

Entanglement Entropy of Disjoint Regions in Excited States : An Operator Method

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1. Introduction

(1) **An operator method** = A new computational method of entanglement entropy (EE) based on the idea of [J. Cardy 2013](#)

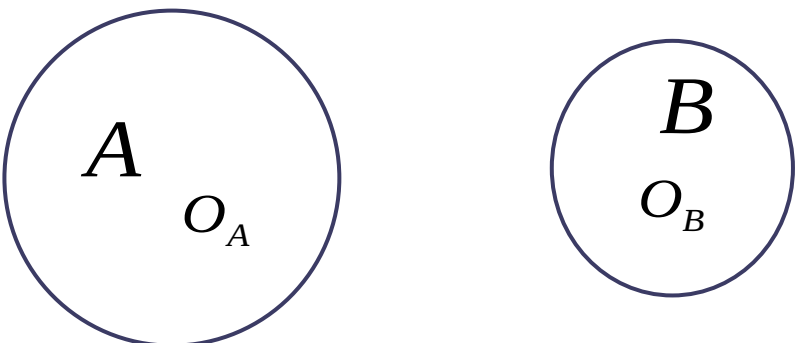
Renyi EE is expressed as a expectation value of a local operator

$$\text{Tr} \rho_{\Omega}^n = \text{Tr}(\rho^{(n)} E_{\Omega})$$

It is useful for Disjoint subsystem
 Excited states

(2) Its application to EE of Disjoint Regions in the vacuum state and Locally Excited States

EE in locally excited states [M. Nozaki, T. Numasawa and T. Takayanagi 2014](#)

$$|\Psi\rangle = N(O_{iA}O_{jB} + O_{i'A}O_{j'B})|0\rangle$$


The diagram consists of two circles. The left circle is labeled 'A' and contains the operator O_A . The right circle is labeled 'B' and contains the operator O_B .

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2. The definition of (Renyi) entanglement entropy and (Renyi) mutual information

We trace out the degrees of freedom of B and consider the reduced density matrix of A.

$$\rho_A = \text{Tr}_B \rho_{\text{tot}}$$

EE is defined as von Neumann entropy.

$$S_A := -\text{tr}_A \rho_A \log \rho_A$$

The Renyi EE is the generalization of EE and defined as

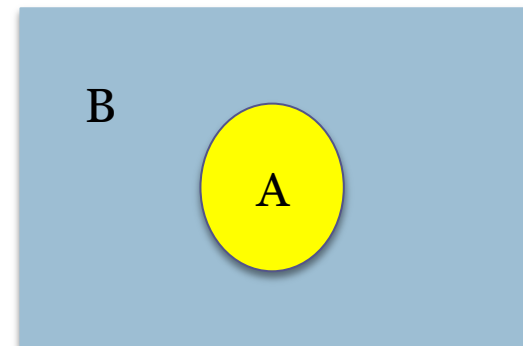
$$S_A^{(n)} := \frac{1}{1-n} \log \text{Tr}(\rho_A^n)$$

$$S_A = \lim_{n \rightarrow 1} S_A^{(n)}$$

Mutual (Renyi) information

$$I(A, B) := S_A + S_B - S_{A \cup B}$$

$$I^{(n)}(A, B) := S_A^{(n)} + S_B^{(n)} - S_{A \cup B}^{(n)}$$



3. An operator method in EE

We consider the general scalar field in $(d+1)$ dimensional spacetime and do not specify its Hamiltonian.

We consider n copies of the scalar fields and the j -th copy of the scalar field is denoted by $\{\phi^{(j)}\}$.

Thus the total Hilbert space, $H^{(n)}$, is the tensor product of the n copies of the Hilbert space,

$$H^{(n)} = H \otimes H \cdots \otimes H$$

where H is the Hilbert space of one scalar field.

We define the density matrix $\rho^{(n)}$ in $H^{(n)}$ as

$$\rho^{(n)} = \rho \otimes \rho \cdots \otimes \rho$$

where ρ is an arbitrary density matrix in H .

