Entanglement & C-theorems

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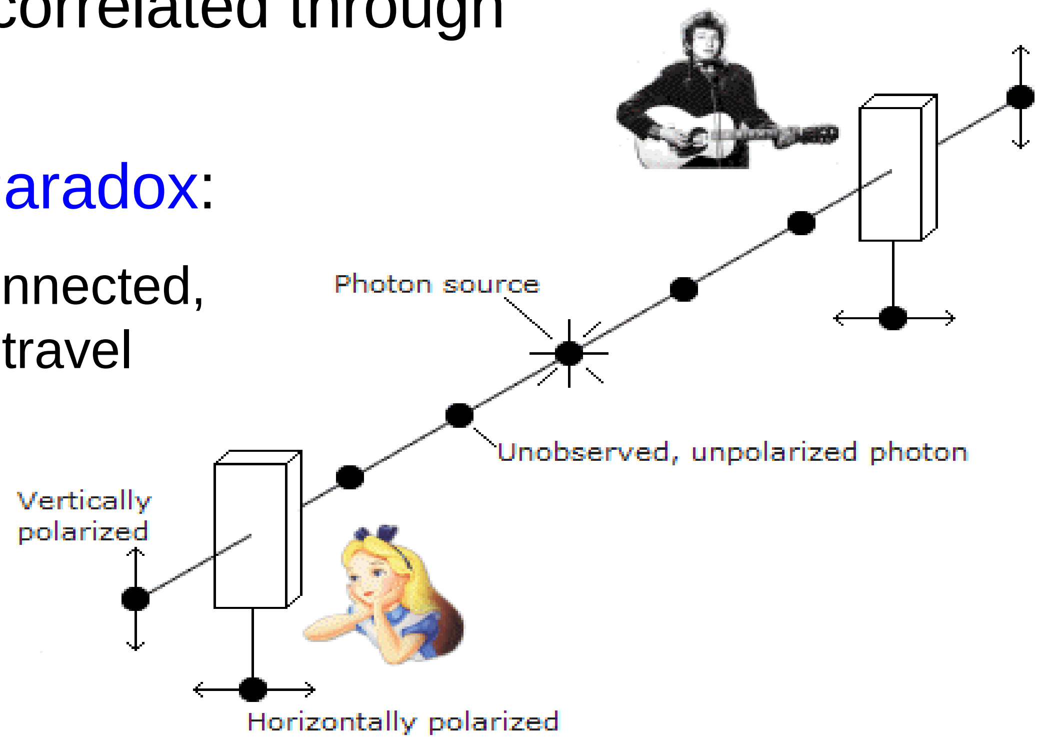
(with Sinha; Casini, Huerta & Yale)

Quantum Entanglement

- different subsystems are correlated through global state of full system



Einstein-Podolsky-Rosen Paradox:

- properties of pair of photons connected, no matter how far apart they travel

"*spukhafte Fernwirkung*" = spooky action at a distance

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\Big)$$

compare: $$|\psi'\rangle = \frac{1}{2}\Big(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle\Big)$$

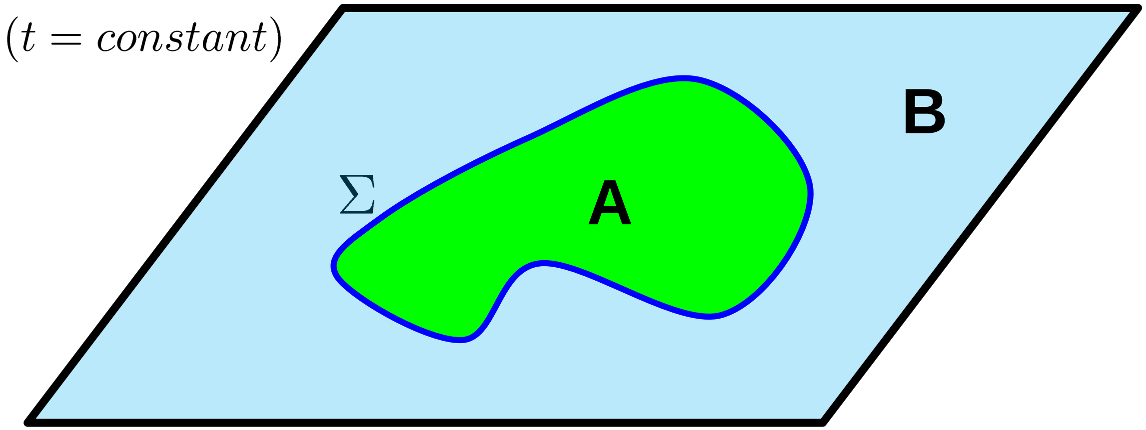
$$= \frac{1}{2}\Big(|\uparrow\rangle + |\downarrow\rangle\Big) \otimes \Big(|\uparrow\rangle + |\downarrow\rangle\Big) \longrightarrow$$ **No Entanglement!!**

$$|\psi''\rangle = \frac{1}{2}\Big(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\Big) \longrightarrow$$ **Entangled!!**

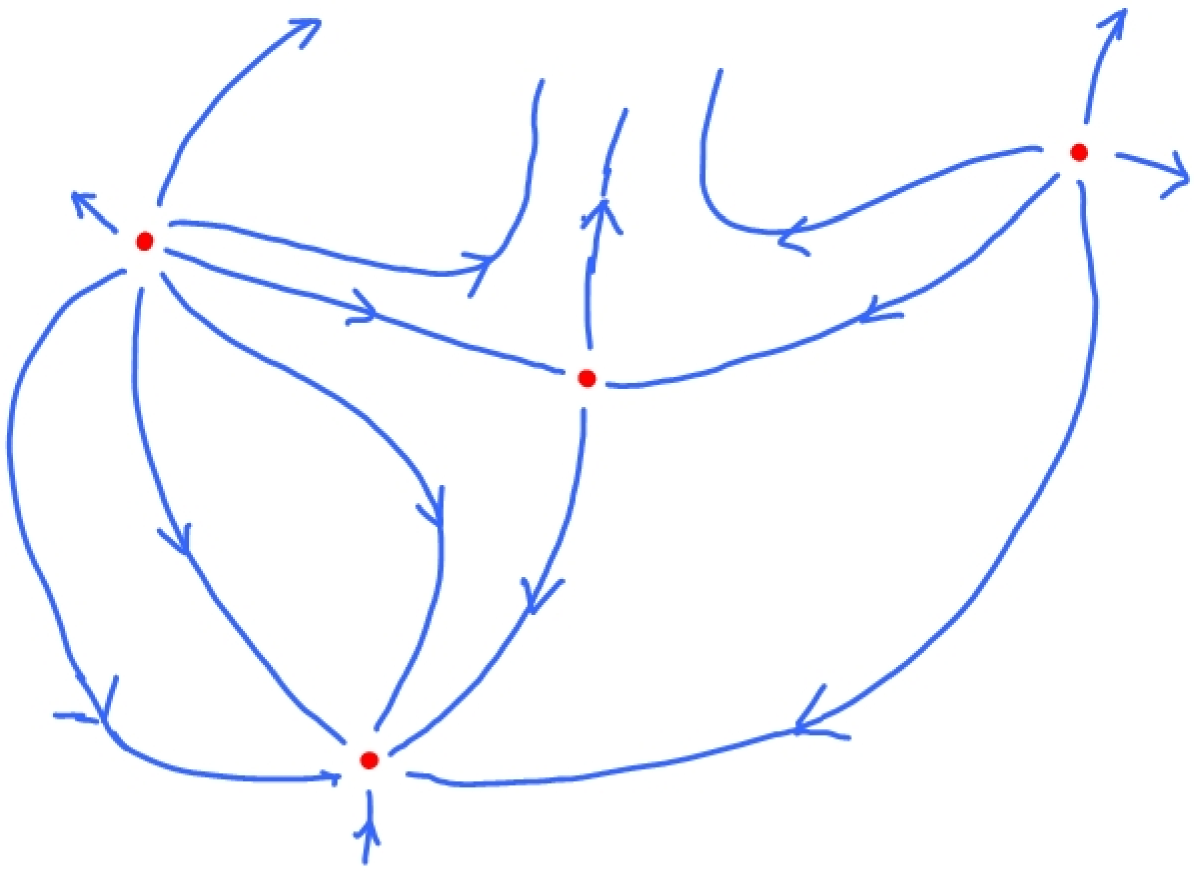
Entanglement Entropy

- general tool; divide quantum system into two parts and use entropy as measure of correlations between subsystems
- in QFT, typically introduce a (smooth) boundary or entangling surface Σ which divides the space into two separate regions
- integrate out degrees of freedom in "outside" region
- remaining dof are described by a density matrix ρ_A

→ calculate von Neumann entropy: $S_{EE} = -Tr\left[\rho_A \, \log \rho_A\right]$



RG flows:



Renormalization Group:

mathematical apparatus that allows systematic investigation of the changes of a physical system as viewed at different *distance/energy scales*"

Zamolodchikov's c-theorem (1986):

- renormalization-group (RG) flows can seen as one-parameter motion

$$\frac{d}{dt} \equiv -\beta^i(g)\,\frac{\partial}{\partial g^i}$$

in the space of (renormalized) coupling constants $\{g^i,\ i = 1, 2, 3, \cdots\}$ with beta-functions as "velocities"

- for unitary, Lorentz-inv. QFT's in two dimensions, there exists a positive-definite real function of the coupling constants $C(g)$:

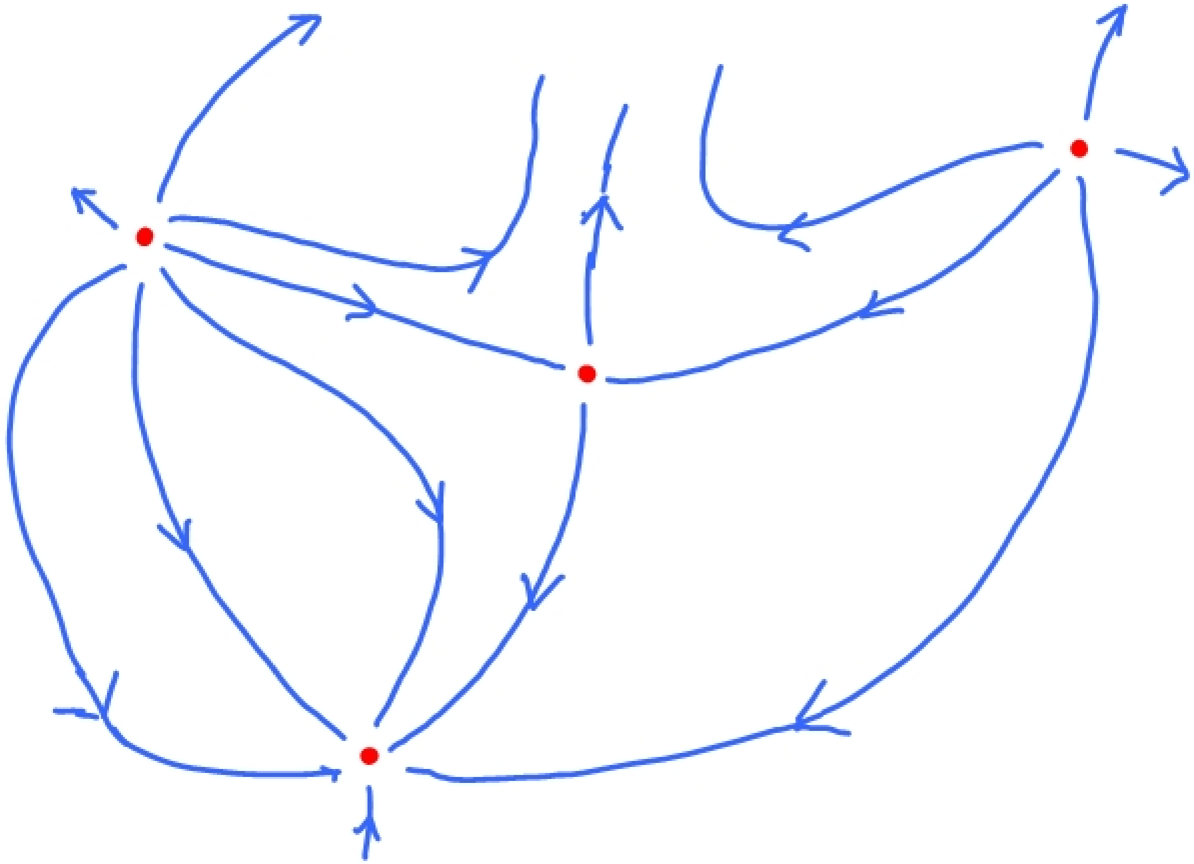
  1. monotonically decreasing along flows: $\frac{d}{dt}C(g) \le 0$
  2. "stationary" at fixed points $g^i = (g^*)^i$:

$$\beta^i(g^*) = 0 \longleftrightarrow \frac{\partial}{\partial g^i}C(g) = 0$$

  3. at fixed points, it equals central charge of corresponding CFT

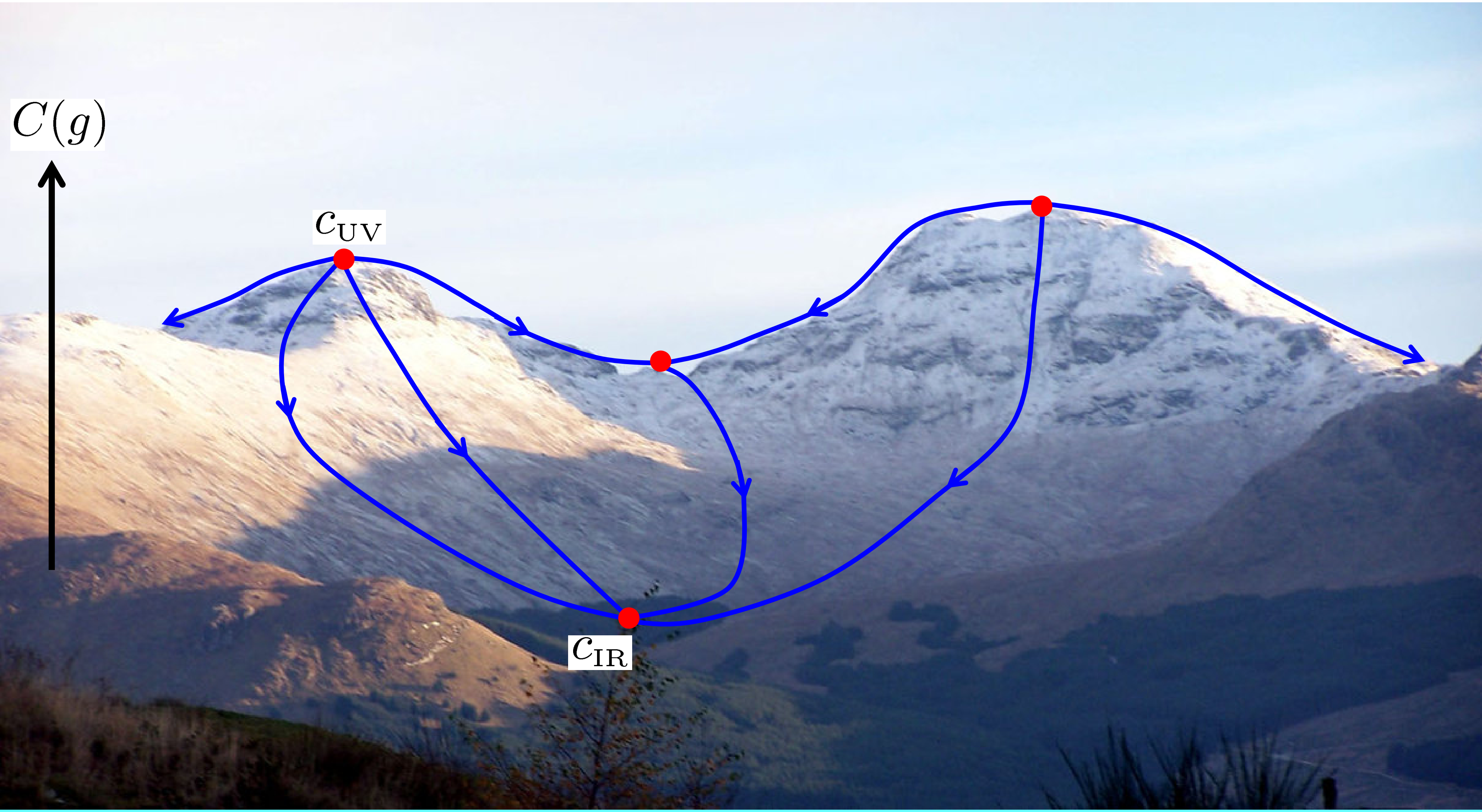
$$C(g^*) = c$$

Zamolodchikov's C-function adds a dimension to RG flows:



BECOMES

Zamolodchikov's C-function adds a dimension to RG flows:



Simple consequence for any RG flow in d=2: $c_{\mathrm{UV}} > c_{\mathrm{IR}}$

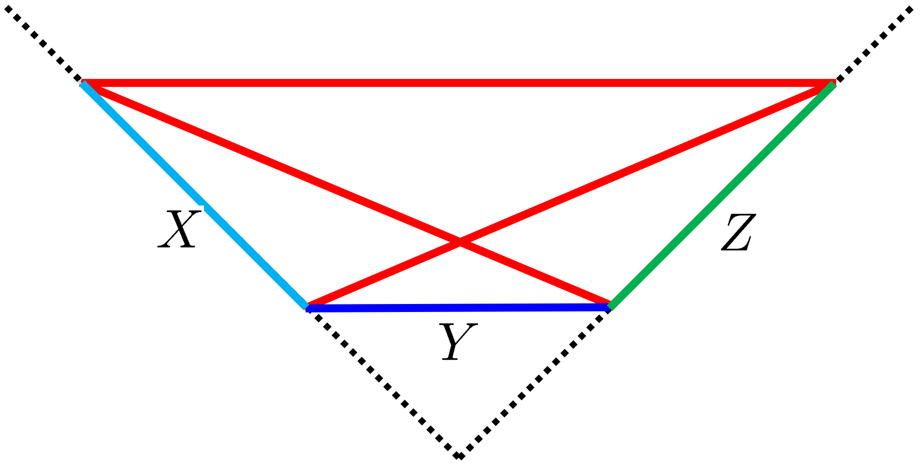
Entanglement & c-theorem?

- Preskill '99: "Quantum information and physics: some future directions"

  $\longrightarrow$ QI may provide new insight into RG flows & c-theorem

- Casini & Huerta '04: reformulate c-theorem for d=2 RG flows in terms of **entanglement entropy** using unitarity, Lorentz inv. and **strong subaddivity inequality**:

$$S(X \cup Y \cup Z) - S(X \cup Y) - S(Y \cup Z) + S(Y) \leq 0$$



(Casini & Huerta '04)

RG flows Meet Entanglement:

- c-theorem for d=2 RG flows can be established using unitarity, Lorentz invariance and **strong subadditivity inequality**:

$$S(X\cup Y\cup Z)-S(X\cup Y)-S(Y\cup Z)+S(Y)\leq 0$$

- for d=2 CFT: $S_{\rm CFT}=\frac{c}{3}\log(\ell/\delta)+a_0$ (Holzhey, Larsen & Wilczek) (Calabrese & Cardy)

$\longrightarrow$ isolate central charge with: $3\,\ell\;\partial_\ell S_{\rm CFT}(\ell)=c$

- in general, define: $C(\ell)=3\,\ell\;\partial_\ell S(\ell)$

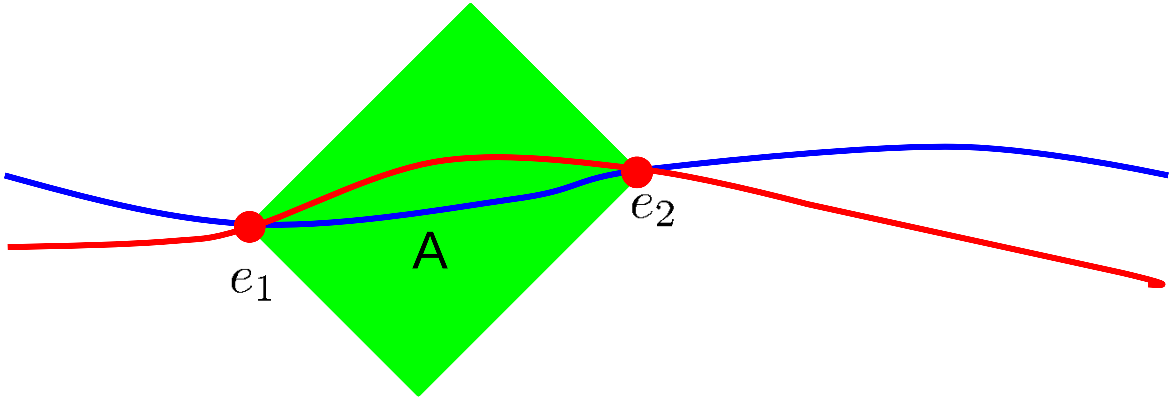
$\longrightarrow$ $C_{\rm CFT}(\ell)=c$

$\longrightarrow$ ℓ appears as proxy for energy scale

(Casini & Huerta '04)

RG flows Meet Entanglement:

- interval A with endpoints e_1 and e_2 on some Cauchy surface



- by causality, ρ_A describes physics in causal diamond

- by unitarity, $S(e_1,e_2)$ independent of details of Cauchy surface

- by translation invariance (in vacuum), $S(e_1,e_2)$ only depends on proper distance between e_1 and e_2

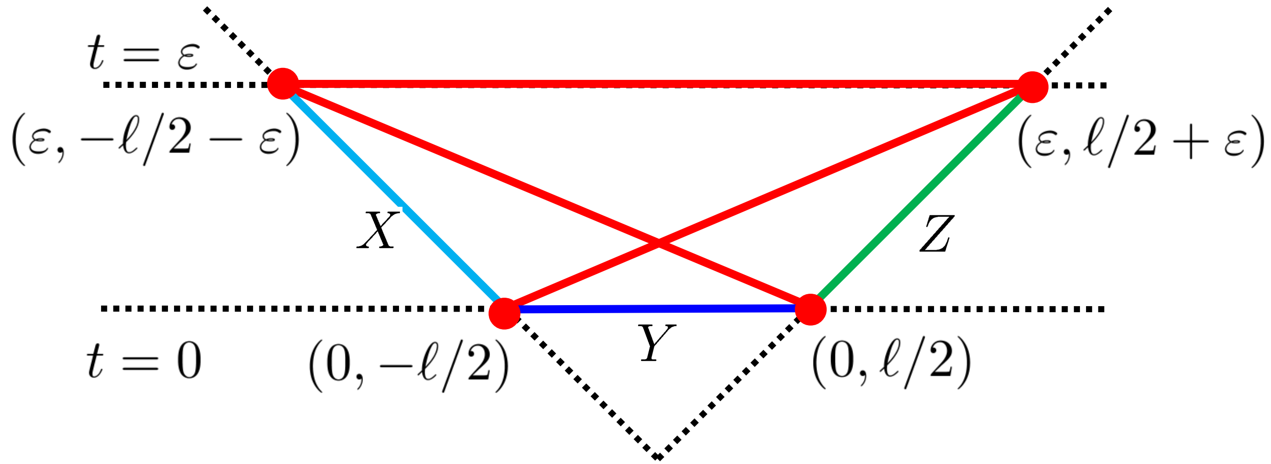
$$\ell_{12} = \left[(x_2 - x_1)^2 - (t_2 - t_1)^2 \right]^{1/2}$$

