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Quantum Entanglement of Local Operators

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1. arXiv:1401.0539 [hep-th]
2. arXiv:1405.5875 [hep-th]
3. arXiv:1405.5946 [hep-th]
4. arXiv:1507.04352[hep-th]
5. arXiv:150x.xxxxx

Introduction

Recently, (Renyi) entanglement entropy ((R)EE) has a center of wide interest in a broad array of theoretical physics.

- It is useful to study the distinctive features of various quantum state in condensed matter physics. (***Quantum Order Parameter***)
- (Renyi) entanglement entropy is expected to be an important quantity which may shed light on the mechanism behind the AdS/CFT correspondence .(***Gravity*** $\leftrightarrow$ ***Entanglement***)

Introduction

Recently, (Renyi) entanglement entropy ((R)EE) has a center of wide interest in a broad array of theoretical physics.

- In the lattice gauge theory, it is expected that entanglement entropy is a new order parameter which helps us study QCD more.
- (R)EE is expected to be entropy in non-equilibrium system.

Introduction

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- In the lattice gauge theory, it is expected that entanglement entropy is a new order parameter which helps us study QCD more.
- (R)EE is expected to be entropy in non-

It is important to study the properties of (Renyi) entanglement entropy.

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In this work, we investigate the time dependent property of (Renyi) entanglement entropy.

The Definition of (Renyi) Entanglement Entropy

- Definition of Entanglement Entropy

We divide the total Hilbert space into A and B: $H_{tot} = H_A \otimes H_B$.

The reduced density matrix ρ_A is defined by $\rho_A \equiv Tr_B \rho_{tot}$

This means the D O F in B are traced out.

The entanglement entropy is defined by von Neumann entropy S_A.

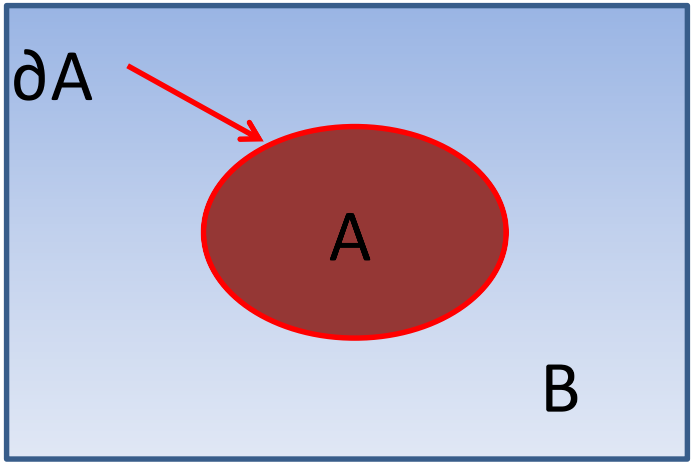
(Renyi) Entanglement Entropy (REE)

$$\boxed{S_A^{(n)} = \frac{\log \mathrm{tr}[\rho_A^n]}{1-n}}$$

n→1

Entanglement Entropy (EE)

$$\boxed{S_A = -\mathrm{tr}_A \rho_A \log \rho_A}$$



on a certain time slice

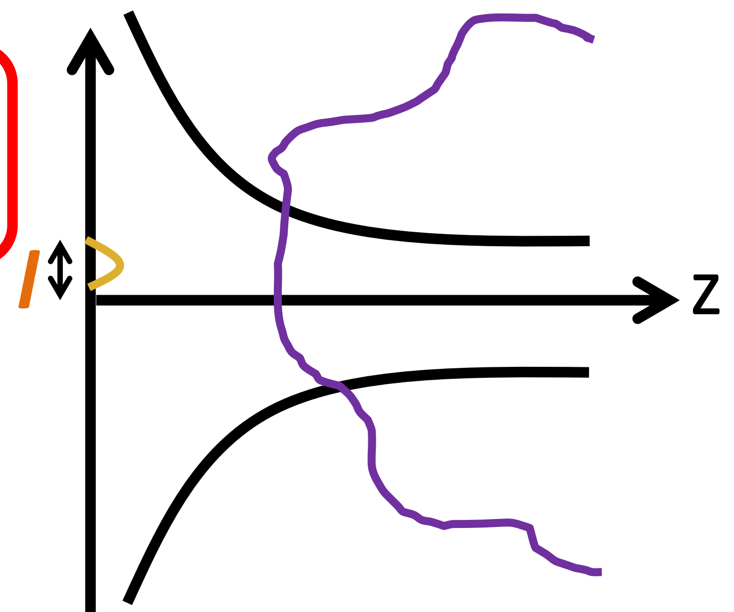
Motivation

Previously, we studied the property of EE for the subsystem whose size (l) is ***very small*** in d CFT.

l << (The Excess of Energy Density)$^{-\frac{1}{d}}$,

$$\boxed{\Delta E_A = \underline{T_{ent}} \Delta S_A}$$

This temperature is universal.



Phys.Rev.Lett. 110 (2013) 9, 091602

Jyotirmoy Bhattacharya, MN, Tadashi Takayanagi, Tomonori Ugajin

JHEP 1308 (2013) 060

David D. Blanco, Horacio Casini, Ling-Yan Hung, Robert C. Myers

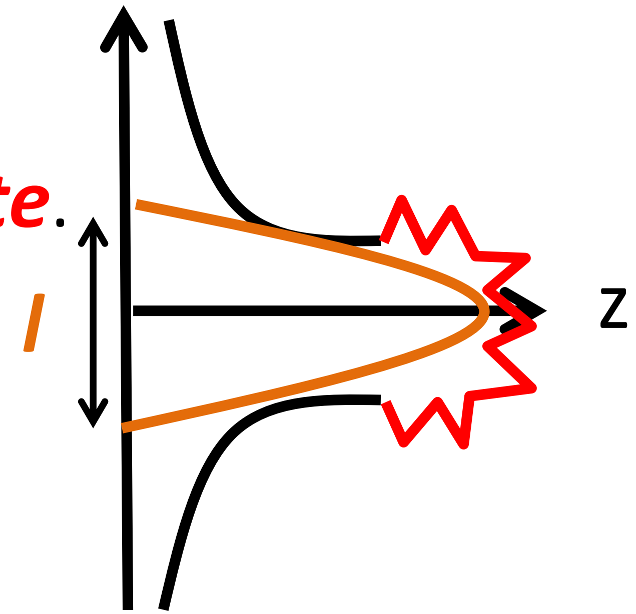
Setup

We study the property of **(R)EE** for

1. The size of subsystem is ***infinite***.

   A half of the total system:

$$x^1 \geq 0$$

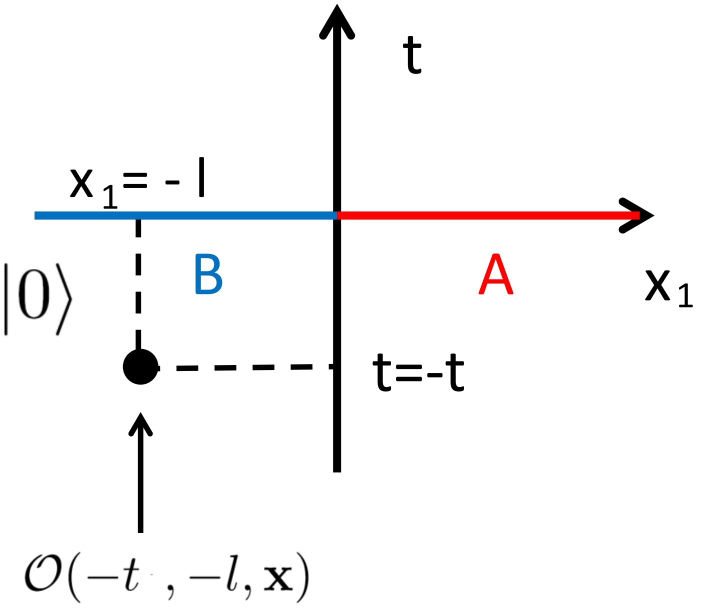


Setup

We study the property of **(R)EE** for

2. A state is defined by acting a local operator on the ground state:

$$|\Psi\rangle = \mathcal{N}\mathcal{O}(t=-t, x^1=-l, \mathbf{x})\,|0\rangle$$



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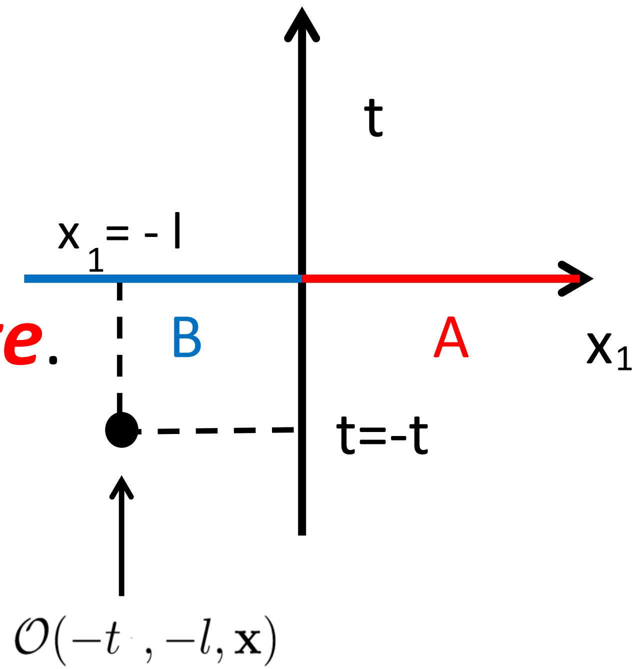

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   A half of the total system:

$$x^1 \geq 0$$

2. A state is defined by acting a local operator on the ground state:

$$|\Psi\rangle = \mathcal{N}\mathcal{O}(t, x^i)\,|0\rangle$$

Motivation

We would like to focus on the time evolution of the (R)EE.

