsilon}{2\ell} + \frac{1}{4} + \frac{8\mu^2}{9} + \frac{\mu^4}{15}\right)$$

EE from Generalized Gravitational Entropy

(Uhlemann, AK)

Probe EE from GGE

Lewkowycz and Maldacena give a “derivation” of the RT prescription:

- Uses replica trick
- Generalizes CHM: EE derived from on-shell action.
- Probe brane can be included.

LM procedure, step 1

$$S_{EE} = \lim_{n\to 1} n^2\, \partial_n S(n)$$

On-shell action on suitable geometry.

For spherical entangling surface, background AdS this geometry is hyperbolic black hole (CHM) even for massive flavor branes.

LM procedure, step 2

$$\frac{d}{dn} = \frac{d\,g_{\mu\nu}}{dn}\frac{\delta}{\delta g_{\mu\nu}}$$

property of the
n-covering geometry
(hyperbolic bh)

allows use of
equations of motion!

LM procedure, step 2

$$\frac{d}{dn}S(n) = \int(EOM) + \text{boundary term}$$

This gives the RT entropy

Option 1: Follow this same procedure including the brane. EOM only satisfied if linearized backreaction is accounted for. Gives back double integral.

Using LM for probe EE

Option 2: Use non-backreacted metric

- Calculate full on-shell action for probe
- Does not reduce to boundary terms
- But no backreaction needed

(Uhlemann, AK)

Using LM for EE

- Need probe embedding in hyperbolic bh
- Obtained from CHM change of coordinates
- But: constant mass in flat space turns into position and time dependent mass
- On shell action complicated 3d integral.

$$ds^2 \to \Omega^2 ds^2 \qquad\qquad m \to \Omega^{-1}\, m$$

Upshot: gives same answer, but ends up being at least as cumbersome as double integral.

Application: flavored N=4 at finite density.

(Chang, Uhlemann, AK)

EE for strongly correlated fluid

Holographic quantum fluid (D3/D7 at finite density):

- Finite density = wordvolume electric field

(Kobayashi, Mateos, Matsuura, Thomson, Myers)

- analytic solution for probe brane known

(O'Bannon, AK)

- Interesting properties

Properties of holographic D7 fluid

- finite T=0 entropy density

  (Kobayashi, Mateos, Matsuura, Thomson, Myers)

- Heat capacity scales as T^6

  (Son, Starinets, AK)

- zero sound mode

  (Son, Starinets, AK)

- moduli space w/o SUSY

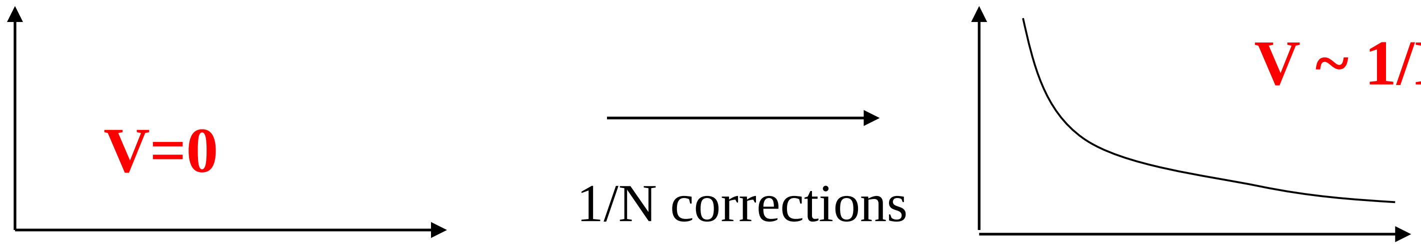
  (Ammon, Jensen, Kim, O'Bannon; Chang, AK)

- perturbatively stable

  (Ammon, Erdmenger, Lin, O'Bannon, Shock)

The long lived large N fluid

Many of these properties are presumably large N artifacts.