(Casini & Huerta '04)

RG flows Meet Entanglement:

- apply strong subaddivity inequality in following geometry:

$$S(X\cup Y\cup Z)-S(X\cup Y)-S(Y\cup Z)+S(Y)\le 0$$



$$S(Y)=S(\ell)\,,\quad S(X\cup Y\cup Z)=S(\ell+2\varepsilon)$$

$$S(X\cup Y)=S(Y\cup Z)=S(\sqrt{\ell(\ell+2\varepsilon)})$$

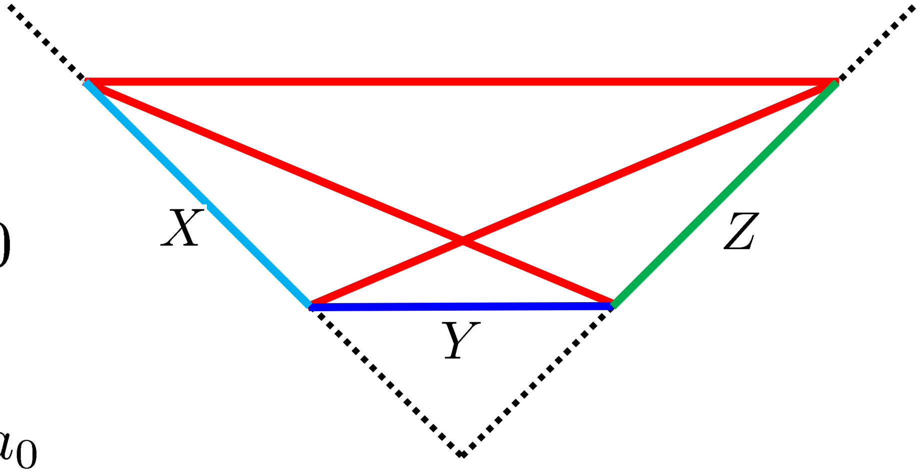
SSA $\longrightarrow$ $S(\ell+2\varepsilon)+S(\ell)-2S(\sqrt{\ell(\ell+2\varepsilon)})\le 0$

$\varepsilon\to 0\ :\quad S''+S'/\ell\le 0 \longrightarrow \partial_\ell(\ell S')\le 0 \longrightarrow \boxed{\partial_\ell C(\ell)\le 0}$

RG flows Meet Entanglement:

- Casini & Huerta '04: reformulate c-theorem for d=2 RG flows in terms of **entanglement entropy** using unitarity, Lorentz inv. and **strong subaddivity inequality**:

$$S(X \cup Y \cup Z) - S(X \cup Y) - S(Y \cup Z) + S(Y) \leq 0$$



- define: $C(\ell) = 3\,\ell\ \partial_\ell S(\ell)$

$$\longrightarrow \quad \partial_\ell C(\ell) \leq 0$$

- for d=2 CFT: $S = \frac{c}{3} \log(\ell/\delta) + a_0$

$$\longrightarrow \quad C_{\rm CFT}(\ell) = c$$

(Calabrese & Cardy)

(Holzhey, Larsen & Wilczek)

- hence it follows that: $\boxed{c_{\rm UV} > c_{\rm IR}}$

C-theorems in higher dimensions??

d=2: $$\langle\, T_\mu{}^\mu \,\rangle = -\frac{c}{12}\, R$$

d=4: $$\langle\, T_\mu{}^\mu \,\rangle = \cancel{\frac{c}{16\pi^2} I_4} - \frac{a}{16\pi^2} E_4 - \cancel{\frac{a'}{16\pi^2} \nabla^2} R$$

where $I_4 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$ and $E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$

- in 4 dimensions, have three central charges: $c,\ a,\ a'$
- do any of these obey a similar "c-theorem" under RG flows? $[\,??\,]_{\rm UV} > [\,??\,]_{\rm IR}$

a -theorem: proposed by Cardy (1988)

- numerous nontrivial examples, eg, perturbative fixed points (Osborn '89), SUSY gauge theories (Anselmi et al '98; Intriligator & Wecht '03)
- holographic field theories with Einstein gravity dual (Freedman et al '99; Giradello et al '98)
- progress stalled; no proof found; . . . .

- past few years have seen a resurgence of interest and rapid progress

C-theorems in higher dimensions??

(RM & Sinha '10)

- RG flows in generalized holographic models with higher curvatures

→ found new holographic c-theorem: $[a_d^*]_{UV} \geq [a_d^*]_{IR}$

$$a_d^* = \frac{\pi^{(d-2)/2} L^{d-1}}{8\Gamma(d/2)\, G_N\, f_\infty^{(d-1)/2}} \left(1 - \frac{2(d-1)}{d-3}\lambda f_\infty - \frac{3(d-1)}{d-5}\mu f_\infty^2\right)$$

where $\alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0$

gravitational couplings

d = spacetime dimension of boundary theory

- compare trace anomaly for CFT's in even dimensions (Deser & Schwimmer)

$$\langle T_\mu{}^\mu \rangle = B_i\,(\text{Weyl invariant})_i - 2(-)^{d/2} A\,(\text{Euler density})_d + \nabla_\mu K^\mu$$

- precisely reproduces coefficient of A-type anomaly: $\boxed{a_d^* = A}$

(Henningson & Skenderis; Nojiri & Odintsov; Blau, Narain & Gava; Imbimbo, Schwimmer, Theisen & Yankielowicz)

→ agrees with Cardy's general conjecture!!

What about odd d??

Entanglement C-theorem conjecture:

(RM & Sinha)

- identify ‘central charge’ with universal contribution in entanglement entropy of ground state of CFT across sphere S^{d-2} of radius R:

$$S_{univ} = \begin{cases} (-)^{\frac{d}{2}-1}\, 4\, a_d^* \, \log(2R/\delta) & \textbf{for even } \boldsymbol{d} \\ (-)^{\frac{d-1}{2}}\; 2\pi\, a_d^* & \textbf{for odd } \boldsymbol{d} \end{cases}$$

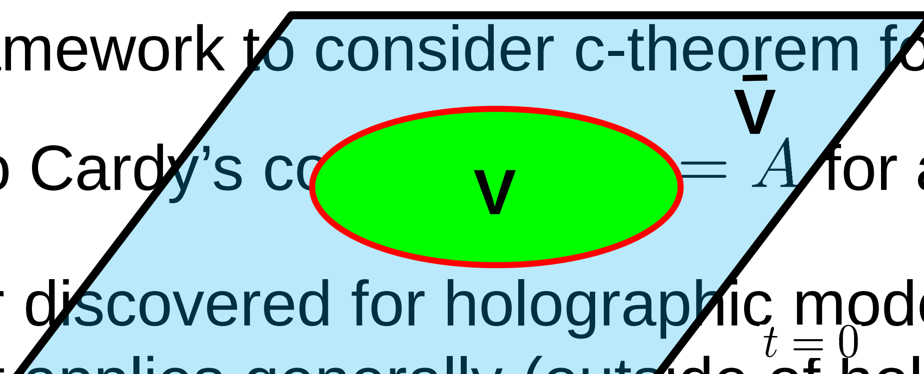
- for RG flows connecting two fixed points

$$(a_d^*)_{UV} \;\geq\; (a_d^*)_{IR}$$

⟶ unified framework to consider c-theorem for **odd** or even *d*

⟶ connect to Cardy's co $= A$ for any CFT in even *d*

⟶ behaviour discovered for holographic model but conjectured that result applies generally (outside of holography)



F-theorem:

(Jafferis, Klebanov, Pufu & Safdi)

- examine partition function for broad classes of 3-dimensional quantum field theories on three-sphere (SUSY gauge theories, perturbed CFT's & O(N) models)
- in all examples, F= – log Z(S^3)>0 and decreases along RG flows

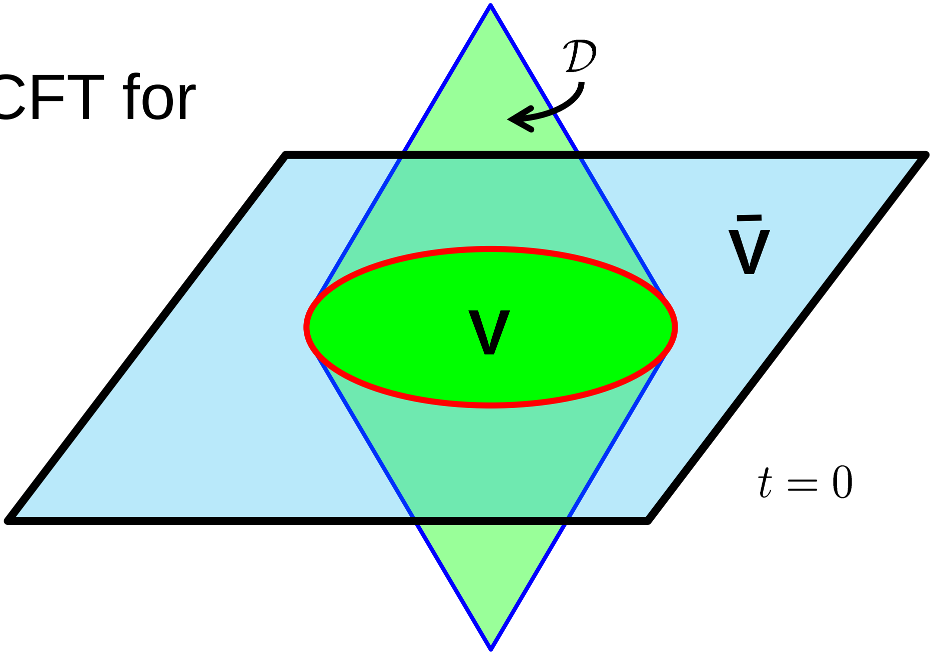
$\longrightarrow$ **conjecture:** $F_{UV} \; > \; F_{IR}$

- also naturally generalizes to higher odd *d*
- **coincides with entanglement c-theorem**

(Casini, Huerta & RM)

- consider S_{EE} of d-dimensional CFT for sphere S^{d-2} of radius R
- conformal mapping:

$$\mathcal{D} \;\rightarrow\; (\text{static patch of})\; dS_d$$

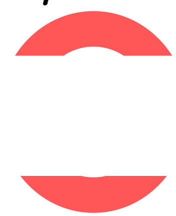


F-theorem:

- **coincides with entanglement c-theorem** (Casini, Huerta & RM)
- consider S_{EE} of d-dimensional CFT for sphere S^{d-2} of radius R
- conformal mapping: $\mathcal{D} \to$ (static patch of) dS_d

curvature ~ 1/R and thermal state: $\rho = \exp[-2\pi R\, H_\tau]/Z$

$$\longrightarrow \quad S_{EE} = S_{thermal}$$



- stress-energy fixed by trace anomaly – vanishes for odd d!
- upon passing to Euclidean time with period $2\pi R$:

$$S_{EE} = \log Z\big|_{S^d} \quad \text{for any odd d}$$

- focusing on renormalized or universal contributions, eg,

$$F_0 = -\log Z\big|_{finite} = -S_{univ} = (-)^{\frac{d+1}{2}} 2\pi\, a_d^* \, .$$