We define $\Delta S_A^{(n)}$ the excess of the (R)EE:

$$\boxed{\Delta S_A^{(n)} = S_A^{(n)Ex} - S_A^{(n)G},}$$

$S_A^{(n)Ex}$: (R)EE for $\hat{\rho}_A$ (Reduced Density Matrix for $|\Psi\rangle = \mathcal{N}\mathcal{O}(t, x^i)\,|0\rangle$)

$S_A^{(n)G}$: (R)EE for the ground state

Field Theory

1. Free massless scalar field theory

   Phys.Rev.Lett. 112 (2014) 111602 MN, Tokiro Numasawa, Tadashi Takayanagi
   JHEP 1410 (2014) 147 MN

2. *U(N)* or *SU(N)* free massless scalar field theory
   in Large N limit

   PTEP 2014 (2014) 093B06 Pawel Caputa, MN, Tadashi Takayanagi

3. Free massless fermionic field theory

   arXiv:1507.04352 [hep-th] MN, Tokiro Numasawa, Shunji Matsuura
   Work in Progress Pawel Caputa, MN, Tokiro Numasawa

4. Charged Renyi Entanglement Entropy (CREE)

   Work in Progress Pawel Caputa, MN, Tokiro Numasawa

5. Holographic field theory

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   JHE

2.

   F

3. F

$$\Delta S_A^{(n)}$$

At late time

Some Constants

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   JHEP 1410 (2014) 147 MN

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$$\Delta S_A^{(n)}$$

→ **Logarithmically grows**

At late time

3.

   Work in Progress Pawel Caputa, MN, Tokiro Numasawa

4. Charged Renyi Entanglement Entropy (CREE)

   Work in Progress Pawel Caputa, MN, Tokiro Numasawa

5. Holographic field theory

   PTEP 2014 (2014) 093B06 Pawel Caputa, MN, Tadashi Takayanagi

Results in scalar field

$\Delta S_A^{(n)}$ At Late Time(t >> *l*)

F.T.	Operator	$n \geq 2$	$n = 1$
Free Massless Scalar	$:(\partial^m \phi)^k:$	$\frac{1}{1-n}\log\left(\frac{1}{2^{nk}}\sum_{j=0}^{k}({}_kC_j)^n\right)$	$k\cdot\log 2 - \frac{1}{2^k}\sum_{j=0}^{k}{}_kC_j\log{}_kC_j$
U(N),SU(N) Free Massless Scaler	$Tr(\phi_1 + i\phi_2)^J$	$\frac{2n-1}{n-1}\cdot\log 2$ ***J=2***	$\log\left(2\sqrt{2}N\right)$ ***J=2***
AdS/CFT	Gauge invariant Operator $\mathcal{O}_\Delta$	$\frac{4n\Delta}{d(n-1)}\log t$ $1 \ll \Delta \ll c$	$\frac{c}{6}\log t$ ***2d CFT*** $\Delta \simeq c$

Results in femionic field

$\Delta S_A^{(n)}$ At Late Time(t >> l)

F.T.	Operator	$n \geq 2$	$n = 1$
Free Massless Fermion	ψ_a	$\frac{1}{1-n}\log\left[A_1 + A_2\right]$ $A_1 = \left(\frac{(\gamma^t\gamma^1)_{aa}+2}{4}\right)^n,$ $A_2 = \left(\frac{-(\gamma^t\gamma^1)_{aa}+2}{4}\right)^n$	$\frac{1}{4}((\gamma^t\gamma^1)_{aa}-2)\log(2-(\gamma^t\gamma^1)_{aa})$ $-\frac{1}{4}((\gamma^t\gamma^1)_{aa}+2)\log((\gamma^t\gamma^1)_{aa}+2)$ $+\log(4)$
U(N),SU(N) **Free Massless Fermion**	$\bar{\psi}\psi$	$\frac{2n-1}{n-1}\log 2 + \frac{n}{1-n}\log\left(\frac{3}{4}\right)$	Unknown
CREE in 2d CFT	$\bar{\psi}\psi$	$\frac{1}{1-n}\log\frac{\cosh n\mu q}{2^{n-1}(\cosh\mu q)^n}$	$\log(2\cosh(\mu q))$ $-\mu q\tanh(\mu q)$

Results in scalar field

$\Delta S_A^{(n)}$ At Late Time(t >> l)

<table>
<tr><th>F.T.</th><th>Operator</th><th>$n \geq 2$</th><th>$n = 1$</th></tr>
<tr><td>Free Massless Scalar</td><td>$:(\partial^m \phi)^k:$</td><td colspan="2" rowspan="2">Constant</td></tr>
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</table>

Results in femionic field

$\Delta S_A^{(n)}$ At Late Time(t >> *l*)

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Results in scalar field

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AdS/CFT	Gauge invariant Operator $\mathcal{O}_\Delta$	***Logarithmically Grows***	

Results in scalar field

$\Delta S_A^{(n)}$ At Late Time(t >> *l*)

Not Depend on m!!

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AdS/CFT	Gauge invariant Operator $\mathcal{O}_\Delta$	$\frac{4n\Delta}{d(n-1)}\log t$ $1 \ll \Delta \ll c$	$\frac{c}{6}\log t$ ***2d CFT*** $\Delta \simeq c$

Results in scalar field

$\Delta S_A^{(n)}$ At Late Time($t \gg l$)

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AdS/CFT	Gauge invariant Operator $\mathcal{O}_\Delta$	$\frac{4n\Delta}{d(n-1)}\log t$ $1 \ll \Delta \ll c$	$\frac{c}{6}\log t$ ***2d CFT*** $\Delta \simeq c$

Results in femionic field

$\Delta S_A^{(n)}$ At Late Time(t >> *l*)

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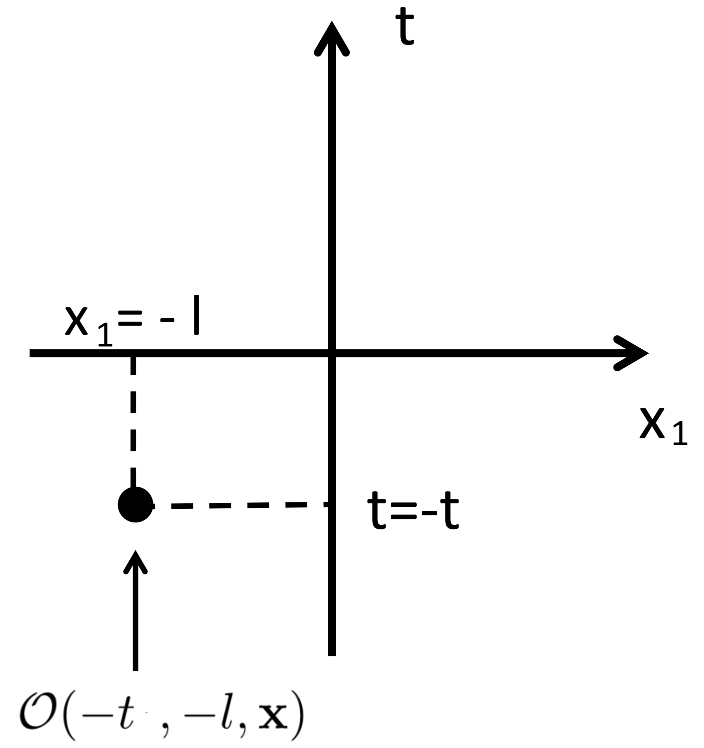
Setup

We consider d+1 dim. QFT.

