Anomalies and Entanglement Entropy

Tatsuma Nishioka (University of Tokyo)

based on a work with A. Yarom (Technion)
(to appear)

Roles of entanglement entropy in QFT

- An order parameter for various phase transitions
  - Confinement/deconfinement (like Polyakov loop) [TN-Takayanagi 06, Klebanov-Kutasov-Murugan 07, · · ·]
  - Classification of phases [Calabrese-Cardy 04, Kitaev-Preskill 06, Levin-Wen 06, · · ·]
- Reconstruction of bulk geometry from entanglement
  - Similarity between MERA and AdS space [Swingle 09, Nozaki-Ryu-Takayanagi 12, · · ·]
  - 1st law of entanglement and linearized Einstein equation of GR [Bhattacharya-Nozaki-Takayanagi-Ugajin 12, Blanco-Casini-Hung-Myers 13, Nozaki-Numasawa-Prudenziati-Takayanagi 13, Lashkari-McDermott-Raamsdonk 13, Faulkner-Guica-Hartman-Myers-Raamsdonk 13, · · ·]

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Renormalization group and entanglement entropy

- Entanglement entropy as a measure of degrees of freedom

- Construct a monotonic function c(Energy) of the energy scale
  - Entropic c-theorem in two dimensions [Casini-Huerta 04]
  - F-theorem in three dimensions [Jafferis-Klebanov-Pufu-Safdi 11, Myers-Sinha 10, Liu-Mezei 12, Casini-Huerta 12]

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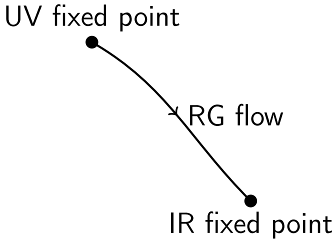

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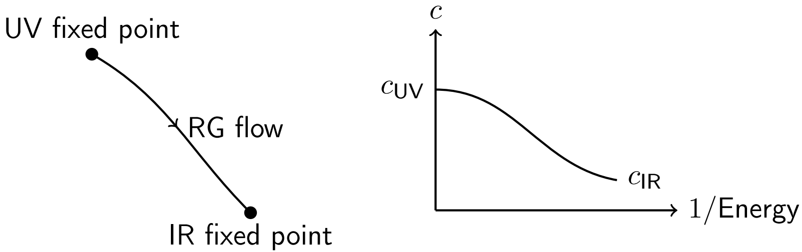

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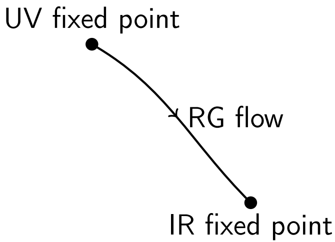

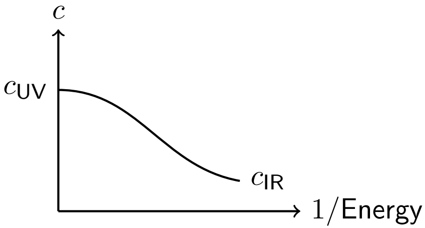

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Universal terms

- Conformal anomalies in even d dimensions: [cf. Myers-Sinha 10]

$$S_A = \frac{c_{d-2}}{\epsilon^{d-2}} + \cdots + c_0 \log \epsilon + \cdots \ , \qquad c_0 \sim \text{central charges}$$

- Relation to sphere partition function $F \equiv (-1)^{\frac{d-1}{2}} \log Z[\mathbb{S}^d]$: [Casini-Huerta-Myers 11]

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- Gravitational anomaly in CFT$_2$ with $c_L \neq c_R$
  [Wall 11, Castro-Detournay-Iqbal-Perlmutter 14]
  (Holography: [Guo-Miao 15, Azeyanagi-Loganayagam-Ng 15])

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How about chiral and (mixed-)gravitational anomalies in other dimensions?