We can express $\text{Tr} \rho_{\Omega}^n$ as

$$\text{Tr} \rho_{\Omega}^n = \text{Tr}(\rho^{(n)} E_{\Omega})$$

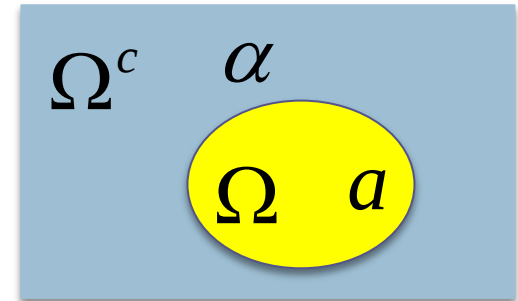
Discrete version

$$\text{Tr} \rho_{\Omega}^n = \text{Tr}(\rho^{(n)} E_{\Omega})$$

$$\begin{aligned} E_{\Omega}(q_B^{(1)'}, \dots, q_B^{(n)'}; q_A^{(1)}, \dots, q_A^{(n)}) &\equiv \langle \{q_B^{(1)'}\}, \dots, \{q_B^{(n)'}\} | E_{\Omega} | \{q_A^{(1)}\}, \dots, \{q_A^{(n)}\} \rangle \\ &\equiv \prod_a \prod_{\alpha} \delta(q_{\alpha}^{(1)} - q_{\beta}^{(1)'}) \delta(q_{\alpha}^{(2)} - q_{\beta}^{(2)'}) \dots \delta(q_{\alpha}^{(n)} - q_{\beta}^{(n)'}) \\ &\quad \times \delta(q_b^{(1)'} - q_a^{(2)}) \delta(q_b^{(2)'} - q_a^{(3)}) \dots \delta(q_b^{(n)'} - q_a^{(1)}). \end{aligned}$$

$$[\hat{q}_A, \hat{p}_B] = i\delta_{AB}, \quad [\hat{q}_A, \hat{q}_B] = [\hat{p}_A, \hat{p}_B] = 0.$$

We rewrite the delta functions as the Fourier integrals and use the following identity



$$\int dq' \int dq |q'\rangle \langle q| = \int dK \exp[iK \hat{p}], \quad \text{and obtain}$$

$$\begin{aligned} E_{\Omega} &= \int \prod_{j=1}^n \prod_{a \in \Omega} \frac{dJ_a^{(j)}}{2\pi} dK_a^{(j)} \exp[i(J_a^{(2)} \hat{q}_a^{(1)} + J_a^{(3)} \hat{q}_a^{(2)} + \dots + J_a^{(n)} \hat{q}_a^{(n-1)} + J_a^{(1)} \hat{q}_a^{(n)})] \\ &\quad \times \exp[i(K_a^{(1)} \hat{p}_a^{(1)} + K_a^{(2)} \hat{p}_a^{(2)} + \dots + K_a^{(n)} \hat{p}_a^{(n)})] \\ &\quad \times \exp[-i(J_a^{(1)} \hat{q}_a^{(1)} + J_a^{(2)} \hat{q}_a^{(2)} + \dots + J_a^{(n)} \hat{q}_a^{(n)})]. \end{aligned}$$

Continuum version

$$\text{Tr} \rho_{\Omega}^n = \text{Tr}(\rho^{(n)} E_{\Omega})$$

$$q \rightarrow \phi \quad p \rightarrow \pi \quad A \rightarrow x$$

$$E_{\Omega} = \int \prod_{j=1}^n \prod_{x \in \Omega} DJ^{(j)}(x) DK^{(j)}(x) \exp \left[i \int d^d x \sum_{l=1}^n J^{(l+1)}(x) \phi^{(l)}(x) \right] \\ \times \exp \left[i \int d^d x \sum_{l=1}^n K^{(l)}(x) \pi^{(l)}(x) \right] \times \exp \left[-i \int d^d x \sum_{l=1}^n J^{(l)} \phi^{(l)} \right]$$

where $\pi(x)$ is a conjugate momenta of $\phi(x)$,
 $[\phi(x), \pi(y)] = i\delta^d(x-y)$ and

$J^{(j)}(x)$ and $K^{(j)}(x)$ exist only in Ω and $J^{(n+1)} = J^{(1)}$
 $J^{(j)}(x)$ and $K^{(j)}(x)$ are auxiliary fields.