We prepare a locally excited state: $|\Psi\rangle = \mathcal{N}\mathcal{O}(-t,-l,\mathbf{x})\,|0\rangle$

$$\mathbf{x} = (x^2, x^3, \cdots, x^d)$$



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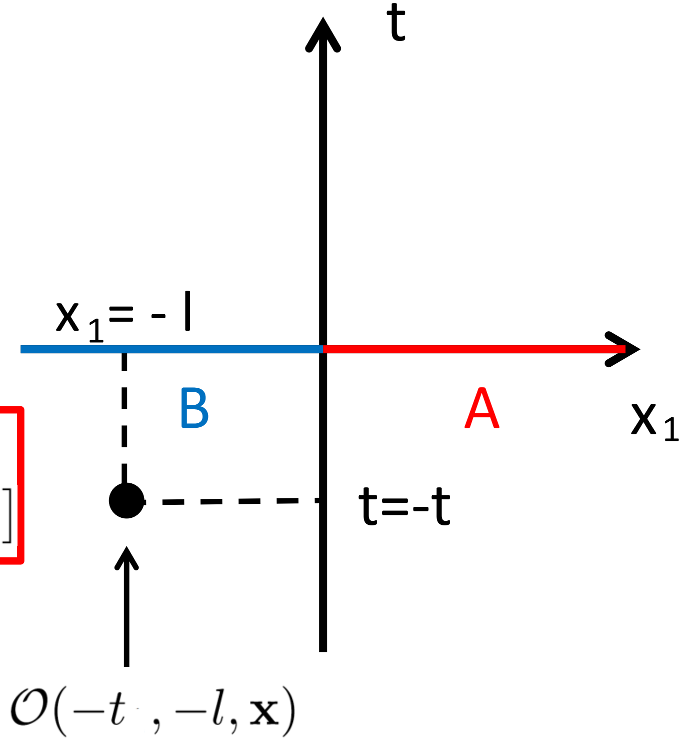
We investigate the time evolution of (Renyi) entanglement entropies.

We choose a half space ($x^1 \geq 0$) as subsystem A.

The reduced density matrix
$\rho_A = \mathcal{N}^{-2} tr_B \left[\mathcal{O}(-t_1,-l,\mathbf{x})|0\rangle\langle 0|\mathcal{O}^\dagger(-t_1,-l,\mathbf{x})\right]$

We will compute $S_A^{(n)} = \dfrac{\log \mathrm{tr}[\rho_A^n]}{1-n}$.



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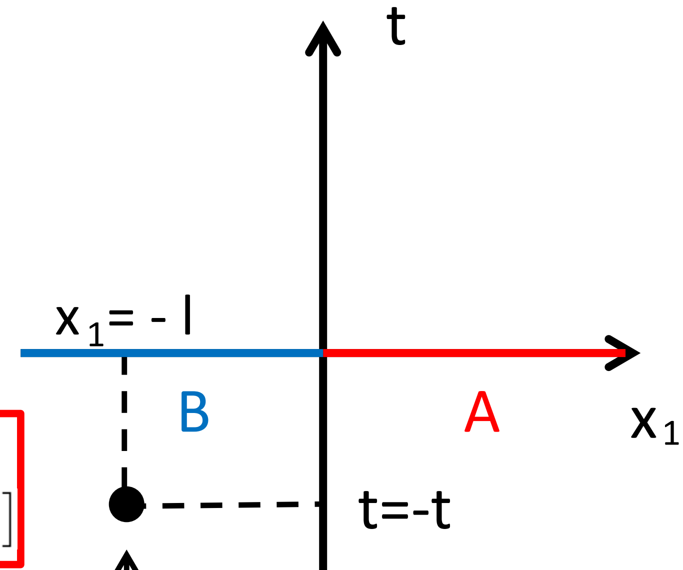

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We will compute $S_A^{(n)} = \dfrac{\log \mathrm{tr}[\rho_A^n]}{1-n}$.

Replica Method !!

How to compute

1. We compute $\Delta S_A^{(n)} = S_A^{(n)Ex} - S_A^{(n)G}$ by path-integral:

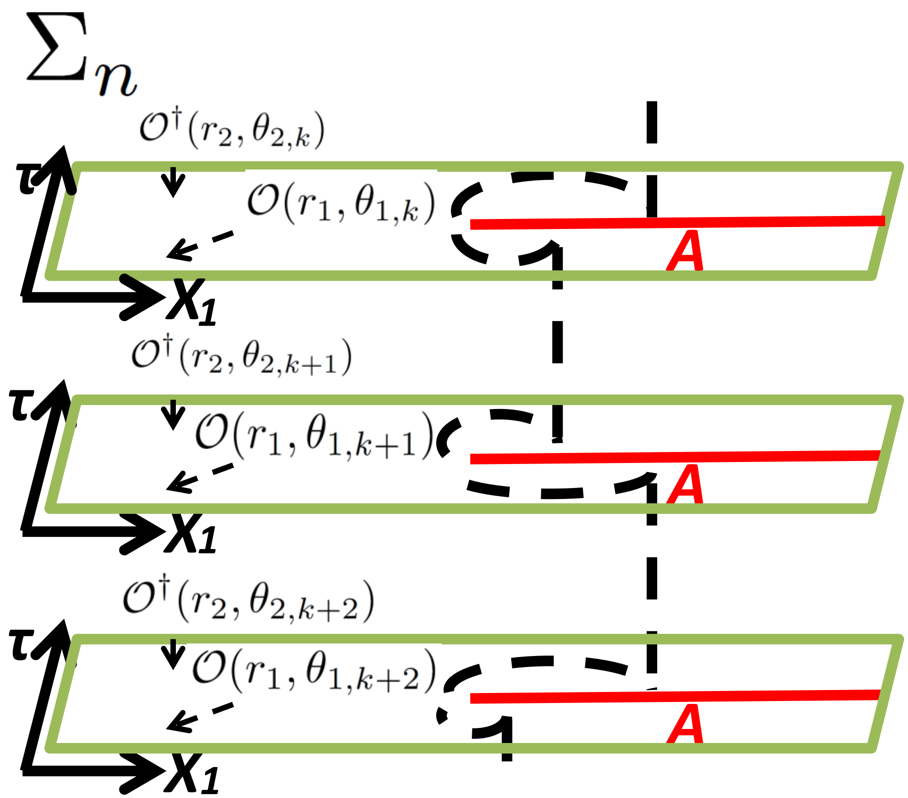
$$\boxed{\begin{aligned}&\Delta S_A^{(n)} = \\ &\frac{1}{1-n}\left(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\right).\end{aligned}}$$

2. After that, we perform an analytic continuation to real time.

$$\boxed{\Delta S_A^{(n)} = S_A^{(n)Ex} - S_A^{(n)G}};$$

$$S_A^{(n)Ex} = \frac{1}{1-n}\log\left[\frac{\int D\Phi \mathcal{O}^\dagger(r_1,\theta_{1,1})\mathcal{O}(r_2,\theta_{2,1})\cdots\mathcal{O}^\dagger(r_1,\theta_{1,n})\mathcal{O}(r_2,\theta_{2,n})e^{-S_n}}{\left(\int D\Phi \mathcal{O}^\dagger(r_1,\theta_{1,1})\mathcal{O}(r_2,\theta_{2,1})e^{-S}\right)^n}\right]$$

$$S_A^{(n)G} = \frac{1}{1-n}\log\left[\frac{Z_n}{Z_1^n}\right]$$



Z_n : The partition function on Σ_n

Z_1 : The partition function on Σ_1

Flat Space

$$\boxed{\Delta S_A^{(n)} = S_A^{(n)Ex} - S_A^{(n)G},}$$

$$S_A^{(n)Ex} = \frac{1}{1-n}\log\left[\frac{\int D\Phi\mathcal{O}^\dagger(r_1,\theta_{1,1})\mathcal{O}(r_2,\theta_{2,1})\cdots\mathcal{O}^\dagger(r_1,\theta_{1,n})\mathcal{O}(r_2,\theta_{2,n})e^{-S_n}}{\left(\int D\Phi\mathcal{O}^\dagger(r_1,\theta_{1,1})\mathcal{O}(r_2,\theta_{2,1})e^{-S}\right)^n}\right]$$