But instability lifetime ~ 1/N: $X = X_0 \, e^{t/N}$

EE of D7 fluid

Results (double integral; strip and ball):

1) $q\, l^3 \ll 1$: small entangling region

$$\Delta\mathcal{S}^{(1)}_{\rm EE} = \frac{t_0}{2d}\frac{V_{S^{d-2}}}{4G}\frac{\alpha(d)}{d+1}q^{\frac{d}{d-1}}\ell^d$$

sphere

$$\Delta\mathcal{S}^{(1)}_{\rm EE} = \frac{t_0}{2d}\frac{V_{d-2}}{4G}\ell_\star^2 q^{\frac{d}{d-1}}\frac{\sqrt{\pi}\alpha(d)\Gamma(\frac{1}{d-1})}{(d+1)\Gamma(\frac{1}{2}+\frac{1}{d-1})}\,,$$

ball

EE of D7 fluid

Results (double integral; strip and ball):

1) $q\, l^3 \ll 1$: small entangling region

$$T_{\rm ent}\delta S_{\rm EE}(A) = \delta E(A)$$

T universal, shape dependent.
Perfect agreement with general results.

(Bhattacharya, Nozaki, Takayanagi, Ugajin)

EE of D7 fluid

Results (double integral; strip and ball):

1) $q\, l^3 \gg 1$: large entangling region

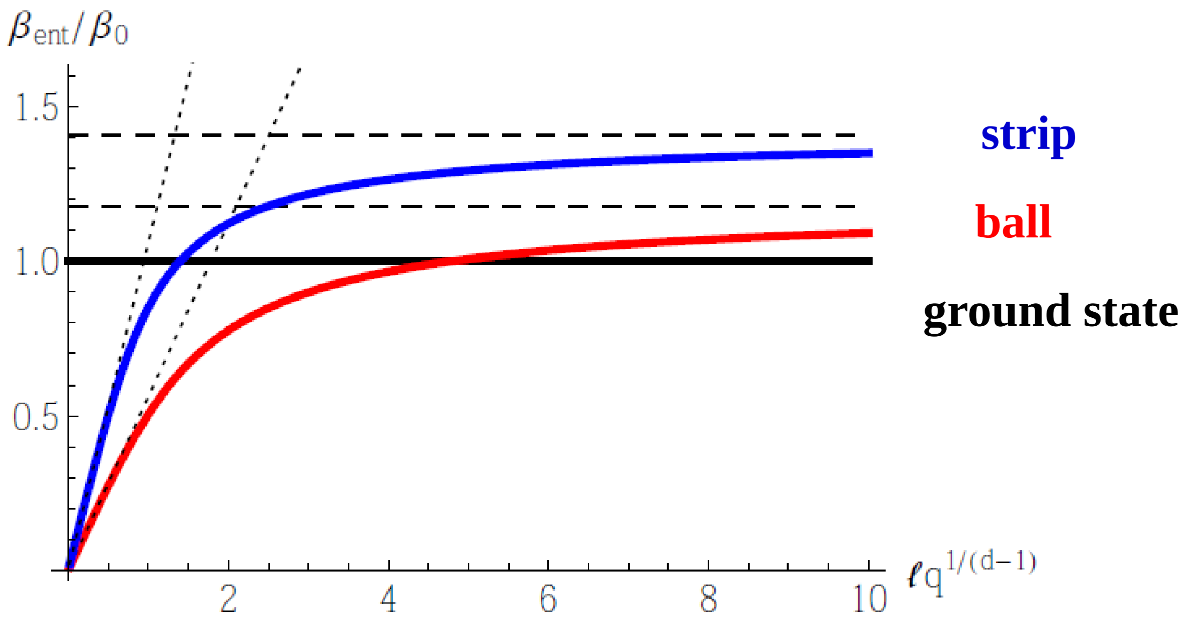
Volume law for EE!!!

Is this just the T=0 grounds state entropy?