Thus E_{Ω} is a operator at Ω .

General properties of E_Ω

(1) Symmetry:

$$E_\Omega(\phi^{(1)}, \dots, \phi^{(n)}, \pi^{(1)}, \dots, \pi^{(n)}) = E_\Omega(-\phi^{(1)}, \dots, -\phi^{(n)}, -\pi^{(1)}, \dots, -\pi^{(n)}).$$

(2) Locality: when $\Omega = A \cup B$ and $A \cap B = \emptyset$

$$E_{A \cup B} = E_A E_B$$

(3) For n arbitrary operators F_j ($j = 1, 2, \dots, n$) on H ,

$$\text{Tr}(F_1 \otimes F_2 \otimes \dots \otimes F_n \cdot E_\Omega) = \text{Tr}(F_{1\Omega} F_{2\Omega} \dots F_{n\Omega}),$$

$$\text{where } F_{j\Omega} = \text{Tr}_{\Omega^c} F_j$$

This is the generalization of $\text{Tr} \rho_\Omega^n = \text{Tr}(\rho^{(n)} E_\Omega)$

(4) The cyclic property:

$$\text{Tr}(F_1 \otimes F_2 \otimes \dots \otimes F_n \cdot E_\Omega) = \text{Tr}(F_2 \otimes F_3 \otimes \dots \otimes F_n \otimes F_1 \cdot E_\Omega)$$

(5) The relation between E_{Ω} and E_{Ω^c} for pure states:

$$\langle \psi_1 | \langle \psi_2 | \dots \langle \psi_n | E_{\Omega} | \phi_1 \rangle | \phi_2 \rangle \dots | \phi_n \rangle = [\langle \phi_2 | \langle \phi_3 | \dots \langle \phi_n | \langle \phi_1 | E_{\Omega^c} | \psi_1 \rangle | \psi_2 \rangle \dots | \psi_n \rangle]^*$$

where $|\phi_j\rangle$ and $|\psi_j\rangle$ are arbitrary pure states.

By using (5), we can obtain the following basic property for a pure state $\rho = |\psi\rangle\langle\psi|$

$$\text{Tr} \rho_{\Omega}^n = \langle \Psi^{(n)} | E_{\Omega} | \Psi^{(n)} \rangle = [\langle \Psi^{(n)} | E_{\Omega^c} | \Psi^{(n)} \rangle]^* = \text{Tr} \rho_{\Omega^c}^n.$$

So (5) is the generalization of $\text{Tr} \rho_{\Omega}^n = \text{Tr} \rho_{\Omega^c}^n$ for a pure state $\rho = |\psi\rangle\langle\psi|$

(5) is a remarkable property because it relates E_{Ω} with E_{Ω^c} which exists on Ω^c , the complement of Ω .

In particular, the properties (1), (2), and (5) are useful for computing EE of excited states.

3. Application to free scalar fields

For free scalar fields, it is useful to represent the operator E_{Ω} as **the normal ordered operator**.

$$E_{\Omega} = \int \prod_{j=1}^n \prod_{x \in \Omega} DJ^{(j)}(x) DK^{(j)}(x) : \exp \left[i \sum_{l=1}^n \int d^d x ((J^{(l+1)} - J^{(l)})\phi^{(l)} + K^{(l)}\pi^{(l)}) \right] : \exp[-\tilde{S}]$$

$$\begin{aligned} \tilde{S} \equiv \sum_{l=1}^n & \left[\int d^d x d^d y \left[\frac{1}{4} K^{(l)}(x) W(x-y) K^{(l)}(y) + \frac{1}{4} (J^{(l+1)} - J^{(l)})(x) W^{-1}(x-y) (J^{(l+1)} - J^{(l)})(y) \right] \right. \\ & \left. + \frac{i}{2} \int d^d x K^{(l)}(x) (J^{(l+1)} + J^{(l)})(x) \right]. \end{aligned}$$

$$[\phi^+(x), \phi^-(y)] = \langle 0 | \phi(x) \phi(y) | 0 \rangle \equiv \frac{1}{2} W^{-1}(x-y),$$

$$[\pi^+(x), \pi^-(y)] = \langle 0 | \pi(x) \pi(y) | 0 \rangle \equiv \frac{1}{2} W(x-y)$$

By expanding the exponential in the normal ordered product and performing the Gauss integral of J and K, we can rewrite the E_{Ω} as a series of operators.