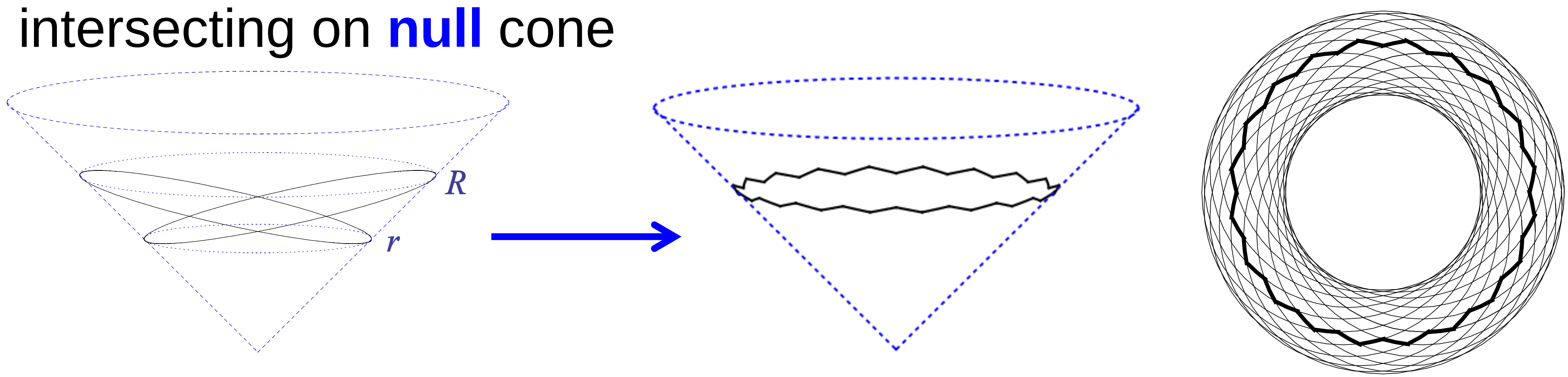
(Casini & Huerta ‘12)

Entanglement proof of F-theorem:

- F-theorem for d=3 RG flows established using unitarity, Lorentz invariance and **strong subadditivity**

$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j)) + S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + \ldots + S(\cap_i X_i)$$

- geometry more complex than d=2: consider many circles intersecting on **null** cone



- no corner contribution from intersection in null plane
- define: $C(R) = RS'(R) - S(R)$
- for d=3 CFT: S(R) = $\frac{2¼R}{\pm} c_0$ ¡ 2¼a_3 $\longrightarrow$ $C_{\text{CFT}}(R) = 2\pi a_3$
- with SSA and “continuum” limit $\longrightarrow$ $\partial_R C(R) \leq 0$
- hence $C(R)$ decreases monotonically and $[a_3]_{\text{UV}} > [a_3]_{\text{IR}}$

A beautiful story but why is universal term in S_{EE} universal?

$$S_{univ} = \begin{cases} (-)^{\frac{d}{2}-1}\, 4\, a_d^*\, \log(2R/\delta) & \textbf{for even } \boldsymbol{d} \\ (-)^{\frac{d-1}{2}}\; 2\pi\, a_d^* & \textbf{for odd } \boldsymbol{d} \end{cases}$$

(Schwimmer & Theisen)

- QFT intution: log divergences define physical cuts but finite *p* polynomials subject to renormalization ambiguities

$\longrightarrow$ even d seems okay but odd d might be problematic?

recall d=2 CFT: $S_{uni} = \frac{\mathbf{c}}{3} \log\left(\frac{C}{\pi\,\delta} \sin\frac{\pi\ell}{C}\right)$ (Calabrese & Cardy) (Holzhey, Larsen & Wilczek)

d=4 CFT: (Solodukhin)

$$S_{uni} = \log(R/\delta)\, \frac{1}{2\pi} \int_\Sigma d^2x\sqrt{h}\, \left[\mathbf{c}\left(C^{ijkl}\, \tilde{g}^{\perp}_{ik}\, \tilde{g}^{\perp}_{jl} - K^{i\,b}_{a} K^{i\,a}_{b} + \frac{1}{2} K^{i\,a}_{a} K^{i\,b}_{b}\right) - \mathbf{a}\,\mathcal{R}\right]$$

d=2m CFT (with symmetry): (RCM & Sinha)

$$S_{uni} = \log\,(R/\delta)\;2\pi \int_\Sigma d^{d-2}x\,\sqrt{h}\;\frac{\partial\langle T_\lambda{}^\lambda\rangle}{\partial R^{\mu\nu}{}_{\rho\sigma}}\;\hat{\varepsilon}^{\,\mu\nu}\;\hat{\varepsilon}_{\rho\sigma}$$

Why is universal term in S_{EE} universal?

$$
S_{univ} = \begin{cases} (-)^{\frac{d}{2}-1}\, 4\, a_d^*\, \log(2R/\delta) & \textbf{for even } \boldsymbol{d} \\ (-)^{\frac{d-1}{2}}\, 2\pi\, a_d^* & \textbf{for odd } \boldsymbol{d} \end{cases}
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- QFT intution: log divergences define physical cuts but finite *p* polynomials subject to renormalization ambiguities

$\longrightarrow$ even d seems okay but odd d might be problematic?

- shifting $\delta \to \delta' = \delta + \alpha\, m\delta^2,$ constant term polluted by UV data

$\longrightarrow$ sure but no scales in CFT, so no scale m!!

$\longrightarrow$ scales from RG flow can appear in final S_{EE}!!

(eg, Hertzberg & Wilczek; Banerjee)

S(R)S(R)=2¼R $\dfrac{2¼R}{\pm}$ +c$_0$m c$_1$ 2¼a$_3$ 2¼a$_3$

Why is universal term in S_{EE} universal?

$$S_{univ} = \begin{cases} (-)^{\frac{d}{2}-1}\, 4\, a_d^*\, \log(2R/\delta) & \textbf{for even } \boldsymbol{d} \\ (-)^{\frac{d-1}{2}}\, 2\pi\, a_d^* & \textbf{for odd } \boldsymbol{d} \end{cases}$$

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$\longrightarrow$ scales from RG flow can appear in final S_{EE}!!

(eg, Hertzberg & Wilczek; Banerjee)

- in regulators, tension between Lorentz inv. and unitarity

$\longrightarrow$ latter emerge in $\delta \to 0$ limit, but regulator exposed in S_{EE}

“Renormalized” Entanglement Entropy: (Liu & Mezei)

- divergences determined by local geometry of entangling surface with covariant regulator, eg,

$$S = c_0(\mu_i\delta)\frac{R^{d-2}}{\delta^{d-2}} + c_2(\mu_i\delta)\frac{R^{d-4}}{\delta^{d-4}} + \cdots + (-)^{\frac{d-1}{2}}\, 2\pi\, a_d(\mu_i\delta) + O(\delta/R)$$

- can isolate finite term with appropriate manipulations, eg,

d=3: $\mathcal{S}_3(R) = RS'(R) - S(R)$ ⟵ c-function of Casini & Huerta

d=4: $\mathcal{S}_4(R) = R^2S''(R) - RS'(R)$

(unfortunately, holographic experiments indicate $\mathcal{S}_d(R)$ are **not** good C-functions for *d*>3 – not monotonic)

- approach demands special class of regulators: “covariant”

⟶ is result artifact of choosing “nice” regulator??

- if a_d is physical, we should be able to use any regularization which defines the continuum QFT

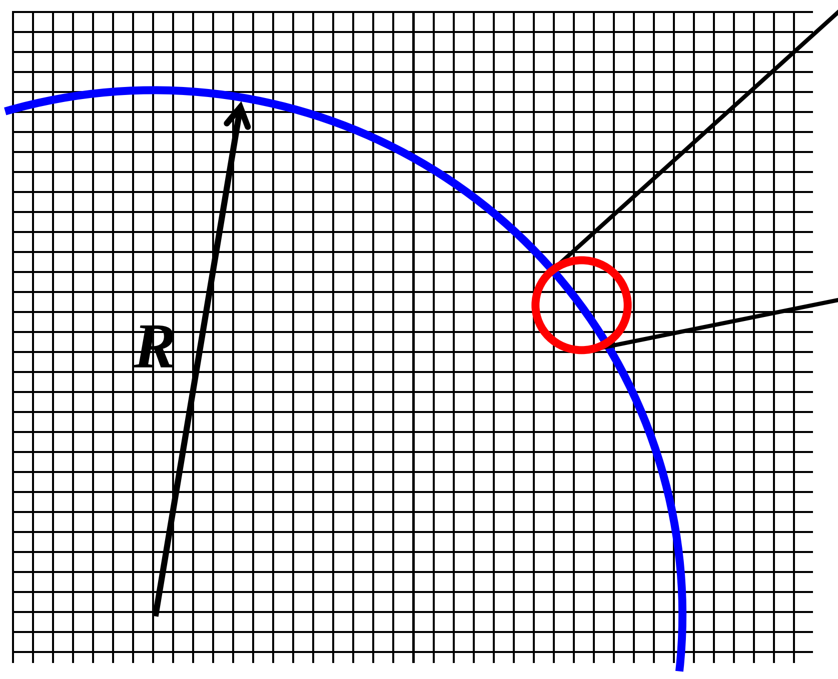
• consider defining a_3 in presence of lattice regulator

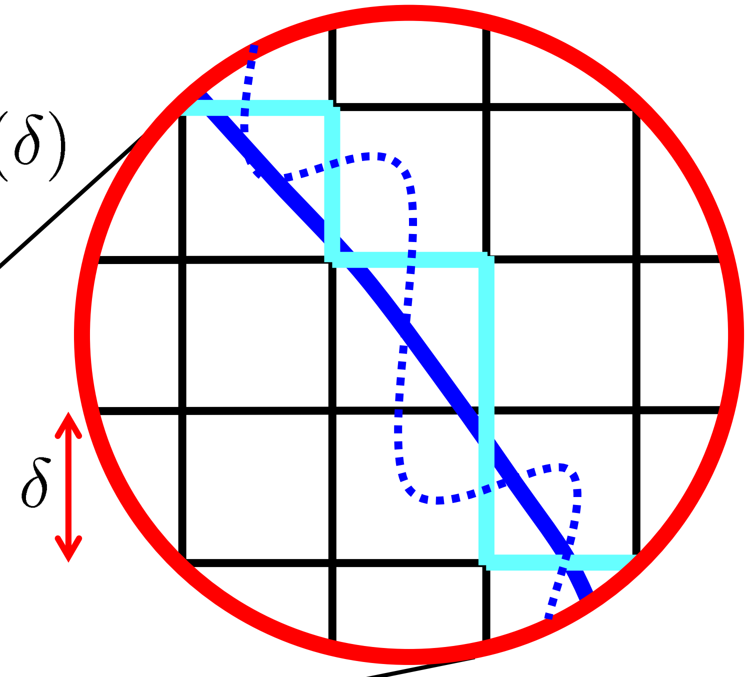
$$d = 3: \quad \mathrm{S(R)} = \frac{2¼\mathrm{R}}{±}\, c_0 \;¡\; 2¼a_3$$

• circumference always uncertain to $O(\delta)$

$$\mathrm{R} \;!\; \mathrm{R}^{0} = \mathrm{R} + ®±;$$

$\longrightarrow$ a_3 always polluted by UV





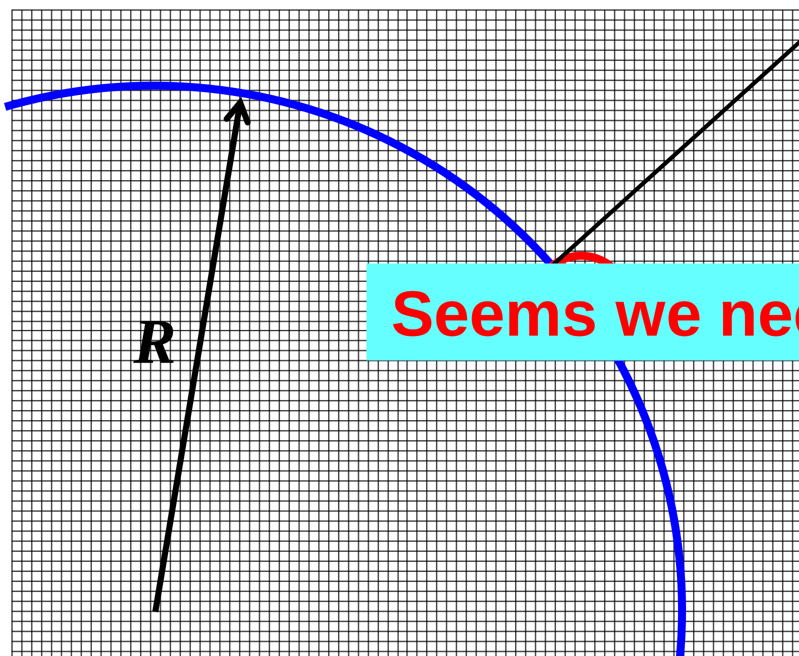
- consider defining a_3 in presence of lattice regulator