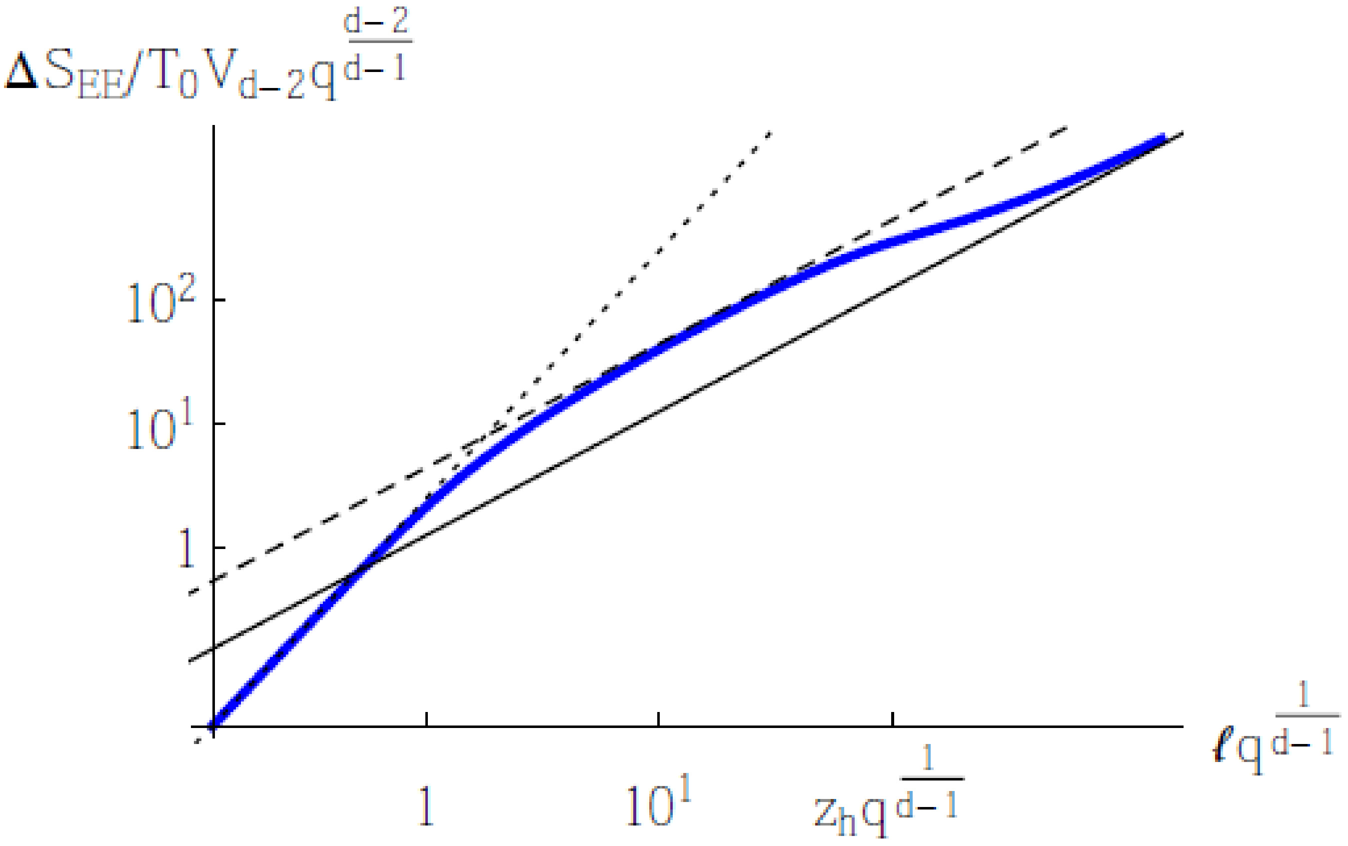
EE for D7 fluid

No! EE density larger than thermal