3-2 The case n=2

$$\phi_{\pm} = \frac{1}{\sqrt{2}}(\phi^{(1)} \pm \phi^{(2)}), \quad \pi_{\pm} = \frac{1}{\sqrt{2}}(\pi^{(1)} \pm \pi^{(2)}), \quad J_{\pm} = \frac{1}{\sqrt{2}}(J^{(1)} \pm J^{(2)}), \quad K_{\pm} = \frac{1}{\sqrt{2}}(K^{(1)} \pm K^{(2)}).$$

$$E_{\Omega} = \int DJ_- DK_- : \exp \left[i \int d^d x (-2J_- \phi_- + K_- \pi_-) \right] : \\ \times \exp \left[\int d^d x d^d y \left(-\frac{1}{4} K_-(x) W(x-y) K_-(y) - J_-(x) W^{-1}(x-y) J_-(y) \right) \right]$$

$$W(x-y) = \begin{pmatrix} W(x_{\Omega} - y_{\Omega}) & W(x_{\Omega} - y_{\Omega^c}) \\ W(x_{\Omega^c} - y_{\Omega}) & W(x_{\Omega^c} - y_{\Omega^c}) \end{pmatrix} \equiv \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

$$W^{-1}(x-y) = \begin{pmatrix} W^{-1}(x_{\Omega} - y_{\Omega}) & W^{-1}(x_{\Omega} - y_{\Omega^c}) \\ W^{-1}(x_{\Omega^c} - y_{\Omega}) & W^{-1}(x_{\Omega^c} - y_{\Omega^c}) \end{pmatrix} \equiv \begin{pmatrix} D & E \\ E^T & F \end{pmatrix}$$

$$\langle J_-(x) J_-(y) \rangle \equiv \frac{\int DJ_- J_-(x) J_-(y) e^{-\int d^d x d^d y J_-(x) W^{-1}(x-y) J_-(y)}}{\int DJ_- e^{-\int d^d x d^d y J_-(x) W^{-1}(x-y) J_-(y)}} = \frac{1}{2} D^{-1}(x-y)$$

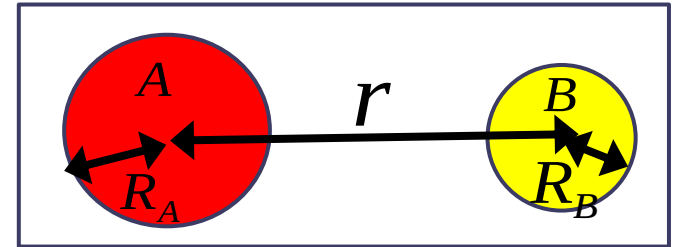
$$\langle K_-(x) K_-(y) \rangle \equiv \frac{\int DK_- K_-(x) K_-(y) e^{-\int d^d x d^d y \frac{1}{4} K_-(x) W(x-y) K_-(y)}}{\int DK_- e^{-\int d^d x d^d y \frac{1}{4} K_-(x) W(x-y) K_-(y)}} = 2A^{-1}(x-y).$$

$$E_{\Omega} = \text{Tr} \rho_{0\Omega}^2 \left[1 - 2 \int d^d x d^d y \langle - (x) J_- (y) \rangle : \phi_- (x) \phi_- (y) : \right. \\ \left. - \frac{1}{2} \int d^d x d^d y \langle K_- (x) K_- (y) \rangle : \pi_- (x) \pi_- (y) : + \dots \right].$$