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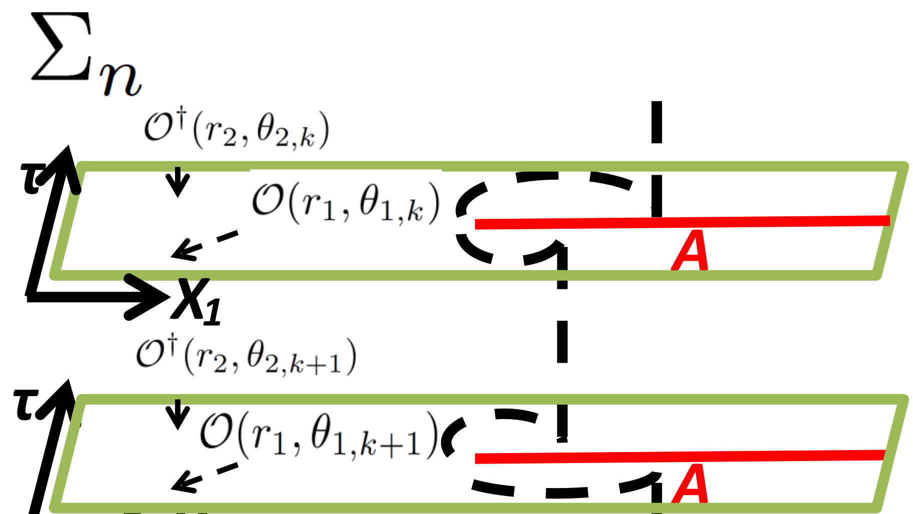
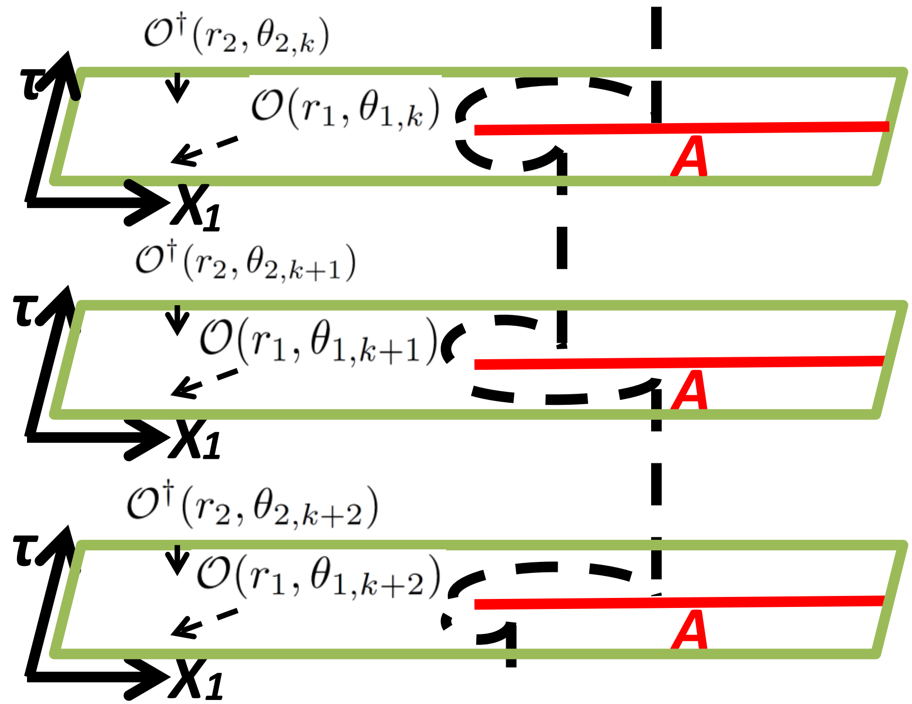


Z_n : The partition function on Σ_n

$$\Delta S_A^{(n)} = \frac{1}{1-n}\log\left[\frac{\int D\Phi\mathcal{O}^\dagger\left(r_1,\theta_{1,1}\right)\mathcal{O}\left(r_2,\theta_{2,1}\right)\cdots\mathcal{O}\left(r_1,\theta_{1,n}\right)^\dagger\mathcal{O}\left(r_2,\theta_{2,n}\right)}{Z_n}\right]$$

$$-\frac{1}{1-n}\log\left[\frac{\left(\int D\Phi\mathcal{O}^\dagger(r_1,\theta_{1,1})\mathcal{O}(r_2,\theta_{2,1})\right)^n}{Z_1^n}\right]$$

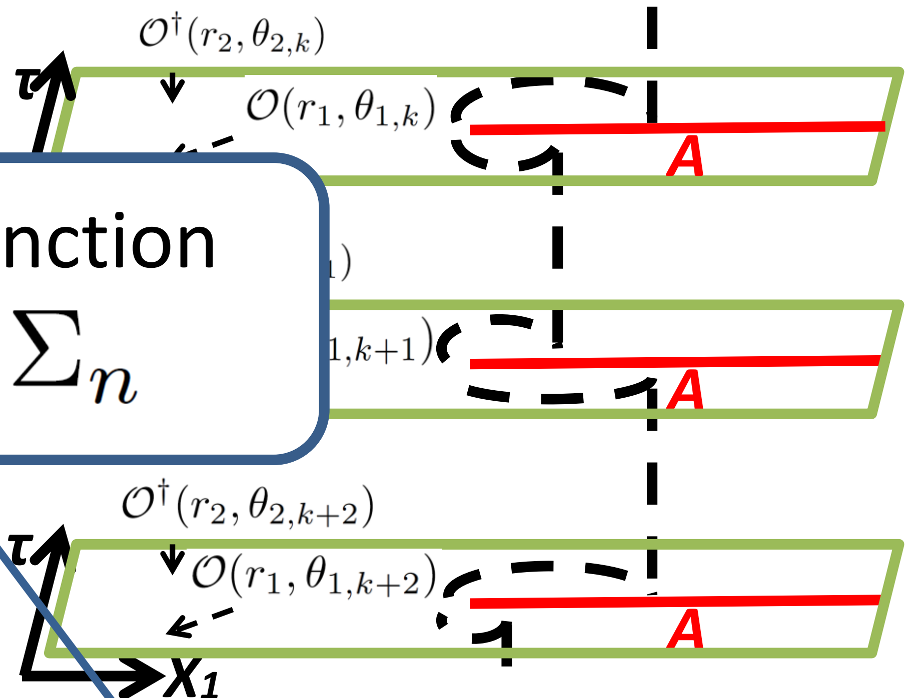
Replica Method



$$\theta_{1,i} = \theta_1 + 2\pi(i-1)$$
$$\theta_{2,i} = \theta_2 + 2\pi(i-1)$$

$$\Delta S_A^{(n)} = \frac{1}{1-n}\Big(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\Big).$$

Replica Method



2n-Point Function on Σ_n

$$\theta_{1,i} = \theta_1 + 2\pi(i-1)$$
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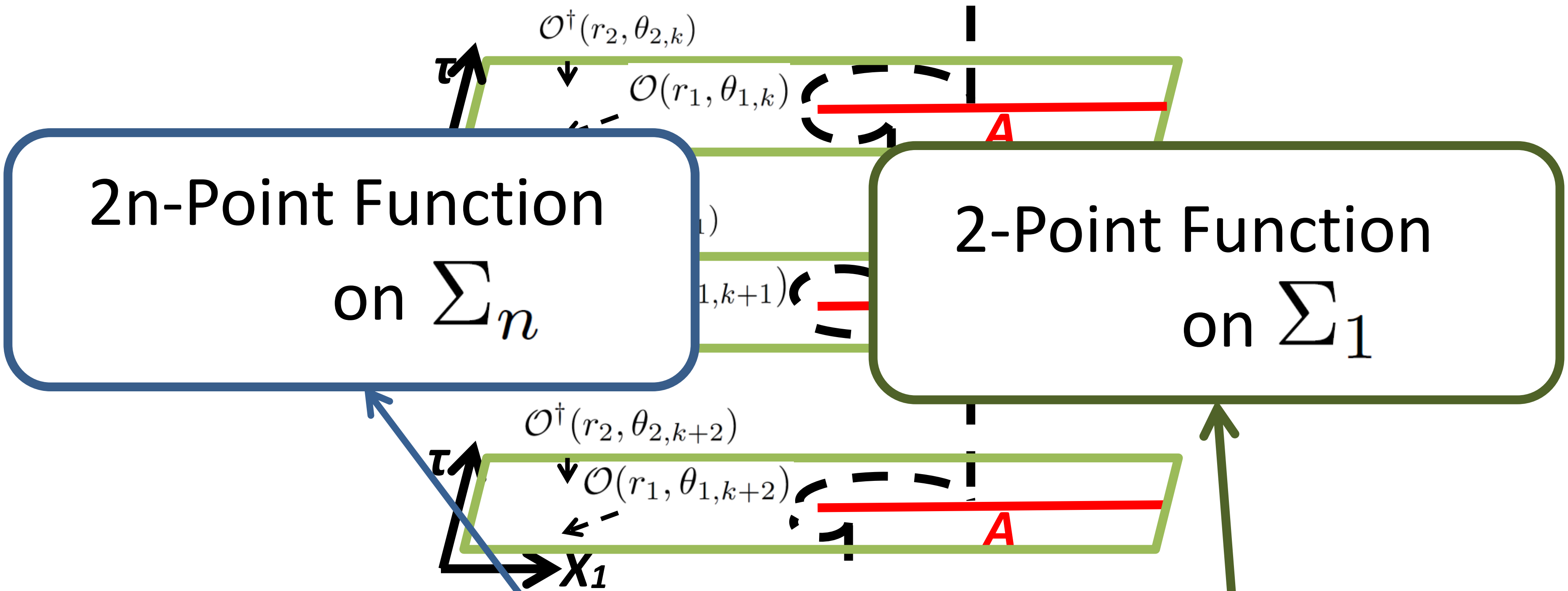
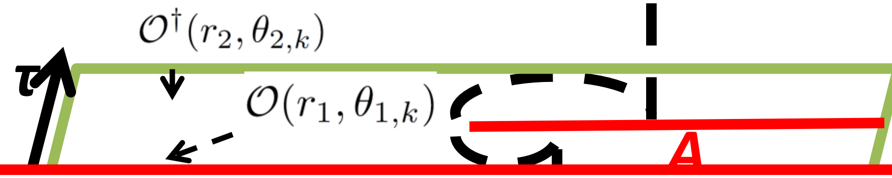
Replica Method