entropy density

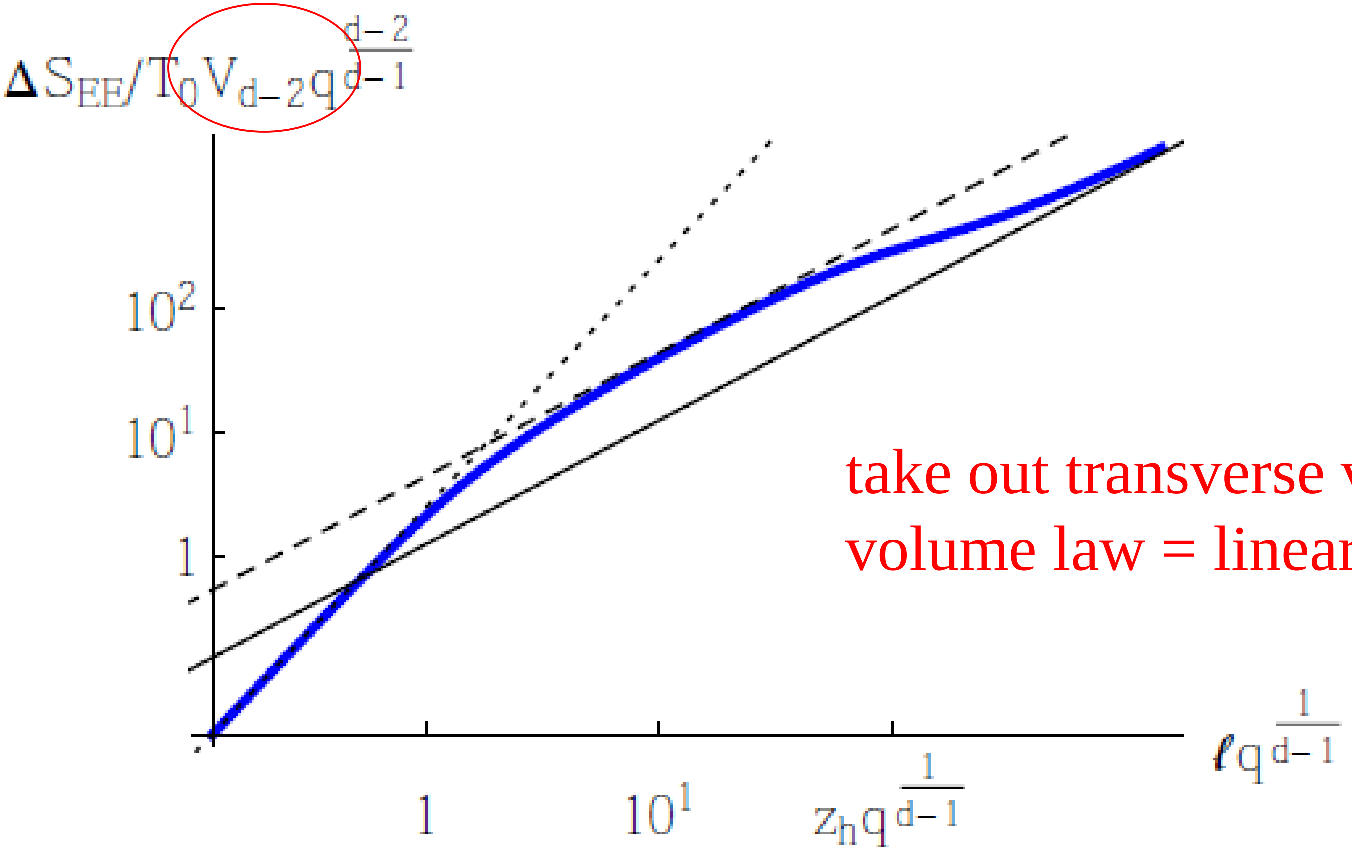
EE density shape dependent.