The mutual Renyi information $I^{(2)}(A, B)$ for the vacuum state

$$\frac{\text{Tr} \rho_{0A \cup B}^2}{\text{Tr} \rho_{0A}^2 \text{Tr} \rho_{0B}^2} = \frac{\langle 0^{(2)} | E_A E_B | 0^{(2)} \rangle}{\langle 0^{(2)} | E_A | 0^{(2)} \rangle \langle 0^{(2)} | E_B | 0^{(2)} \rangle} \simeq 1 + \frac{1}{2} C_A C_B (W^{-1}(r))^2,$$

$$C_{A(B)} = \int d^d x d^d y \langle J_- (x) J_- (y) \rangle.$$



$$\langle 0^{(2)} | : \phi_-^2 (x) :: \phi_-^2 (y) : | 0^{(2)} \rangle = 2(\langle 0 | \phi(x) \phi(y) | 0 \rangle)^2 = \frac{1}{2} (W^{-1}(x - y))^2.$$

$$W^{-1}(x - y) = \frac{B_d}{|x - y|^{d-1}}, \quad r \gg R_{A,B} \quad I^{(2)}(A, B) \simeq \frac{1}{2} C_A C_B \frac{B_d^2}{r^{2d-2}}.$$

The expression of $C_{A(B)}$ is useful for numerical computation. There is another expression of $C_{A(B)}$ which is useful for analytical computation. [J.Cardy 2013](#)

4.EE of Disjoint Regions in Locally Excited States

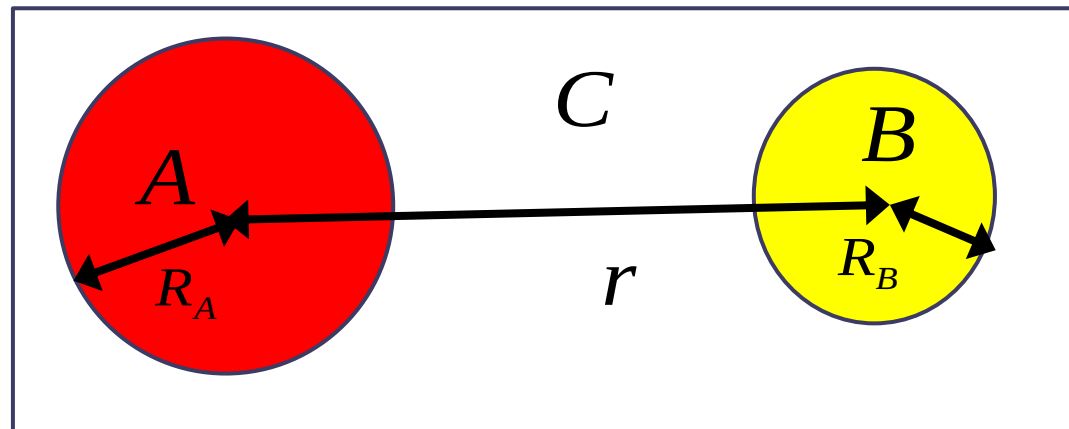
EE in locally excited states [M. Nozaki, T. Numasawa and T. Takayanagi 2014](#)

We consider the mutual Renyi information $I^{(n)}(A, B) := S_A^{(n)} + S_B^{(n)} - S_{A \cup B}^{(n)}$ of disjoint compact spatial regions A and B in the locally excited states.

$$|\Psi\rangle = N(O_{iA}O_{jB} + O_{i'A}O_{j'B})|0\rangle$$

where

$$\langle 0|O_{iA}^\dagger O_{i'A}|0\rangle = \langle 0|O_{jB}^\dagger O_{j'B}|0\rangle = 0.$$



$I^{(n)}(A, B)$ in the general QFT which
has a mass gap m

$$m^{-1} \ll r, R_A, R_B$$

$$|\Psi\rangle = N(O_{iA}O_{jB} + O_{i'A}O_{j'B}) |0\rangle$$

$$I^{(n)}(A, B) = \frac{2}{n-1} \ln \frac{(x+y)^n}{x^n + y^n}$$

$$I(A, B) = 2 \left[\ln(x+y) - \frac{1}{x+y} (x \ln x + y \ln y) \right]$$

$$x \equiv \langle 0 | O_{iA}^\dagger O_{iA} | 0 \rangle \langle 0 | O_{jB}^\dagger O_{jB} | 0 \rangle, \quad y \equiv \langle 0 | O_{i'A}^\dagger O_{i'A} | 0 \rangle \langle 0 | O_{j'B}^\dagger O_{j'B} | 0 \rangle$$