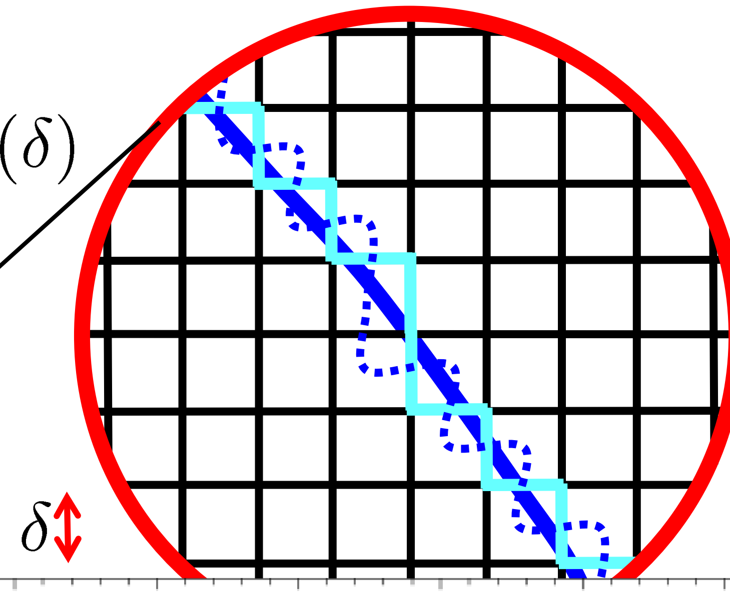
$$d=3:\ S(R) = \frac{2\pi R}{\delta}\,c_0 - 2\pi a_3$$

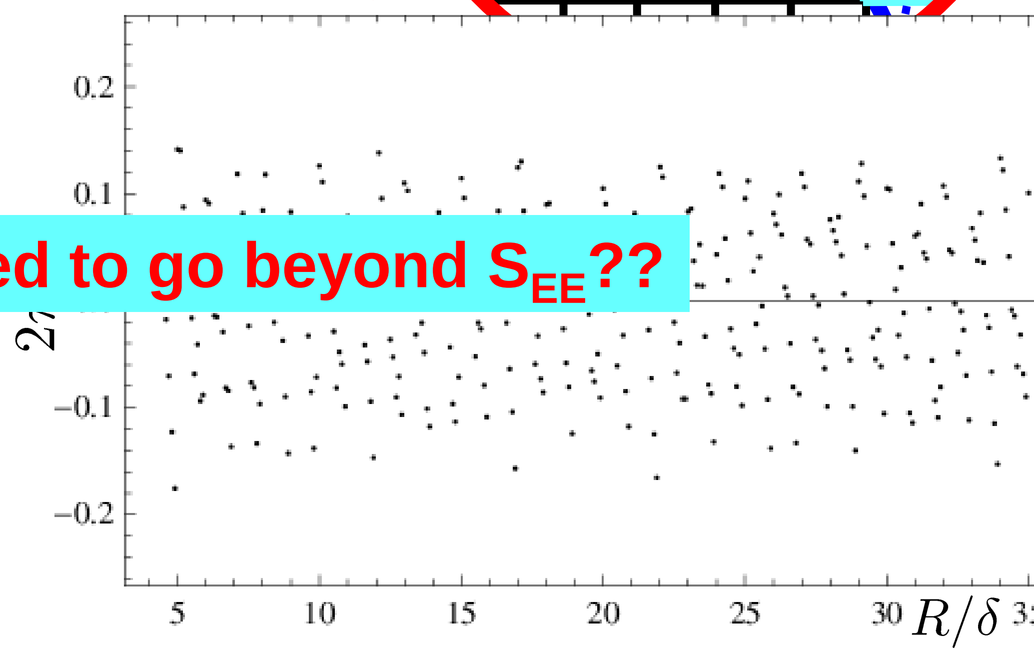
- circumference **always** uncertain to $O(\delta)$

$$R \to R' = R + \alpha\delta$$

$\longrightarrow$ a_3 **always** polluted by UV







Seems we need to go beyond S_{EE}??

Criteria to properly establish c-theorem:

1. C-function must be dimensionless, well-defined quantity, which is independent of the regularization scheme

   $\longrightarrow$ computable with any regulator

2. C-function must be intrinsic to fixed point of interest

   $\longrightarrow$ independent of details of RG flows

3. C-function must decrease monotonically along any RG flows connecting a UV fixed point to an IR fixed point

- S_{EE} seems to fail to satisfy criteria 1 & 2
- **alternate choice? alternate measure of entanglement?**

Mutual Information:

- another measure of entanglement between two systems
- for non-intersecting regions A and B:

$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

- can be defined without reference to S_{EE} (Araki; Narnhofer)
- bounds correlators between A and B (Wolf, Verstraete, Hastings & Cirac)

$$I(A,B) \geq \frac{|\langle \mathcal{O}_A\, \mathcal{O}_B \rangle_c|^2}{2\|\mathcal{O}_A\|^2\,\|\mathcal{O}_B\|^2}$$

- **finite!** UV divergences in S(A) and S(B) canceled by S(A U B)
- if c-function defined with mutual information

$\longrightarrow$ criterion 1 will automatically be satisfied

$\longrightarrow$ criterion 2 & 3 will be satisfied with further care

C-function from Mutual Information:

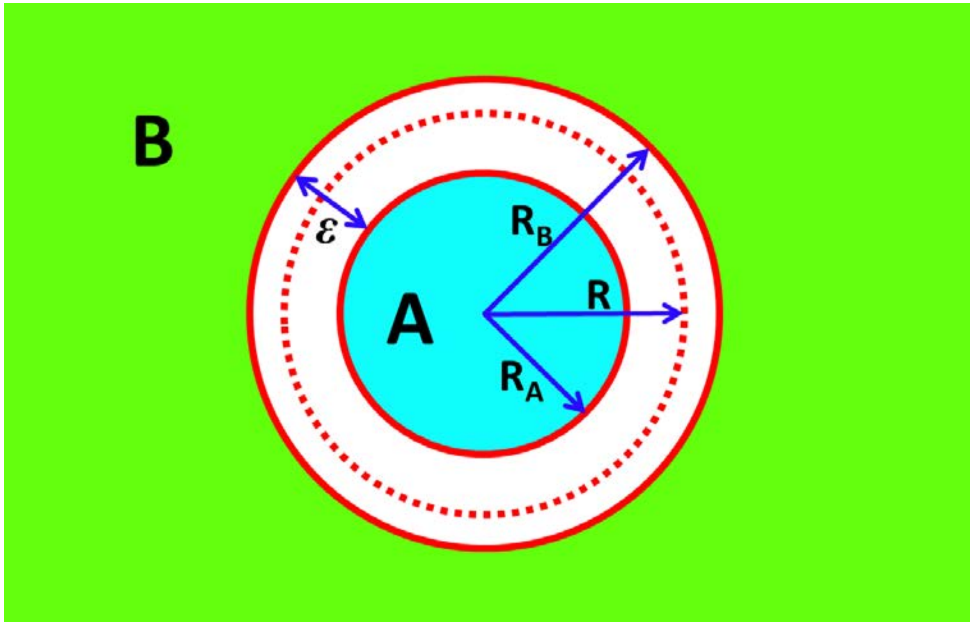
$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

- consider following geometry:

$$R_A = R - \left(\frac{1}{2} - \alpha\right)\varepsilon$$

$$R_B = R + \left(\frac{1}{2} + \alpha\right)\varepsilon$$

or $$R = \frac{R_A + R_B}{2} - \alpha\,\varepsilon$$



- using $S(A) = S(\bar{A})$ for pure state:

$$I(A;B) = S(A) + S(\bar{B}) - S(\overline{A \cup B})$$

two disks ~ R

narrow annulus

- work in continuum: $R \gg \varepsilon \ggg \delta$ (R and ε are macro scales)

C-function from Mutual Information:

$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

- consider following geometry:

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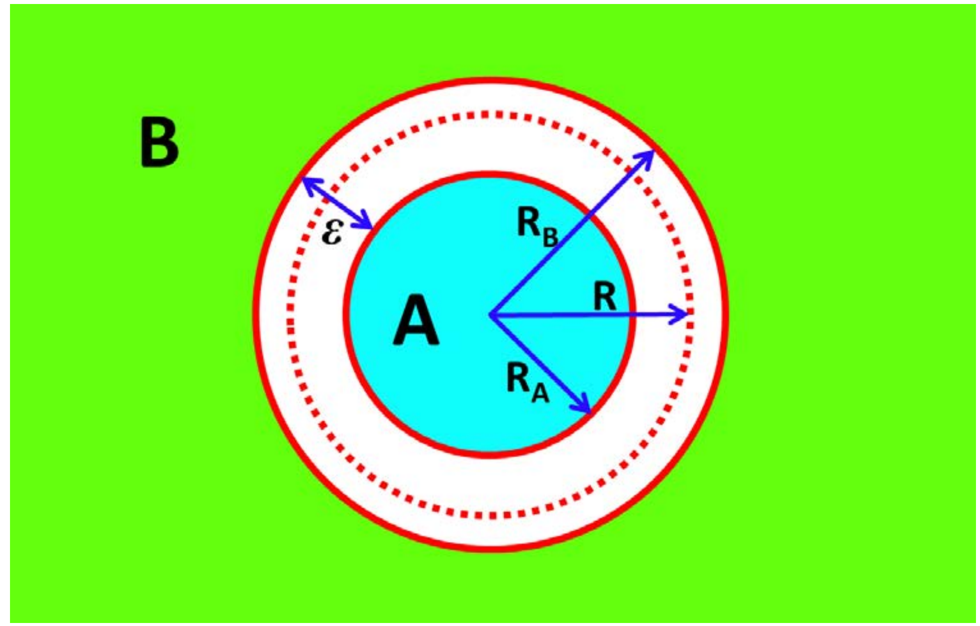


- work in continuum: $R \gg \varepsilon \ggg \delta$ (R and ε are macro scales)
- mutual information takes form:

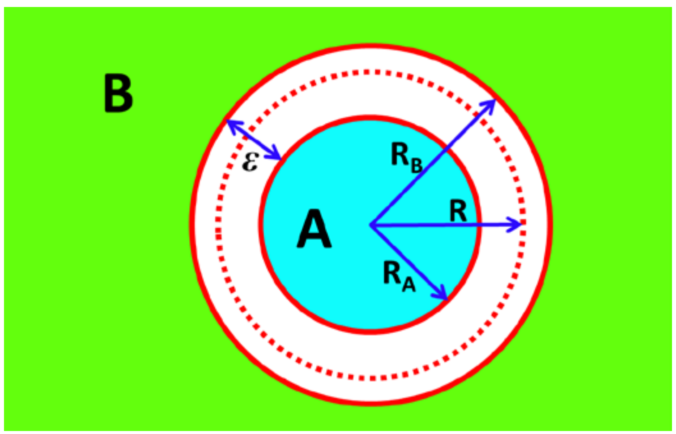
$$I(A,B) = 2\pi R\left[\frac{c_0}{\varepsilon} + c_1\,\varepsilon\right] - 4\pi a_3 + O(\varepsilon/R)$$