Outline

1. Entanglement entropy in QFT
2. Weyl anomalies
3. Consistent gravitational anomalies
4. Other anomalies

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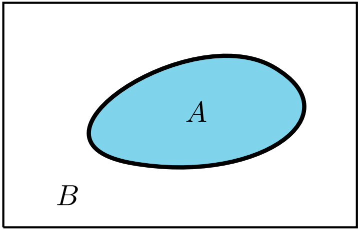
Definition of entanglement entropy

- Divide a system to A and $B = \bar{A}$: $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$



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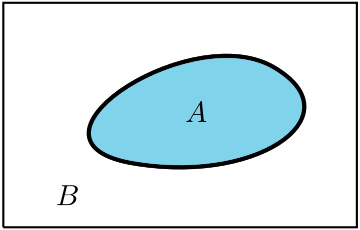

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- Rewrite the definition

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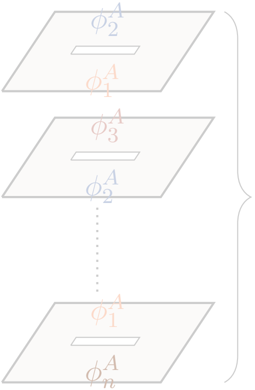

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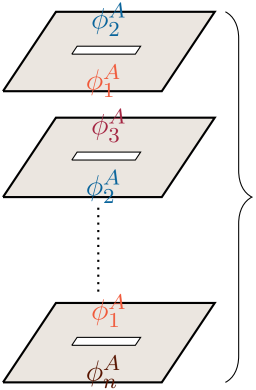

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Replica trick

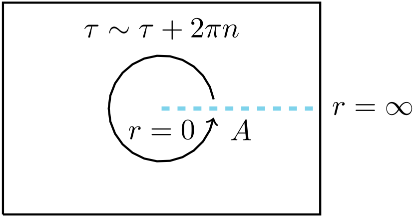
Entanglement entropy

$$S_A = \lim_{n\to 1} \frac{W_n - nW_1}{n-1}$$

- $W_n = -\log Z[\mathcal{M}_n]$: the Euclidean partition function
- $\mathcal{M}_n$: the n-fold cover with a surplus angle $2\pi(n-1)$ around the entangling surface $\Sigma \equiv \partial A$

- Suppose A is a semi-infinite line in two dimensions

$$A = \{0 \le r < \infty,\ \tau = 0\}$$

$\mathcal{M}_n \quad = $ 

Strategy

- Use a partition function W reproducing flavor and (mixed-)gravitational anomalies

- Evaluate W on the n-fold cover $\mathcal{M}_n$ that is an S^1 fibration over an entangling region A

- Calculate the variation of the entanglement entropy with W

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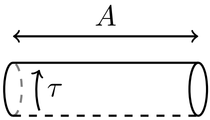

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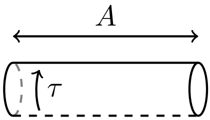

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1 Entanglement entropy in QFT

2 Weyl anomalies

3 Consistent gravitational anomalies

4 Other anomalies

Weyl anomaly

- CFT is classically invariant under the Weyl rescaling $\delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu}$ because of $T^\mu{}_\mu = 0$:

$$\delta_\sigma W = \frac{1}{2}\int d^dx\sqrt{g}\,\sigma\,T^\mu{}_\mu = 0$$

- The Weyl anomalies in even d dimensions:

$$T^\mu{}_\mu = 2(-1)^{d/2}\,a\,E - \sum_i c_i\,I_i$$

E: the Euler density, I_i: Weyl invariants in d dimensions
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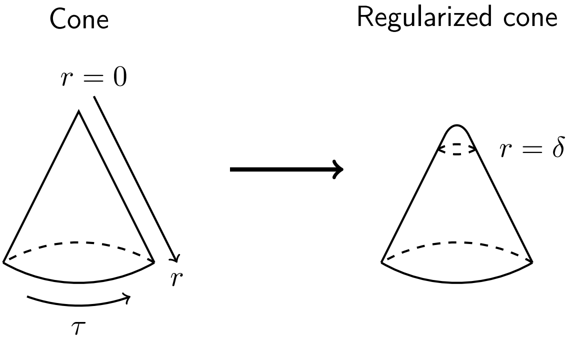