EE at finite temperature (strip)

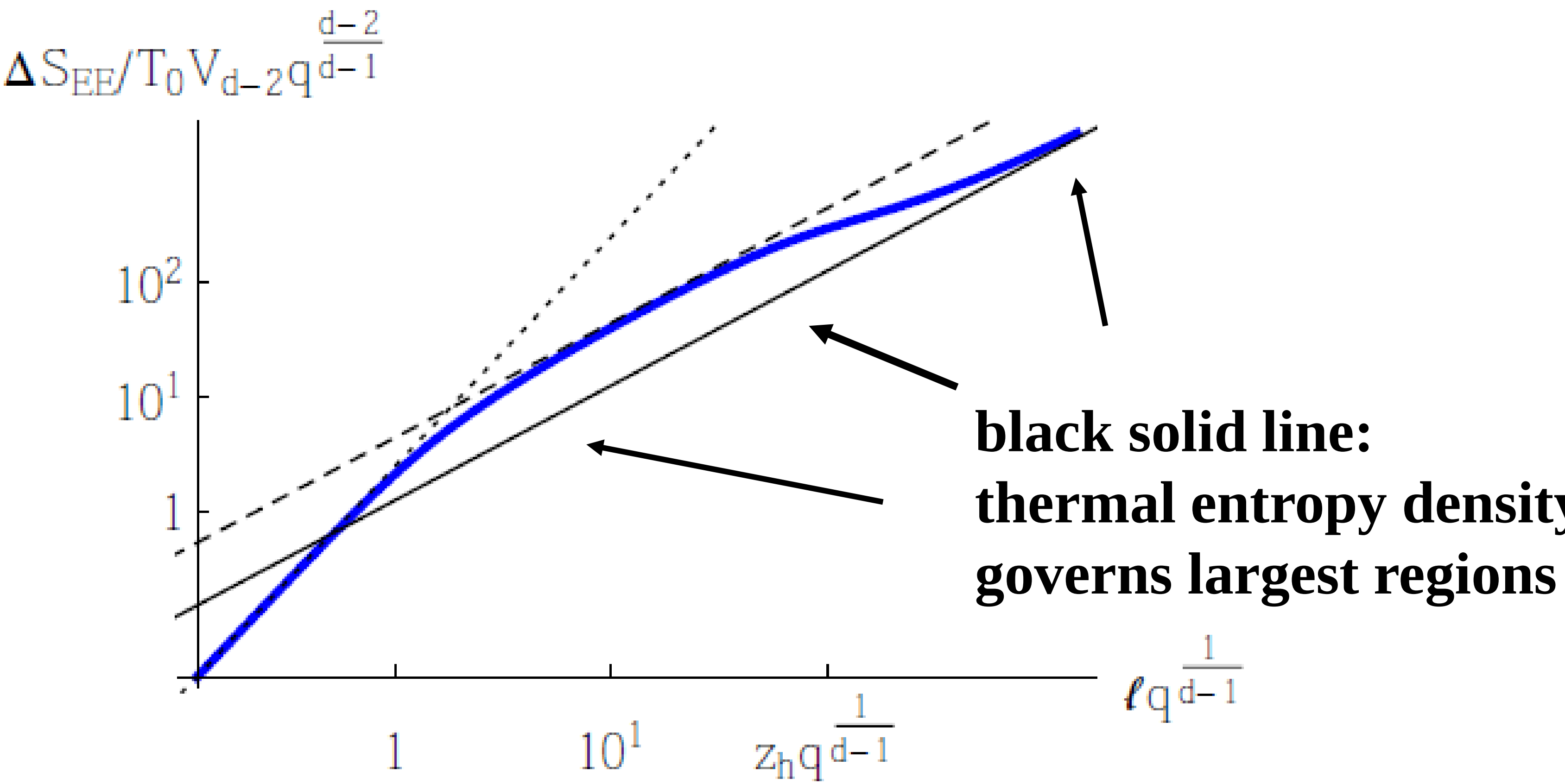


EE at finite temperature (strip)