C-function from Mutual Information:

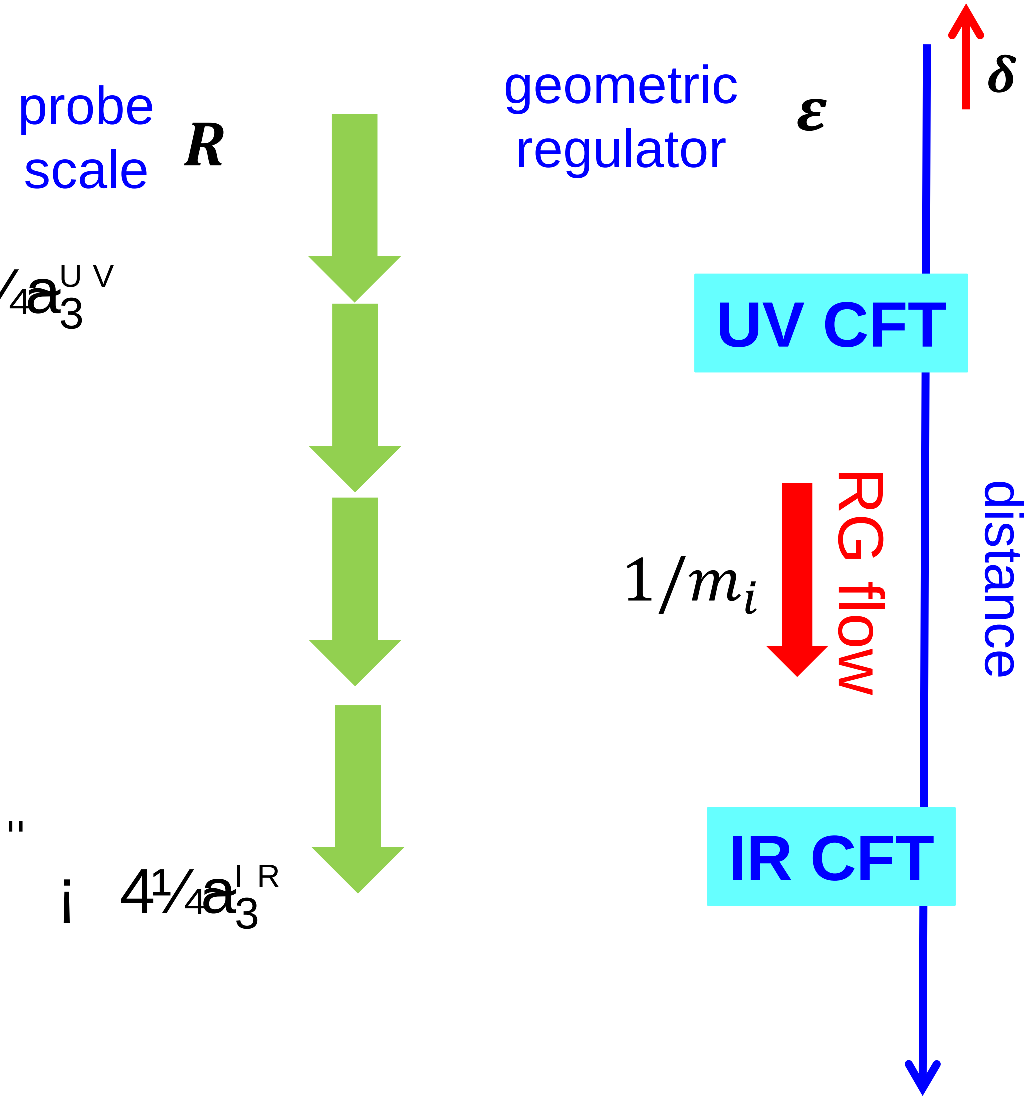
- mutual information "regulates" entanglement entropy of disk
- work with renormalized QFT in continuum limit (R A " A A ±)

- Strategy:

$$I(A;B) = 2\pi R\,\frac{c_0}{\varepsilon} - 4\pi a_3^{UV}$$



$$I(A;B) = 2\pi R\left[\frac{c_0}{\varepsilon} + c_1(m_i)\right] - 4\pi a_3^{IR}$$



C-function from Mutual Information:

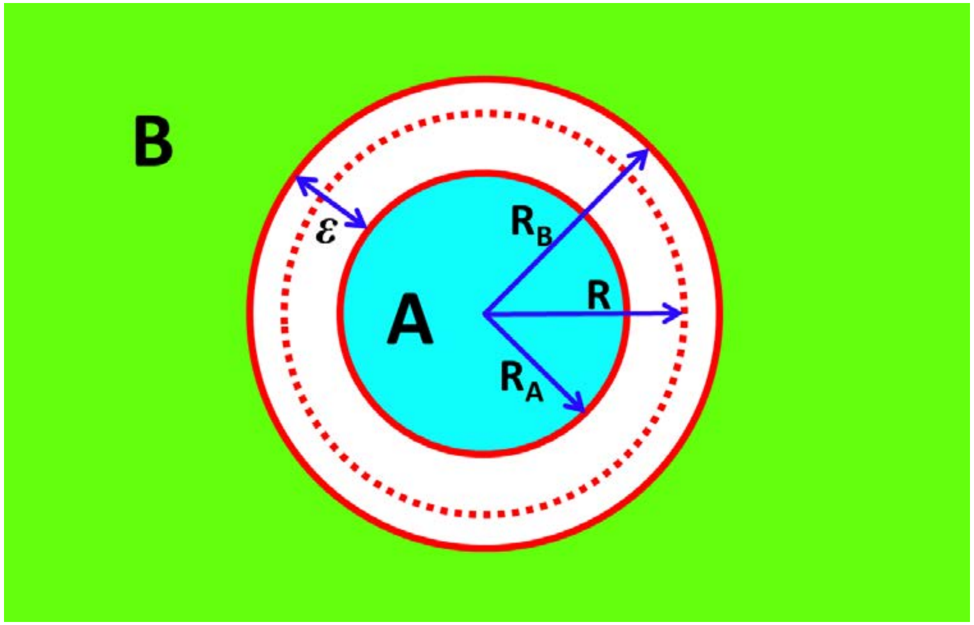
$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

- consider following geometry:

$$R_A = R - \left(\frac{1}{2} - \alpha\right)\varepsilon$$

$$R_B = R + \left(\frac{1}{2} + \alpha\right)\varepsilon$$

or $$R = \frac{R_A + R_B}{2} - \alpha\,\varepsilon$$



- work in continuum: $R \gg \varepsilon \ggg \delta$ (R and ε are macro scales)
- mutual information takes form:

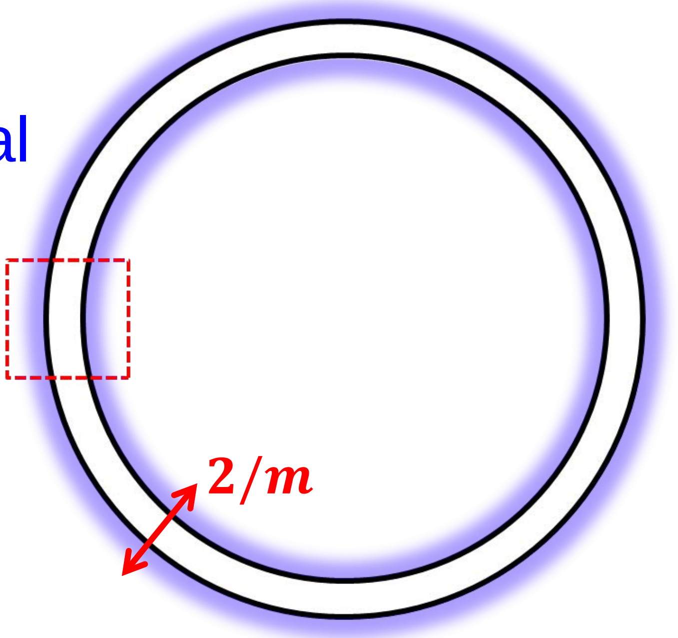
$$I(A;B) = 2\pi R\left[\frac{c_0}{\varepsilon} + c_1\right] - 4\pi a_3 + O(\varepsilon/R)$$

- ambiguity: $\alpha \to \alpha' = \alpha + \delta\alpha,\ \tilde{a}_3 \to \tilde{a}_3' = \tilde{a}_3 + \tilde{c}_0\,\delta\alpha$

- criterion?? is a_3 intrinsic to fixed point??

UV independence of a_3:

- can we choose α such that a_3 is independent of higher scales?
- consider probing at IR critical point where *m*, lowest mass scale in RG flow:: $R \gg 1/m \gg \varepsilon$
- correlations near boundary **non**conformal
- high energy contribution to I(A,B): local and extensive



$$I(A,B)_{HE} = 2\pi R \left(\sigma_0 + \frac{\sigma_1}{R} + \frac{\sigma_2}{R^2} + \cdots\right)$$

- can we choose α to eliminate σ_1??
- for general strip (with small curvatures):

$$I(A,B)_{HE} = \int ds\ \left(\sigma_0 - \sigma_1\, \mathbf{n}\cdot\partial_s \mathbf{t} - \sigma_2\, \mathbf{t}\cdot\partial_s^2 \mathbf{t} + \cdots\right)$$

- σ_1 must vanish if reflection symmetry $\longrightarrow$ $\alpha = 0$

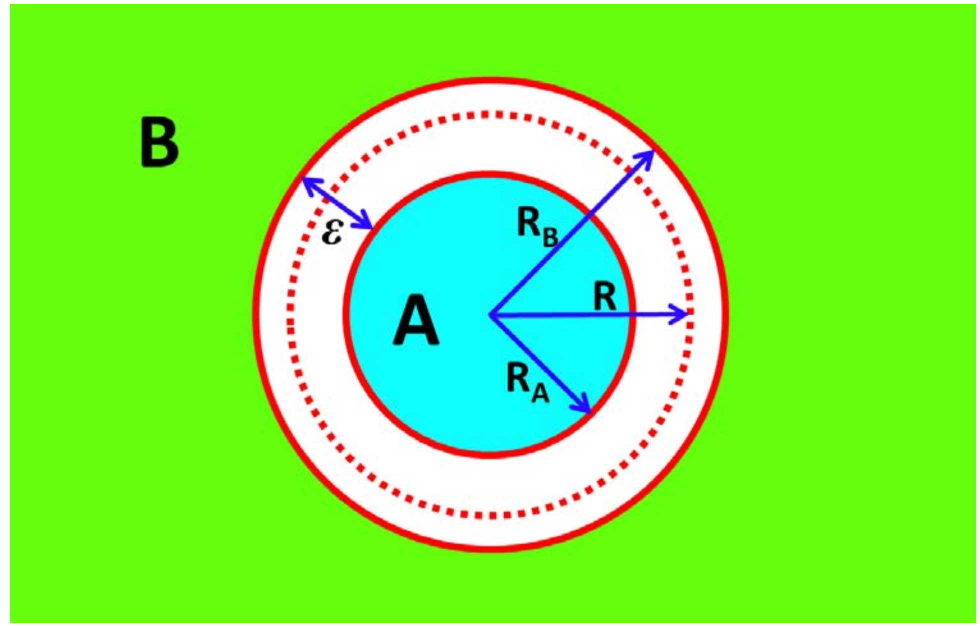
C-function from Mutual Information:

$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

- consider following geometry:

$$R_A = R - \varepsilon/2$$
$$R_B = R + \varepsilon/2$$

or $$R = \frac{R_A + R_B}{2}$$



- in regime: $R \gg \varepsilon \ggg \delta$
- mutual information takes form:

$$I(A;B) = 2\pi R\left[\frac{\tilde{c}_0}{\varepsilon} + \tilde{c}_1\right] - 4\pi\tilde{a}_3 + O(\varepsilon/R)$$

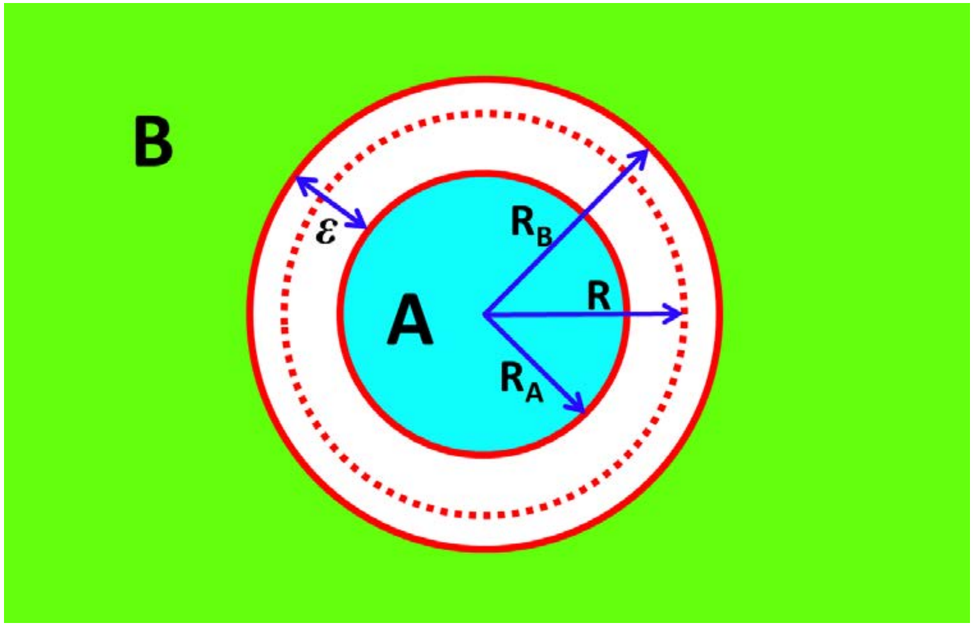
- fixing $\alpha = 0$ ensures $\tilde{a}_3$ is intrinsic to fixed point

$\longrightarrow$ criteria 1 and 2 are satisfied!!

C-function from Mutual Information:

- consider following geometry:

$$R = \frac{R_A + R_B}{2}$$

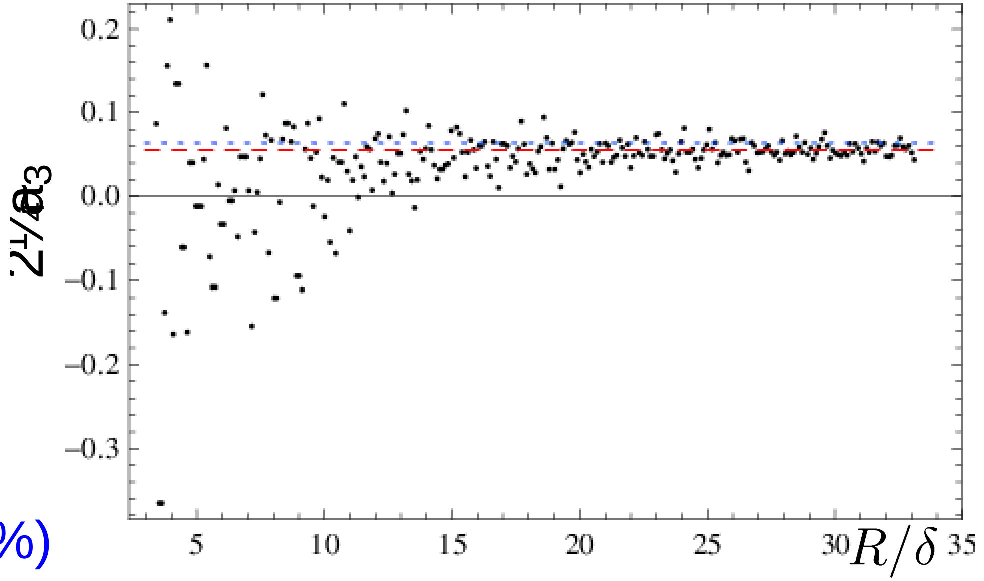


- in regime: $R \gg \varepsilon \ggg \delta$

- calculate for a free scalar on a square lattice:

$$4\pi a_3 \simeq 0.110$$

$$(4\pi a_3)^{\text{scalar}} = \frac{1}{4}\left[\log 2 - \frac{3\zeta(3)}{2\pi^2}\right] \simeq 0.127$$



($R:\varepsilon:\delta = 33:6:1$, result good to 15%)

Criteria to properly establish c-theorem:

1. C-function must be dimensionless, well-defined quantity, which is independent of the regularization scheme

   → computable with any regulator

2. C-function must be intrinsic to fixed point of interest

   → Independent of details of RG flows

3. C-function must decrease monotonically along any RG flows connecting a UV fixed point to an IR fixed point

- defining $\tilde{a}_3$ with mutual information & fixing $\alpha = 0$ ensures criteria 1 and 2 are satisfied; **must still consider criterion 3**
- monotonic flow follows as in entropic proof of F-theorem

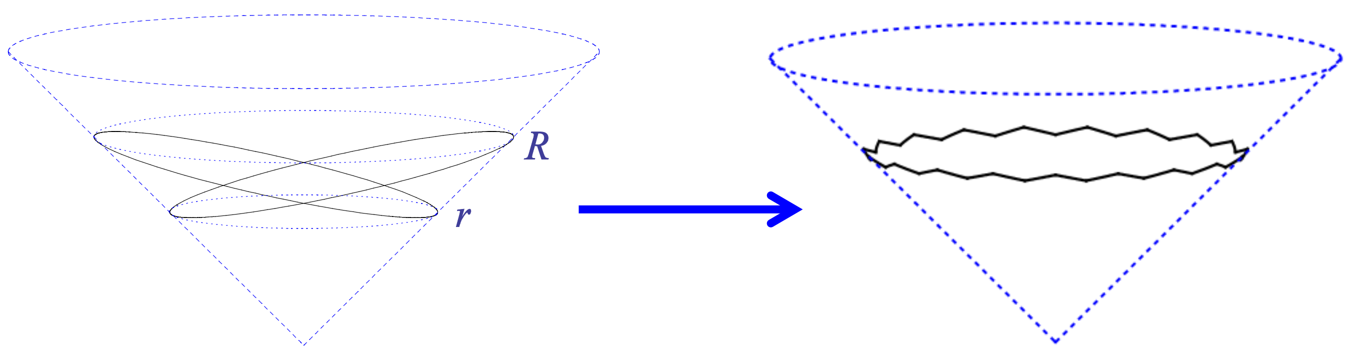
Entanglement proof of F-theorem:

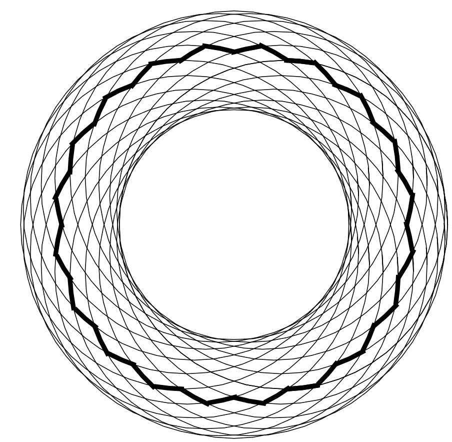
(Casini & Huerta '12)

- F-theorem for d=3 RG flows established using unitarity, Lorentz invariance and **strong subadditivity**

$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j)) + S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + \ldots + S(\cap_i X_i)$$

- geometry more complex than d=2: consider many circles intersecting on **null** cone



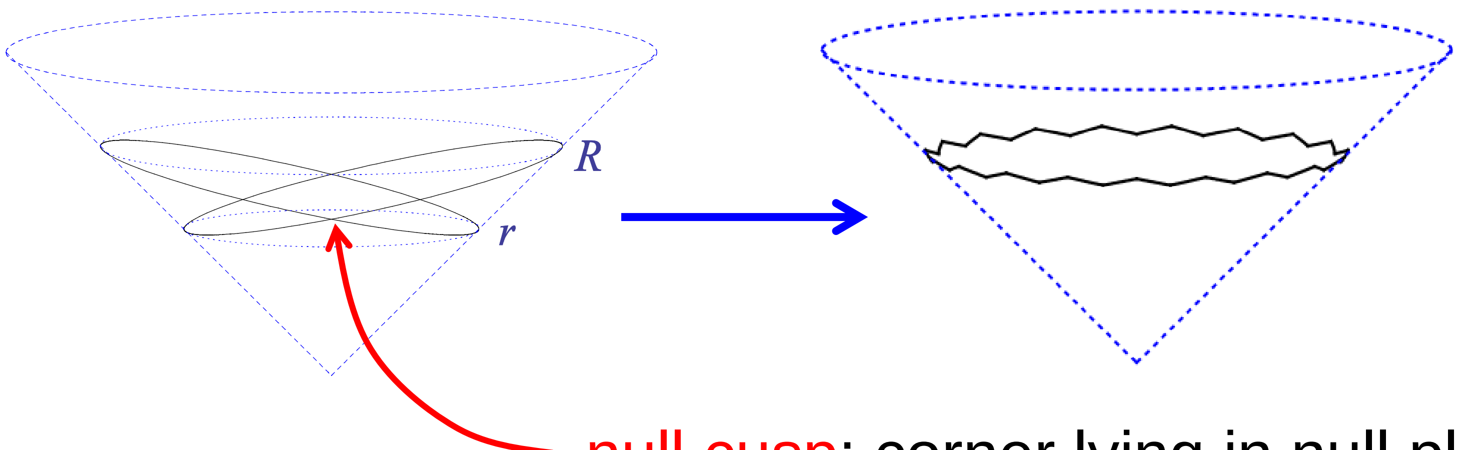


- no corner contribution from intersection in null plane
- define: $C(R) = RS'(R) - S(R)$
- for d=3 CFT: $S(R) = \frac{2\pi R}{\epsilon} c_0 - 2\pi a_3$ $\longrightarrow$ $C_{\rm CFT}(R) = 2\pi a_3$
- with SSA and "continuum" limit $\longrightarrow$ $\partial_R C(R) \leq 0$
- hence $C(R)$ decreases monotonically and $[a_3]_{\rm UV} > [a_3]_{\rm IR}$

Entanglement proof of F-theorem:

- key ingredients:

a) unitary & Lorentz invariant regularization of EE defined on regions with smooth boundaries except for "null cusps"
b) regulated EE satisfies strong subaddivity for sets whose union and intersection only generates more "null cusps"
c) wiggly circles have EE which approaches that of circle with same perimeter as the number of null cusps goes to ∞



null cusp: corner lying in null plane

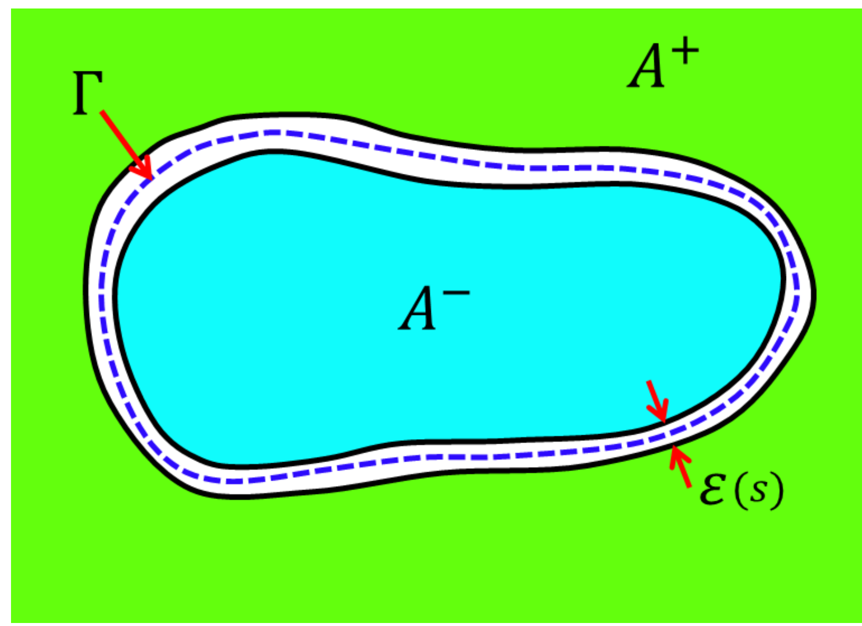
t_1 ¡ t_2 = ∀ with ∀¢∀ = 0

Entanglement proof of F-theorem:

- mutual information approach satisfy these key ingredients?
- consider region A with smooth boundary Γ
- expand boundary: $\Gamma_{\pm} = \Gamma \pm \frac{1}{2}\, \varepsilon(s)\, \hat{\boldsymbol{n}}(s)$

$$I(A^+, A^-) = \tilde{c}_0 \oint_\Gamma {}^{ds}\!/_{\varepsilon(s)} + I_0(A) + O(\varepsilon)$$

- regulated EE: property of A;
  independent of framing



eg, for circle

$I_0(A) = 2\pi R\, c_1(m_i) - 4\pi a_3$

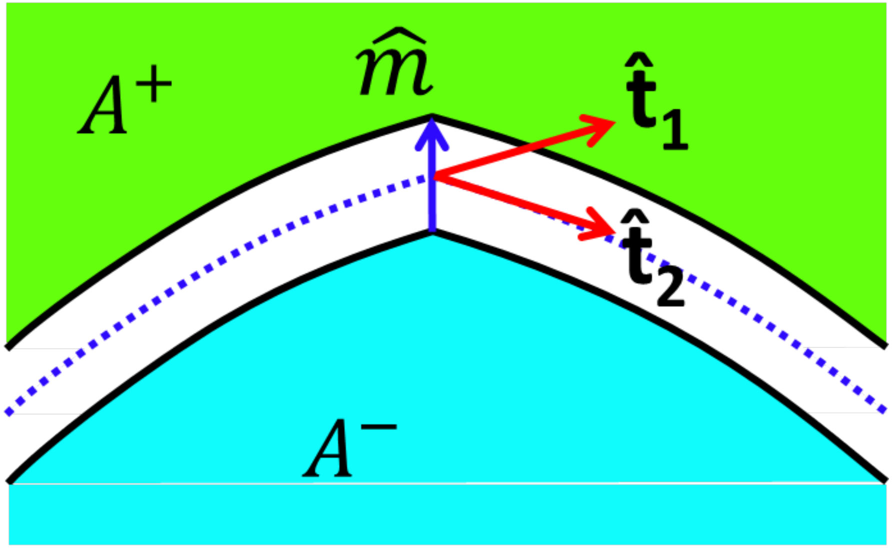
Entanglement proof of F-theorem:

- mutual information approach satisfy these key ingredients? **yes**
- consider region A with ~~smooth~~ boundary Γ **with null cusps**
- expand boundary: $\Gamma_\pm = \Gamma \pm \frac{1}{2}\,\varepsilon(s)\,\hat{\boldsymbol{n}}(s)$

$$I(A^+,A^-) = \tilde{c}_0 \oint_\Gamma {}^{ds}\!/_{\varepsilon(s)} + I_0(A) + \sum f(q_{1i},q_{2i}) + O(\varepsilon)$$

- additional contributions for null cusps characterized by two local invariants:

$\boldsymbol{q_1 = \hat{m}\cdot\hat{t}_1} \qquad \boldsymbol{q_2 = \hat{m}\cdot\hat{t}_2}$



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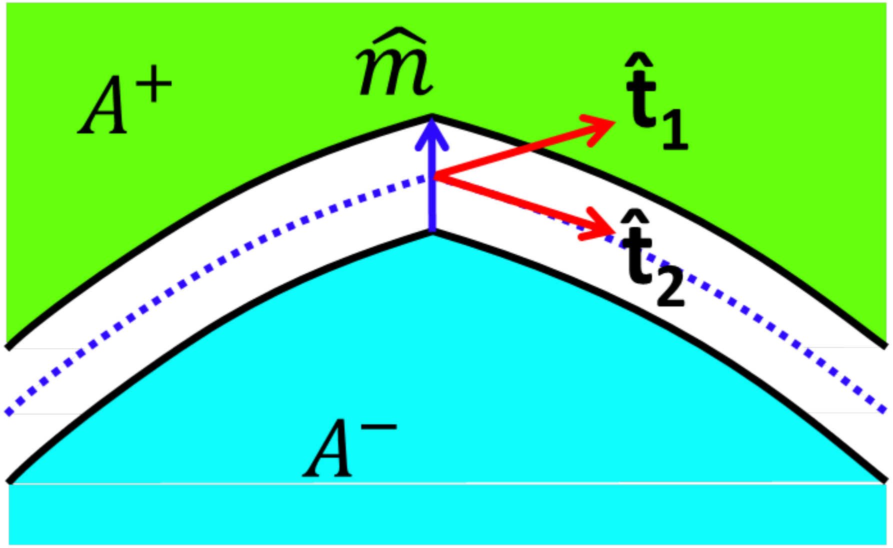

- $I_0(A)$ still satisfies SSA:

$$\mathbf{I_0(A) + I_0(B)} \geq \boldsymbol{I_0(A\cup B) + I_0(A\cap B)}$$

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- monotonic flow follows as in entropic proof of F-theorem

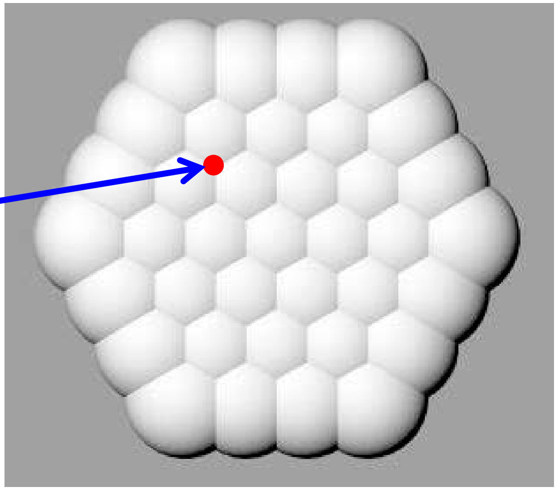
  $\longrightarrow$ have properly established F-theorem in d=3

Beyond d=3:

- is there entropic proof of c-theorem in higher dimensions?

  → need a new idea?

higher dim. intersections lead
to subleading divergences
which trivialize SSA inequality



Beyond d=3:

(Komargodski & Schwimmer;
see also: Luty, Polchinski & Rattazzi)

d=4 a-theorem and Dilaton Effective Action

(Schwimmer & Theisen)

- analyze RG flow as "broken conformal symmetry"
- couple theory to "dilaton" (conformal compensator) and organize effective action in terms of $\hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu}$

  diffeo X Weyl invariant: $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu} \quad \tau \to \tau + \sigma$

- follow effective dilaton action to IR fixed point, eg,

$$S_{anomaly} = -\delta a \int d^4x \sqrt{-g} \Big(\tau E_4 + 4(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) \partial_\mu \tau \partial_\nu \tau - 4(\partial\tau)^2 \Box\, \tau + 2(\partial\tau)^4 \Big)$$

$\delta a = a_{UV} - a_{IR}$: ensures UV & IR anomalies match

- with $g \to \eta$, only contribution to 4pt amplitude with null dilatons:

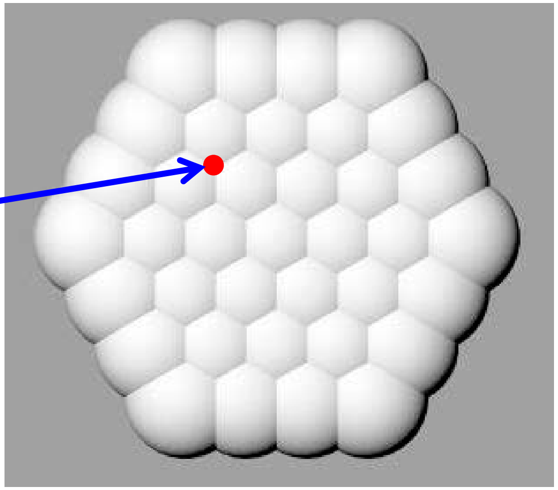
$$S_{anomaly} = 2\,\delta a \int d^4x\, (\partial\tau)^4$$

- dispersion relation plus optical theorem demand: $\delta a > 0$
- no entanglement in sight?

Beyond d=3:

• is there entropic proof of c-theorem in higher dimensions?

→ need a new idea?



higher dim. intersections lead
to subleading divergences
which trivialize SSA inequality

• d=4 a-theorem proved with more “standard” QFT techniques

(Komargodski & Schwimmer)

• hybrid approach proposed (Solodukhin): still needs development

• can c-theorems be proved for higher dimensions? eg, d=5 or 6

→ again, entropic approach needs a new idea

→ dilaton-effective-action approach requires refinement for d=6

(Elvang, Freedman, Hung, Kiermaier, RM & Theisen; Elvang & Olson)

Conclusions and Questions:

- entanglement lends new insights into c-theorems
- using mutual information, properly established d=3 F-theorem
- how much of Zamolodchikov's structure for d=2 RG flows extends higher dimensions?

→ d=3 entropic C-function not always stationary at fixed points

(Klebanov, Nishioka, Pufu & Safdi)

→ same already observed for d=2; special case or generic? need a better C-function?

Zamolodchikov c-theorem (1986):

- renormalization-group (RG) flows can seen as one-parameter motion

$$\frac{d}{dt} \equiv -\beta^i(g)\,\frac{\partial}{\partial g^i}$$

in the space of (renormalized) coupling constants $\{g^i,\ i=1,2,3,\cdots\}$ with beta-functions as “velocities”

- for unitary, Lorentz-inv. QFT’s in two dimensions, there exists a positive-definite real function of the coupling constants $C(g)$:

  1. monotonically decreasing along flows: $\frac{d}{dt}C(g) \le 0$
  2. “stationary” at fixed points $g^i = (g^*)^i$:

$$\beta^i(g^*) = 0 \longleftrightarrow \frac{\partial}{\partial g^i}C(g) = 0$$

  3. at fixed points, it equals central charge of corresponding CFT

$$C(g^*) = c$$

Conclusions and Questions:

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$\longrightarrow$ d=3 entropic C-function not always stationary at fixed points

(Klebanov, Nishioka, Pufu & Safdi)

$\longrightarrow$ same already observed for d=2; special case or generic? need a better C-function?

- does scale invariance imply conformal invariance beyond d=2?

$\longrightarrow$ "more or less" in d=4 (Luty, Polchinski & Rattazzi; Dymarsky, Komargodski, Schwimmer & Theisen)

- further lessons: RG flows and entanglement $\longleftrightarrow$ holography?

SSA $\longrightarrow$ NEC (Bhattacharya etal; Lashkari et al; Lin etal)

- what can entanglement/quantum information really say about RG flows, holography or nonperturbative QFT?