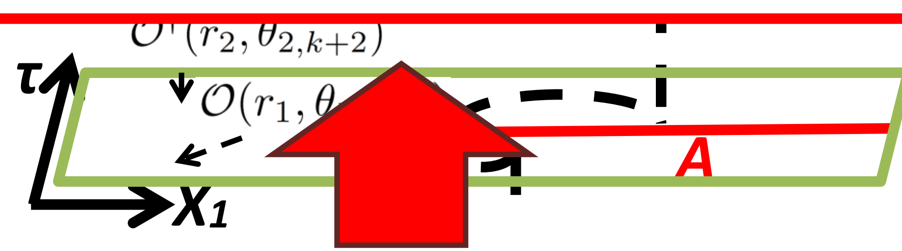
$$\theta_{1,i} = \theta_1 + 2\pi(i-1)$$
$$\theta_{2,i} = \theta_2 + 2\pi(i-1)$$

$$\Delta S_A^{(n)} = \frac{1}{1-n}\Big(\log\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\rangle_{\Sigma_n} - n\log\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\rangle_{\Sigma_1}\Big).$$

Replica Method



This formula holds for any local operators in general QFT in any dimensions.



$$\Delta S_A^{(n)} =$$

$$\theta_{1,i} = \theta_1 + 2\pi(i-1)$$
$$\theta_{2,i} = \theta_2 + 2\pi(i-1)$$

$$\frac{1}{1-n}\left(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\right).$$

Field Theory

1. Free massless scalar field theory

   Phys.Rev.Lett. 112 (2014) 111602 MN, Tokiro Numasawa, Tadashi Takayanagi

   JHEP 1410 (2014) 147 MN

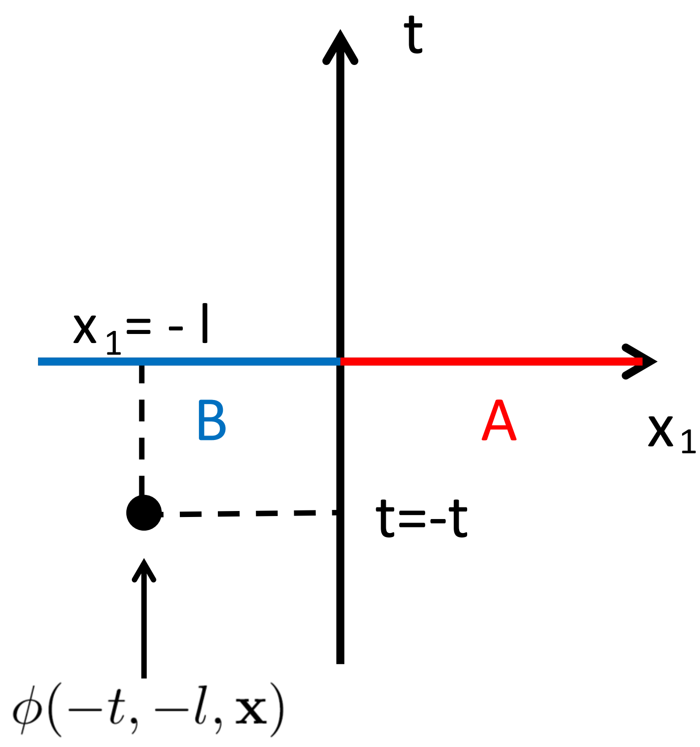
Example

We consider ***free massless scalar*** field theory in ***d+1 dim.***

Especially, we focus on that in ***4 dim.***

We act a local operator $\phi(-t,-l,\mathbf{x})$ on the ground state:

$$|\Psi\rangle = \mathcal{N} \quad \phi(-t,-l,\mathbf{x})\,|0\rangle \ .$$



We measure the (Renyi)

entanglement entropies at t=0.

➡ ***Time evolution!!***

Example

Let's compute $\Delta S_A^{(2)}$ for $|\Psi\rangle = \mathcal{N} \quad \phi(-t,-l,\mathbf{x})\,|0\rangle$

in 4-dimensional free massless scalar field theory.

$$\Delta S_A^{(2)} = -\log\left[\frac{\langle\phi(r_1,\theta_1)\phi(r_2,\theta_2)\phi(r_1,\theta_1+2\pi)\phi(r_2,\theta_2+2\pi)\rangle_{\Sigma_2}}{\langle\phi(r_1,\theta_1)\phi(r_2,\theta_2)\rangle_{\Sigma_1}^2}\right]$$

Green function:

$$\langle\phi\,(r,\theta,\mathbf{x})\,\phi\,(s,\theta',\mathbf{x})\rangle = \frac{1}{8\pi^2\,(r+s)\left(r+s-2\sqrt{rs}\cos\left(\frac{\theta-\theta'}{2}\right)\right)}$$

Example

$$\Delta S_A^{(2)} = -\log\left[\frac{\langle\phi(r_1,\theta_1)\phi(r_2,\theta_2)\phi(r_1,\theta_1+2\pi)\phi(r_2,\theta_2+2\pi)\rangle_{\Sigma_2}}{\langle\phi(r_1,\theta_1)\phi(r_2,\theta_2)\rangle_{\Sigma_1}^2}\right]$$

Green function:

$$\langle\phi\left(r,\theta,\mathbf{x}\right)\phi\left(s,\theta',\mathbf{x}\right)\rangle = \frac{1}{8\pi^2\left(r+s\right)\left(r+s-2\sqrt{rs}\cos\left(\frac{\theta-\theta'}{2}\right)\right)}$$

We compute $\Delta S_A^{(2)}$ by using Green function.

After that, we perform analytic continuation to real time.

Example

$$\Delta S_A^{(2)} = -\log\left[\frac{\langle\phi(r_1,\theta_1)\phi(r_2,\theta_2)\phi(r_1,\theta_1+2\pi)\phi(r_2,\theta_2+2\pi)\rangle_{\Sigma_2}}{\langle\phi(r_1,\theta_1)\phi(r_2,\theta_2)\rangle^2_{\Sigma_1}}\right]$$