We can reproduce these results from the quantum mechanics.

$$|\Psi\rangle_{qm} = N(|i\rangle_A |j\rangle_B + |i'\rangle_A |j'\rangle_B),$$

$$I_{qm}^{(n)}(A, B) = \frac{2}{n-1} \ln \frac{(x_{qm} + y_{qm})^n}{x_{qm}^n + y_{qm}^n},$$

$$O_{i(i')A} |0\rangle \rightarrow |i(i')\rangle_A, \\ O_{j(j')B} |0\rangle \rightarrow |j(j')\rangle_B.$$

$$x_{qm} \equiv \langle i|i\rangle_A \langle j|j\rangle_B, \quad y_{qm} \equiv \langle i'|i'\rangle_A \langle j'|j'\rangle_B$$

The detail of the calculation of $I^{(n)}(A, B)$

We use the properties (2) and (5) of E_Ω .

$$\begin{aligned}\text{Tr} \rho_\Omega^n &= \langle \Psi^{(n)} | E_\Omega | \Psi^{(n)} \rangle \\ &= N^{2n} \langle 0^{(n)} | (O_{iA}^\dagger O_{jB}^\dagger + O_{i'A}^\dagger O_{j'B}^\dagger)^{(1)} \dots (O_{iA}^\dagger O_{jB}^\dagger + O_{i'A}^\dagger O_{j'B}^\dagger)^{(n)} \\ &\quad \times E_\Omega (O_{iA} O_{jB} + O_{i'A} O_{j'B})^{(1)} \dots (O_{iA} O_{jB} + O_{i'A} O_{j'B})^{(n)} | 0^{(n)} \rangle\end{aligned}$$

$$\begin{aligned}\text{Tr} \rho_{A \cup B}^n &= N^{2n} [\langle 0^{(n)} | (O_{iA}^\dagger O_{jB}^\dagger + O_{i'A}^\dagger O_{j'B}^\dagger)^{(1)} \dots (O_{iA}^\dagger O_{jB}^\dagger + O_{i'A}^\dagger O_{j'B}^\dagger)^{(n)} \\ (5) \quad &\quad \times E_{(A \cup B)^c} (O_{iA} O_{jB} + O_{i'A} O_{j'B})^{(1)} \dots (O_{iA} O_{jB} + O_{i'A} O_{j'B})^{(n)} | 0^{(n)} \rangle]^* \\ &\simeq \langle \Psi | \Psi \rangle [\langle 0^{(n)} | E_{(A \cup B)^c} | 0^{(n)} \rangle]^* = \langle 0^{(n)} | E_{A \cup B} | 0^{(n)} \rangle = \langle 0^{(n)} | E_A E_B | 0^{(n)} \rangle \\ mR \gg 1 \quad &\simeq \langle 0^{(n)} | E_A | 0^{(n)} \rangle \langle 0^{(n)} | E_B | 0^{(n)} \rangle = \text{Tr} \rho_{0A}^n \text{Tr} \rho_{0B}^n, \quad (2)\end{aligned}$$