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$$\begin{aligned}\delta_\sigma W_n - n\delta_\sigma W_1 &= \frac{1}{2}\int_{\mathcal{M}_n} \sigma \left[\langle T^\mu{}_\mu\rangle_{\mathcal{M}_n} - \langle T^\mu{}_\mu\rangle_{\mathcal{M}_1}\right] \\ &= (n-1)\sigma \int_\Sigma f(a, c_i, g_{\mu\nu}) + O\left((n-1)^2\right)\end{aligned}$$

because locally $\mathcal{M}_n \simeq \mathcal{M}_1$ away from Σ

Weyl anomaly in entanglement entropy

- In even dimension:

$$\delta_\sigma S_A = \sigma\, c_0 \ , \quad c_0 = \int_\Sigma f(a, c_i, g_{\mu\nu})$$

- It gives the logarithmic divergence: [Holzhey-Larsen-Wilczek 94, Ryu-Takayanagi 06, Solodukhin 08, Myers-Sinha 10]

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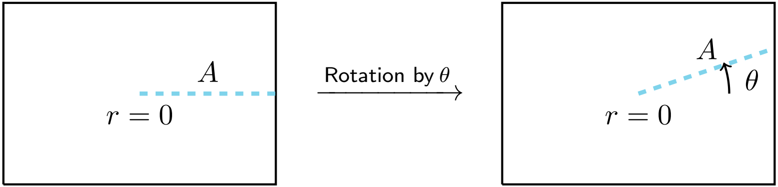

Covariant and consistent gravitational anomalies

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The stress tensor is covariant

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$$\delta W = \int \Omega_{2k}$$

- $2k$-form Ω_{2k} is fixed by $2k+2$-form anomaly polynomials $\mathcal{P}_{2k+2}$

$$\mathcal{P}_{2k+2} = dI_{2k+1} \ , \qquad \delta I_{2k+1} = d\Omega_{2k}$$

I_{2k+1}: Chern-Simons $2k+1$-form

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Anomaly inflow mechanism

- Decompose W into gauge-invariant and anomalous parts

$$W = W_{\text{gauge-inv}} + W_{\text{anom}}$$

- The consistent anomaly for W_{anom} can be fixed by the anomaly inflow mechanism [Callan-Harvey 85, Jensen-Loganayagam-Yarom 12]

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Anomalous partition function in 2d

- Anomaly polynomial in 2d

$$\mathcal{P}_{2d} = c_g \mathrm{tr}(\mathcal{R} \wedge \mathcal{R})$$

- Chern-Simons form, $\mathcal{P}_{2d} = dI_{CS}$:

$$I_{CS} = c_g \,\mathrm{tr}\left[\Gamma \wedge d\Gamma + \frac{2}{3}\Gamma \wedge \Gamma \wedge \Gamma\right]$$

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- For a semi-infinite line

$$\mathcal{M}_n : \quad ds^2 = dr^2 + r^2 d\tau^2 \;, \qquad \tau \sim \tau + 2\pi n$$

- Rotation by θ on $\mathcal{M}_1$ $=$ Rotation by $n\theta$ on $\mathcal{M}_n$

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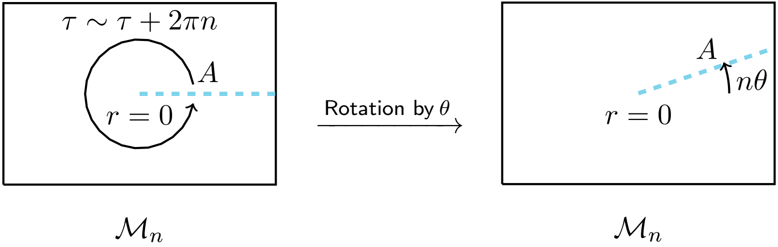