$t < l$

$$\Delta S_A^{(2)} = 0$$

$t \geq l$

$$\Delta S_A^{(2)} = \log\left[\frac{2t^2}{t^2+l^2}\right]$$

Green f

$\langle\phi\,(r,\theta,\mathbf{x}$

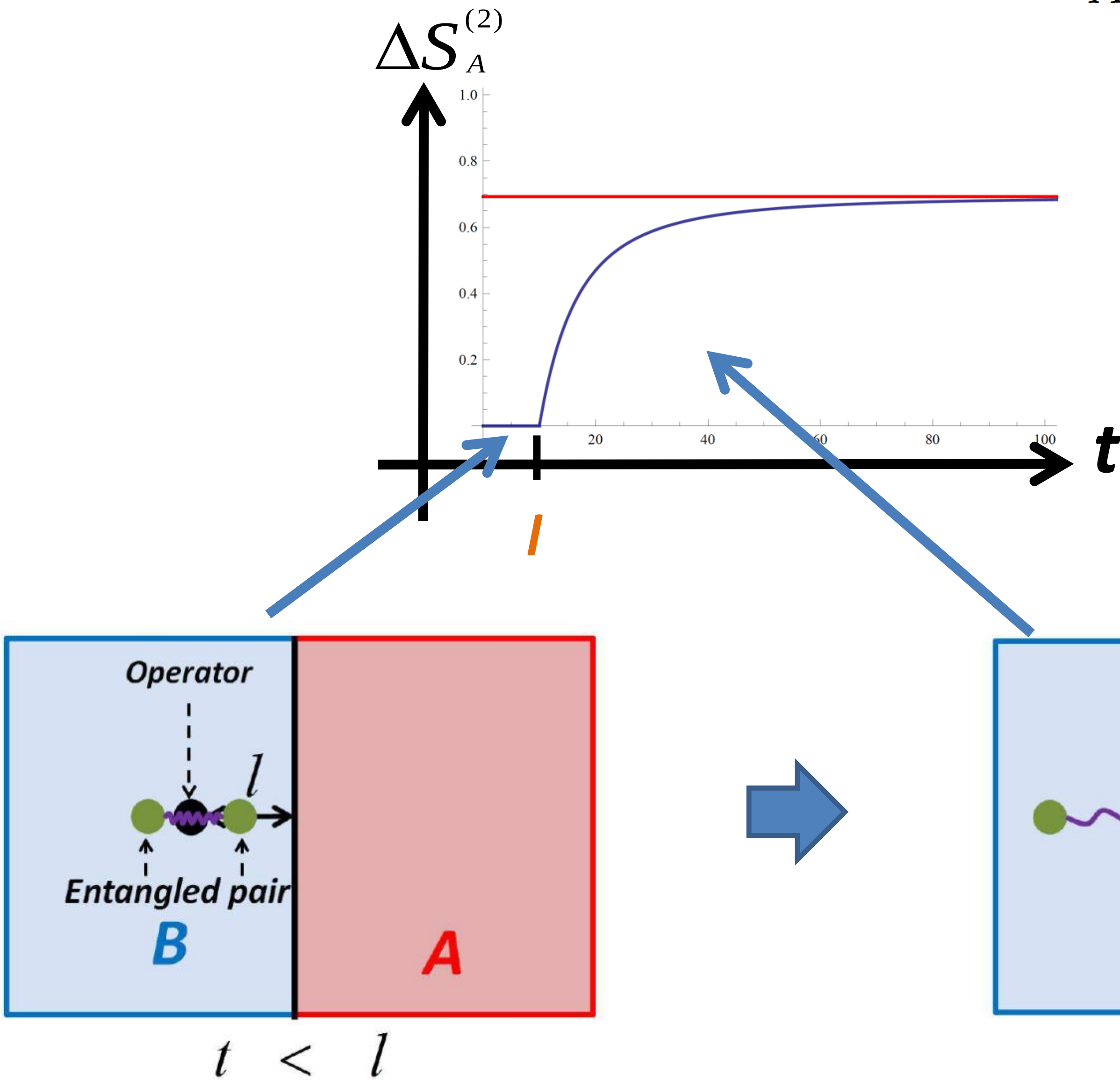
$\cos\left(\frac{\theta-\theta'}{2}\right))$

We compu

by using Green function.

we perform analytic continuation to real time.

Time Evolution of $\Delta S_A^{(2)}$



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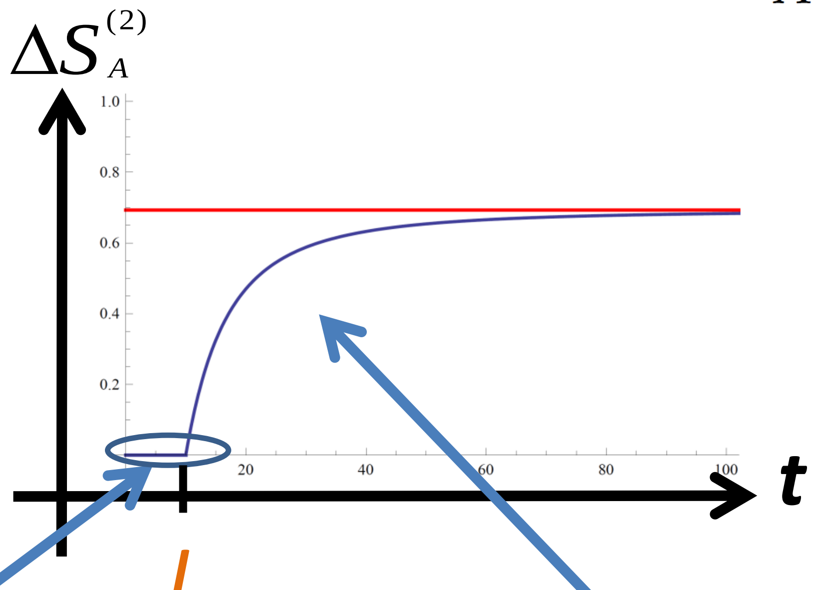
Time Evolution of $\Delta S_A^{(2)}$



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$$\Delta S_A^{(2)} = 0$$

$t \geq l$

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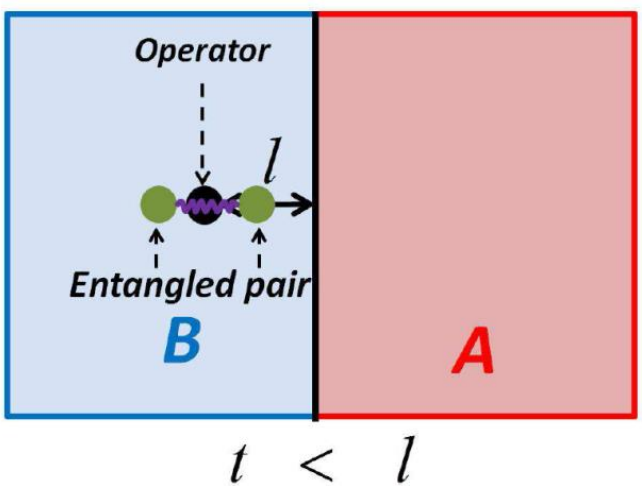
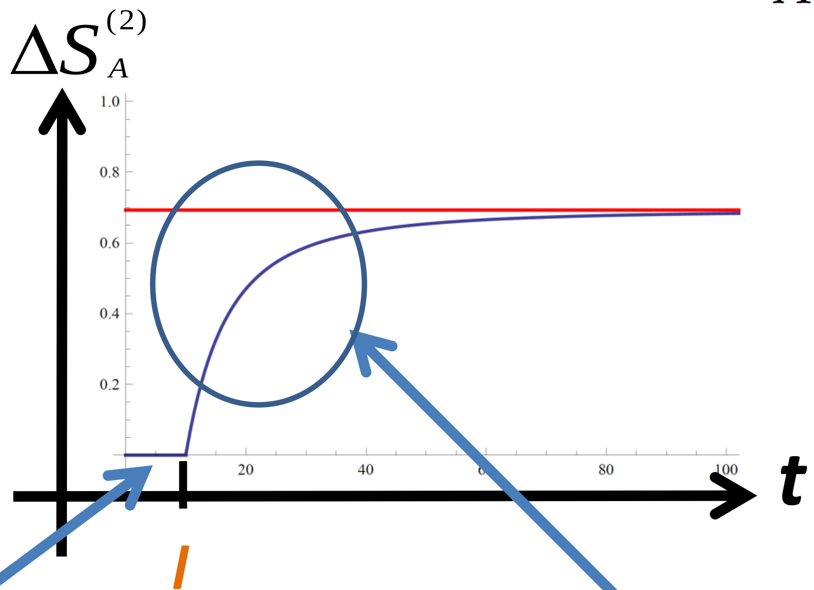


$t < l$

An entangled pair appears.

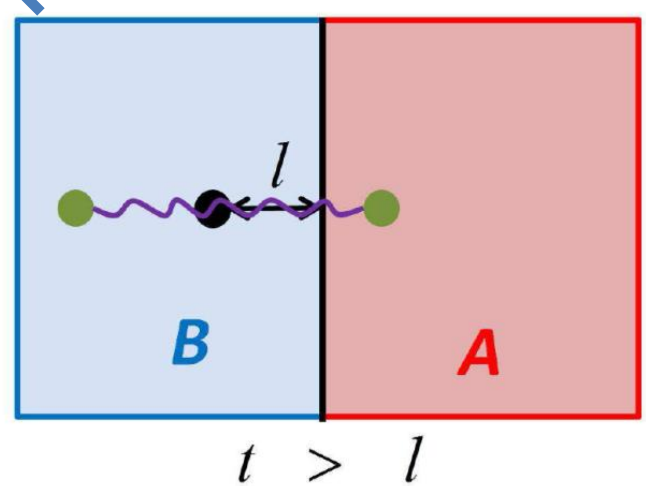
Each of pair is included in the region B.