$$\begin{aligned}&\langle 0^{(n)} | O_{i_1 A}^{(1)\dagger} \dots O_{i_n A}^{(n)\dagger} E_A O_{i_{n+1} A}^{(1)} \dots O_{i_{2n} A}^{(n)} \cdot O_{j_1 B}^{(1)\dagger} \dots O_{j_n B}^{(n)\dagger} O_{j_{n+1} B}^{(1)} \dots O_{j_{2n} B}^{(n)} | 0^{(n)} \rangle \\ mR \gg 1 \quad &\simeq \langle 0^{(n)} | O_{i_1 A}^{(1)\dagger} \dots O_{i_n A}^{(n)\dagger} E_A O_{i_{n+1} A}^{(1)} \dots O_{i_{2n} A}^{(n)} | 0^{(n)} \rangle \langle 0^{(n)} | O_{j_1 B}^{(1)\dagger} \dots O_{j_n B}^{(n)\dagger} O_{j_{n+1} B}^{(1)} \dots O_{j_{2n} B}^{(n)} | 0^{(n)} \rangle \\ &= \langle 0^{(n)} | O_{i_1 A}^{(1)\dagger} \dots O_{i_n A}^{(n)\dagger} E_A O_{i_{n+1} A}^{(1)} \dots O_{i_{2n} A}^{(n)} | 0^{(n)} \rangle \prod_{l=1}^n \langle 0 | O_{j_l B}^\dagger O_{j_{l+n} B} | 0 \rangle \\ &= \langle 0^{(n)} | O_{i_1 A}^{(1)\dagger} \dots O_{i_n A}^{(n)\dagger} E_A O_{i_{n+1} A}^{(1)} \dots O_{i_{2n} A}^{(n)} | 0^{(n)} \rangle \prod_{l=1}^n \delta_{j_l j_{l+n}} \langle 0 | O_{j_l B}^\dagger O_{j_l B} | 0 \rangle\end{aligned}$$

$$\begin{aligned}
\text{Tr} \rho_A^n &= N^{2n} [\langle 0 | O_{jB}^\dagger O_{jB} | 0 \rangle^n \langle 0^{(n)} | O_{iA}^{(1)\dagger} \dots O_{iA}^{(n)\dagger} E_A O_{iA}^{(1)} \dots O_{iA}^{(n)} | 0^{(n)} \rangle \\
&\quad + \langle 0 | O_{jB}^\dagger O_{jB} | 0 \rangle^{n-1} \langle 0 | O_{j'B}^\dagger O_{j'B} | 0 \rangle \\
&\quad \times (\langle 0^{(n)} | O_{i'A}^{(1)\dagger} O_{iA}^{(2)\dagger} \dots O_{iA}^{(n)\dagger} E_A O_{i'A}^{(1)} O_{iA}^{(2)} \dots O_{iA}^{(n)} | 0^{(n)} \rangle + \dots) \\
&\quad + \langle 0 | O_{jB}^\dagger O_{jB} | 0 \rangle^{n-2} \langle 0 | O_{j'B}^\dagger O_{j'B} | 0 \rangle^2 \\
&\quad \times (\langle 0^{(n)} | O_{i'A}^{(1)\dagger} O_{i'A}^{(2)\dagger} O_{iA}^{(3)\dagger} \dots O_{iA}^{(n)\dagger} E_A O_{i'A}^{(1)} O_{i'A}^{(2)} O_{iA}^{(3)} \dots O_{iA}^{(n)} | 0^{(n)} \rangle + \dots) \\
&\quad + \dots + \langle 0 | O_{j'B}^\dagger O_{j'B} | 0 \rangle^n \langle 0^{(n)} | O_{i'A}^{(1)\dagger} \dots O_{i'A}^{(n)\dagger} E_A O_{i'A}^{(1)} \dots O_{i'A}^{(n)} | 0^{(n)} \rangle]. \\
&\quad \langle 0^{(n)} | O_{i_1 A}^{(1)\dagger} \dots O_{i_n A}^{(n)\dagger} E_A O_{i_1 A}^{(1)} \dots O_{i_n A}^{(n)} | 0^{(n)} \rangle
\end{aligned}$$

$$(5) \longrightarrow = [\langle 0^{(n)} | O_{i_2 A}^{(1)\dagger} \dots O_{i_n A}^{(n-1)\dagger} O_{i_1 A}^{(n)\dagger} E_{A^c} O_{i_1 A}^{(1)} \dots O_{i_n A}^{(n)} | 0^{(n)} \rangle]^*$$

$$mR \gg 1 \longrightarrow \simeq [\langle 0^{(n)} | E_{A^c} | 0^{(n)} \rangle \langle 0^{(n)} | O_{i_2 A}^{(1)\dagger} \dots O_{i_n A}^{(n-1)\dagger} O_{i_1 A}^{(n)\dagger} O_{i_1 A}^{(1)} \dots O_{i_n A}^{(n)} | 0^{(n)} \rangle]^*$$