Einstein anomaly

- For a diffeomorphism $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$

$$\delta_\xi S_A = \lim_{n\to 1} \frac{\delta_\xi W_n - n\delta_\xi W_1}{n-1}$$

where

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- Rotation $\tau \to \tau + \theta$, $\xi^\tau = \theta$

Gravitational anomaly for a rotation by angle θ

$$\delta_\theta S_A = 4\pi i\, c_g\, \theta$$

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$$I_{CS} = A \wedge [(1-\alpha) c_m \mathrm{tr}(\mathcal{R} \wedge \mathcal{R})] + \alpha c_m F \wedge j_{GCS}(\Gamma)$$

- Ambiguity in CS form proportional to α

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$$\mathcal{M}_n = \mathcal{C}_n \times \mathcal{N}$$

$\mathcal{C}_n$: two-dimensional cone, $\mathcal{N}$: a (compact) two-manifold

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with $\Phi \equiv \int_{\mathcal{N}} F$

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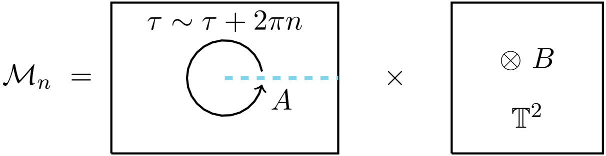
Mixed anomaly

- Take a plane entangling surface



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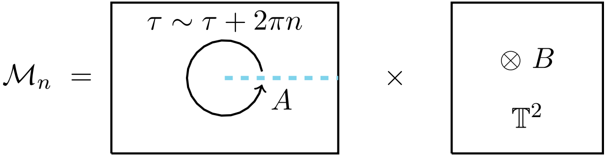

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- Six-dimensional theories have two types of gravitational anomalies and a mixed gauge-gravitational anomaly:

$$\mathcal{P}_{6d} = c_m\, F^2\, \mathrm{tr}\left(\mathcal{R}^2\right) + c_a\, \mathrm{tr}\left(\mathcal{R}^2\right)^2 + c_b\, \mathrm{tr}\left(\mathcal{R}^4\right)$$

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Extract c_m and c_a

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Gravitational and mixed anomalies under the boost

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$$ds^2 = dr^2 + r^2 \left(d\tau + \omega_1(y^1)dy^2 + \omega_2(y^3)dy^4\right)^2 + \sum_{i=1}^{4}(dy^i)^2$$

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Higher dimensions

- Anomaly polynomial in higher dimensions:

$$\mathcal{P}_d = \sum_i c_i\, F^{k_i} \prod_{n=1} \mathrm{tr}\left(\mathcal{R}^{2n}\right)^{m_n^i}$$

where for each i, $2k_i + \sum_n 4nm_n^i = d + 2$

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Outline

1 Entanglement entropy in QFT

2 Weyl anomalies

3 Consistent gravitational anomalies

4 Other anomalies

Gauge anomalies

- $U(1)$ polygon anomaly in $d = 2m$ dimensions

$$\mathcal{P} = cF^m$$

- For $A \to A + d\lambda$,

$$\delta_\lambda W_n = ic\,\lambda \int_{\mathcal{M}_n} F^m$$

- The external gauge field A satisfying the $\mathbb{Z}_n$ replica symmetry

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Summary

- Systematic treatment of anomalies in entanglement entropy using the anomaly inflow mechanism
- Our method is valid for any QFT
- Can be applied as well in higher even dimensions (including mixed anomalies), but how about other anomalies such as parity anomaly in odd dimensions?