Time Evolution of $\Delta S_A^{(2)}$

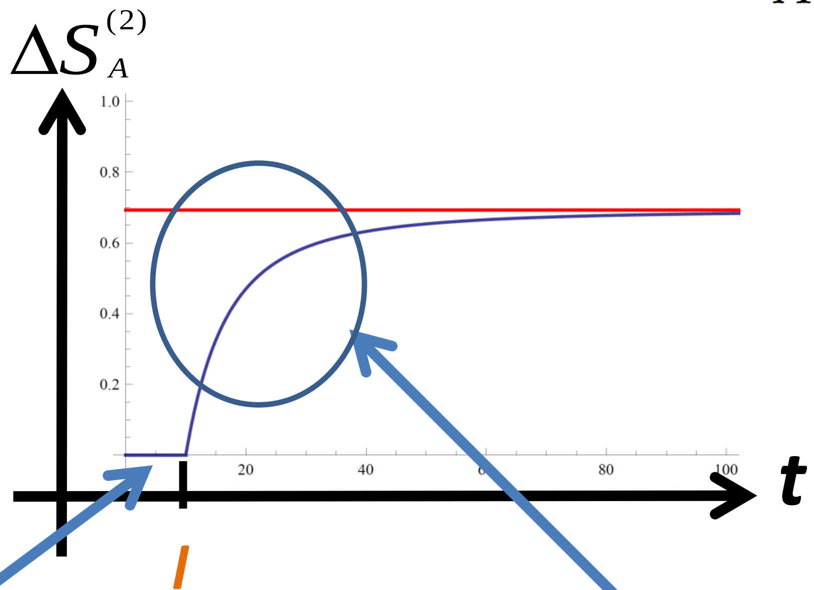


$t \geq l$

In this region, two quanta is included in A and B respectively .



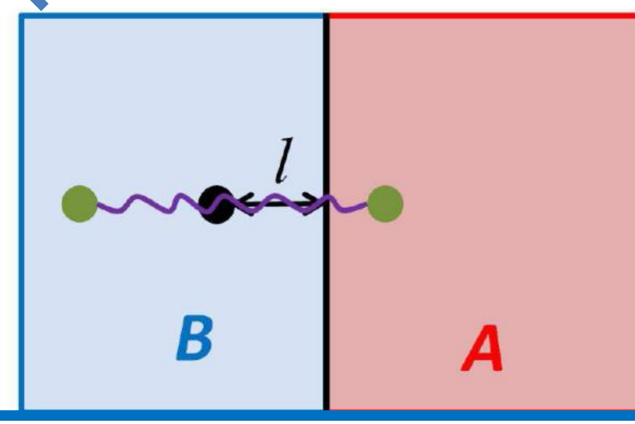
Time Evolution of $\Delta S_A^{(2)}$



$$\Delta S_A^{(2)} = \log\left[\frac{2t^2}{t^2+l^2}\right]$$

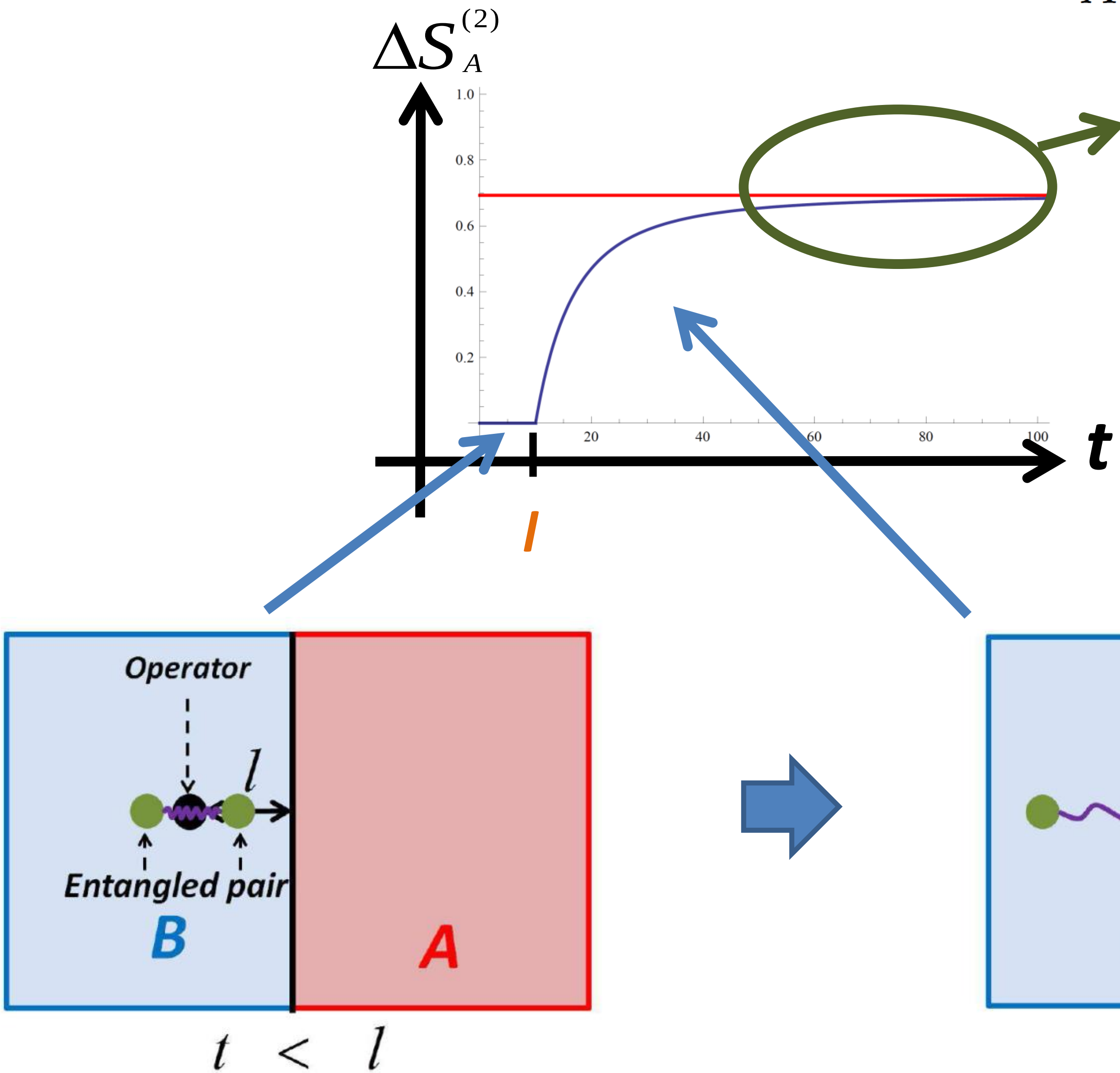
$t \geq l$

In this region, two quanta is included in A and B respectively .



Entanglement between quanta can contribute to $\Delta S_A^{(2)}$.

Time Evolution of $\Delta S_A^{(2)}$



Subsystem

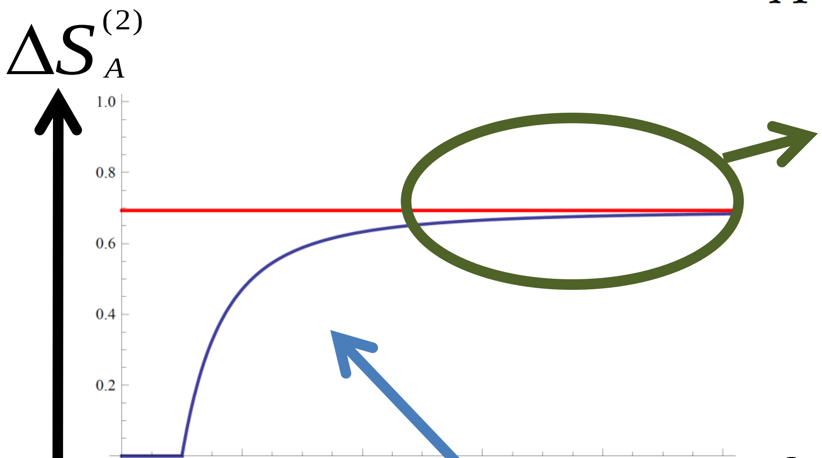
= a half of the

total space

$\Delta S_A^{(2)}$ approaches

Constant!!