$$= \langle 0^{(n)} | E_A | 0^{(n)} \rangle \prod_{l=1}^n \langle 0 | O_{i_l A}^\dagger O_{i_{l+1} A} | 0 \rangle$$

$$= \begin{cases} \text{Tr} \rho_{0A}^n \langle 0 | O_{iA}^\dagger O_{iA} | 0 \rangle^n & \text{for } i = i_1 = i_2 = \dots = i_n \\ \text{Tr} \rho_{0A}^n \langle 0 | O_{i'A}^\dagger O_{i'A} | 0 \rangle^n & \text{for } i' = i_1 = i_2 = \dots = i_n \\ 0 & \text{otherwise} \end{cases}$$

$$(\because) \boxed{\text{Tr} \rho_A^n = \text{Tr} \rho_{0A}^n \cdot N^{2n} [\langle 0 | O_{iA}^\dagger O_{iA} | 0 \rangle^n \langle 0 | O_{jB}^\dagger O_{jB} | 0 \rangle^n + \langle 0 | O_{i'A}^\dagger O_{i'A} | 0 \rangle^n \langle 0 | O_{j'B}^\dagger O_{j'B} | 0 \rangle^n]}.$$

$I^{(n)}(A, B)$ in the free massless scalar field theory

$$|\Psi\rangle = N(O_{iA}O_{jB} + O_{i'A}O_{j'B})|0\rangle$$

we impose the condition that under the sign changing transformation $(\phi, \pi) \rightarrow (-\phi, -\pi)$ the operators O is transformed as

$$O \rightarrow (-1)^{|O|}O,$$

where $|O|=0$ or 1 . We use the property (1) of E_Ω and obtain,

(i) The case $|O_{iA}| = |O_{i'A}|$

$$I^{(n)}(A, B) = I^{(n)}(A, B)|_{r \rightarrow \infty} + O(1/r^{2d-2}),$$

(ii) The case $|O_{iA}| \neq |O_{i'A}|$ and $|O_{jB}| \neq |O_{j'B}|$

$$I^{(n)}(A, B) = I^{(n)}(A, B)|_{r \rightarrow \infty} + O(1/r^{d-1}),$$

For the vacuum state $I^{(n)}(A, B) \approx O(1/r^{2d-2})$

In (ii), the power of $1/r$ is different from that for the vacuum state.

5. Conclusion

We developed the computational method of EE based on the idea that $Tr\rho_{\Omega}^n$ is written as the expectation value of the local operator at Ω .

Operator method:

$$Tr\rho_{\Omega}^n = Tr(\rho^{(n)} E_{\Omega})$$

We found useful expression of this operator and some general properties.

$$E_{\Omega} = \int \prod_{j=1}^n \prod_{x \in \Omega} DJ^{(j)}(x) DK^{(j)}(x) \exp \left[i \int d^d x \sum_{l=1}^n J^{(l+1)}(x) \phi^{(l)}(x) \right] \\ \times \exp \left[i \int d^d x \sum_{l=1}^n K^{(l)}(x) \pi^{(l)}(x) \right] \times \exp \left[-i \int d^d x \sum_{l=1}^n J^{(l)} \phi^{(l)} \right]$$

As applications:

- (1) We obtain a useful expression for the coefficient of $I^{(2)}(A, B)$ for the vacuum state.

$$I^{(2)}(A, B) \simeq \frac{1}{2} C_A C_B \frac{B_d^2}{r^{2d-2}}.$$

$$C_{A(B)} = \int d^d x d^d y \langle J_-(x) J_-(y) \rangle.$$

- (2) For locally excited states

- (2-1) In the massive case, we explicitly compute $I^{(n)}(A, B)$
(2-2) In the massless free case, we obtain the leading r dependence.

The advantages of this methods are as follows:

- (1) We can use ordinary technique in QFT such as OPE and the cluster decomposition property
(2) We can use the general properties of the operator to compute systematically the Renyi entropy for an arbitrary state.

Future work

Application to the operator method to perturbative calculation in an interacting field theory.