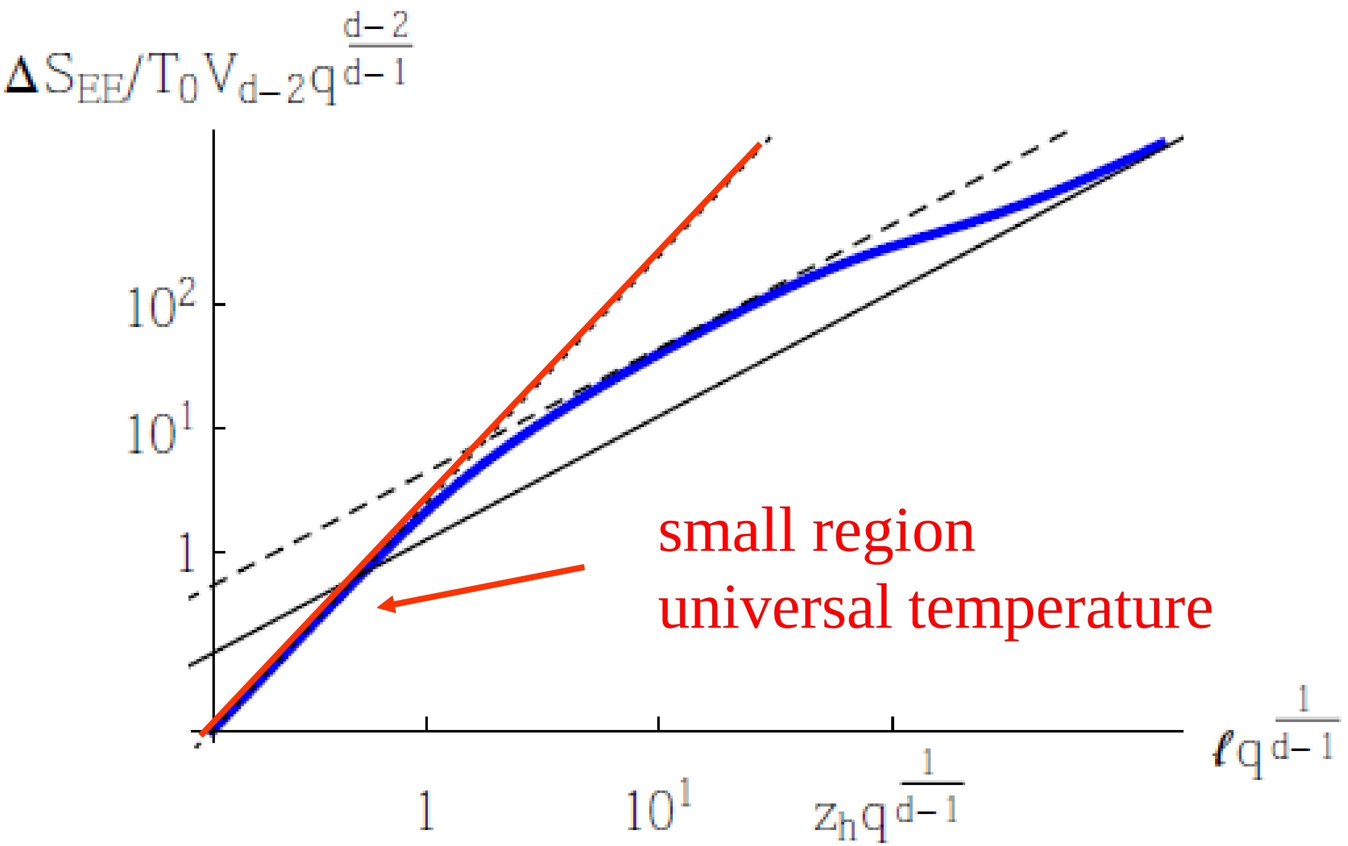


take out transverse volume
volume law = linear in l

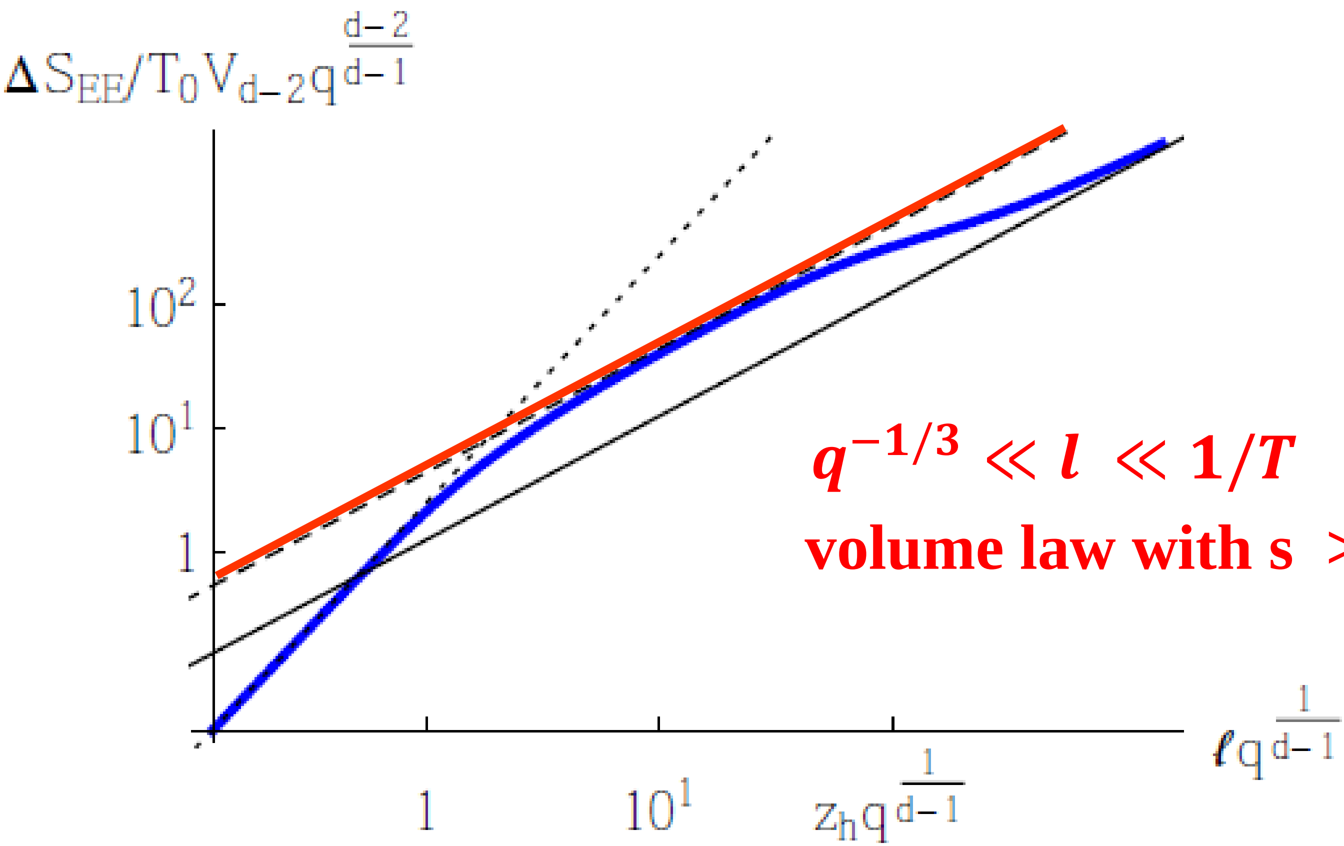
EE at small temperature (strip)



EE at finite temperature (strip)



EE at finite temperature (strip)



$q^{-1/3} \ll l \ll 1/T$

volume law with s > s $_{\text{thermal}}$

EE for D7 fluid

Volume law for large regions:

- very unusual for local QFT
- long range entanglement?
- volume worth of entangled Bell pairs?

Begs for better understanding.

Conclusions

Conclusions

- EE can be calculated for probe branes
- Two methods: double integral or GGE
- In general double integral appears easier
- EE for holographic quantum liquid exhibits puzzling volume law.