Time Evolution of $\Delta S_A^{(2)}$



Subsystem

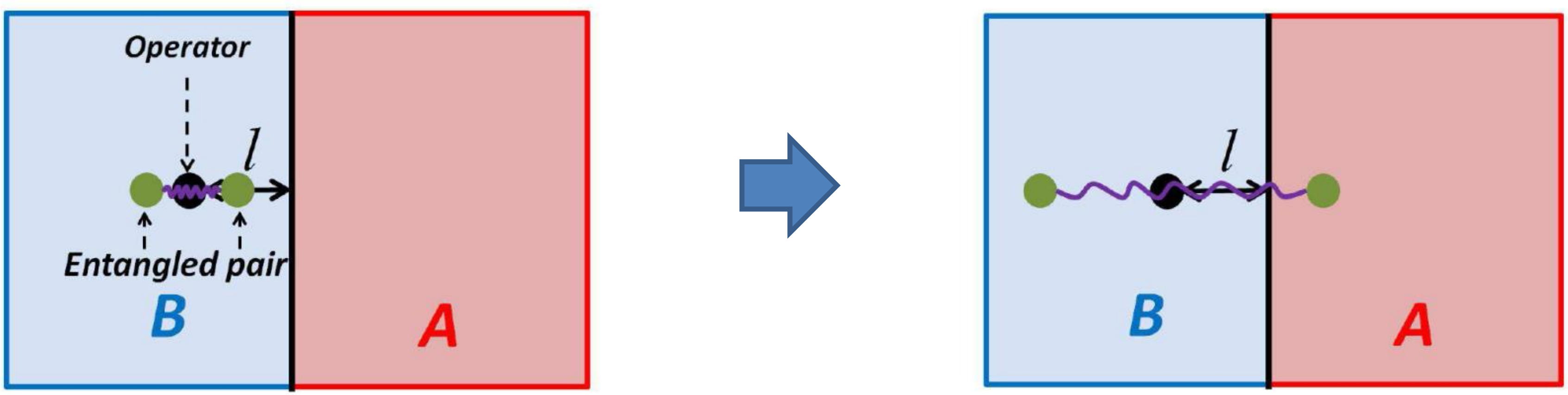
= a half of the

total space

$\Delta S_A^{(2)}$ approaches

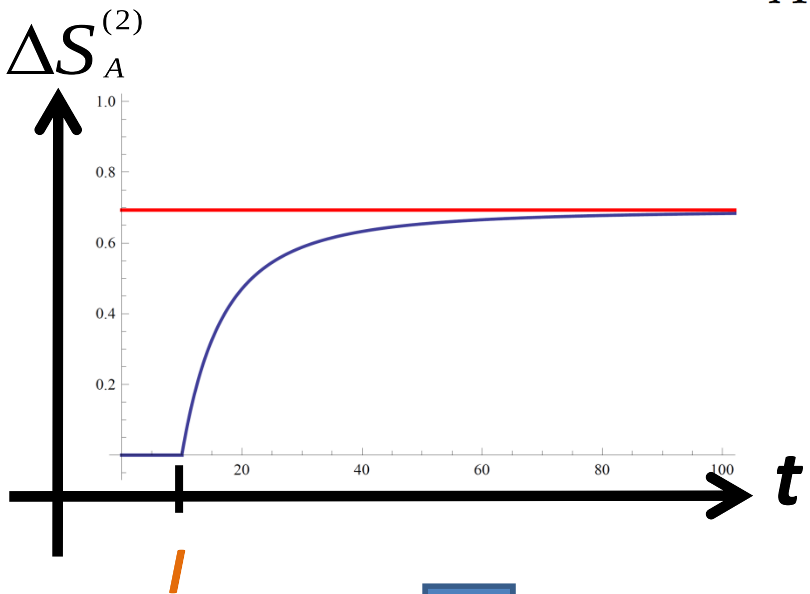
t!!

$\Delta S_A^{(2)}$ ***= Entanglement between Quasi-particles***



$t < l$ $\qquad$ $t > l$

Time Evolution of $\Delta S_A^{(2)}$



$\Delta S_A^{(2)}$ for $|\Psi\rangle = \mathcal{N} \quad \phi(-t,-l,\mathbf{x})\,|0\rangle$ approaches ***constants! !***

(log2)

We call them ***the (Renyi) entanglement entropies of operators*****.**

(Renyi) entanglement entropy of local operators

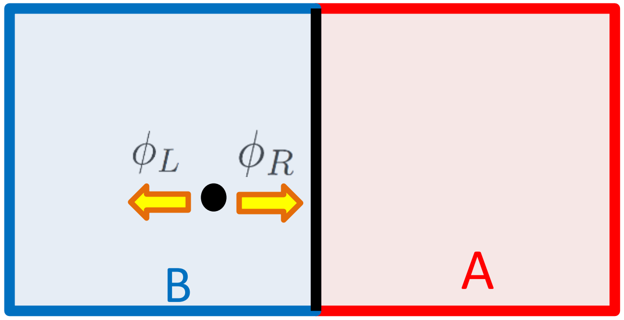
= Entanglement between Quasi-particles

Entangled Pair Interpretation

We derive $\Delta S^{(n)}_{A,k}$ for $|\Psi\rangle = \mathcal{N} : \phi^k(-t,-l,\mathbf{x}) : |0\rangle$ from the entangled pair interpretation.

We decompose ϕ into the left moving mode and the right moving mode,

$$\boxed{\phi = \phi_L + \phi_R}$$



Generalize

In two dimensional CFT, we decompose ϕ into the left moving mode and right moving mode,

$$\phi(z,\bar{z}) = \phi_L(z) + \phi_R(\bar{z})$$

Entangled Pair Interpretation

We derive $\Delta S^{(n)}_{A,k}$ for $|\Psi\rangle = \mathcal{N} : \phi^k(-t, -l, \mathbf{x}) : |0\rangle$ from the entangled pair interpretation.

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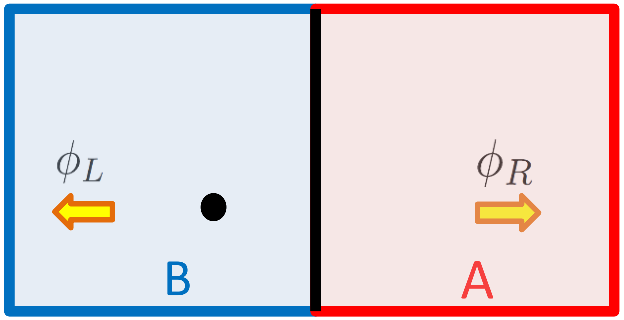

At late time, the d o f in the region B can be identified with the d o f of left moving mode.

Entangled Pair Interpretation

Under this decomposition: $\boxed{\phi = \phi_L + \phi_R}$

$$\boxed{|\Psi\rangle = \mathcal{N} : \phi^k(-t, -l, \mathbf{x}) : |0\rangle}$$

Tracing out the d.o.f in B $\Rightarrow$

$$\rho_A^f = 2^{-k}({}_kC_0 \ , \ {}_kC_1 \ , \ \cdots \ , \ {}_kC_k)$$

Entangled Pair Interpretation

Under this decomposition: $\boxed{\phi = \phi_L + \phi_R}$

$$\boxed{|\Psi\rangle = \frac{1}{2^{\frac{k}{2}}} \sum_{m=0}^{k} \sqrt{{}_kC_m}\, |m\rangle_A \otimes |k-m\rangle_B \,.}$$

Tracing out the d.o.f in B $\Rightarrow$ $\rho_A^f = 2^{-k}\left({}_kC_0\ ,\ {}_kC_1\ ,\ \cdots\ ,\ {}_kC_k\right)$

Entangled Pair Interpretation

Under this decomposition: $\boxed{\phi = \phi_L + \phi_R}$

$|\Psi$

$$\Delta S_A^{(n)f} = \frac{1}{1-n}\log\left(\frac{1}{2^{nk}}\sum_{j=0}^{k}\left({}_kC_j\right)^n\right).$$

$$\Delta S_A = k\cdot\log 2 - \frac{1}{2^k}\sum_{j=0}^{k}{}_kC_j\log{}_kC_j\,.$$

Tr

the d.o.f in B

$\rho_A \quad = \quad ({}_kC_0\,,\ {}_kC_1\,, \quad \ldots \quad ,\ {}_kC_k)$

Entangled Pair Interpretation

Under this decomposition: $\boxed{\phi = \phi_L + \phi_R}$

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$$\Delta S_A^{(n)f} = \frac{1}{1-n} \log \left(\frac{1}{2^{nk}} \sum_{j=0}^{k} \left({}_kC_j\right)^n \right).$$

$$\Delta S_A = k \cdot \log 2 - \frac{1}{2^k} \sum_{j=0}^{k} {}_kC_j \log {}_kC_j \, .$$

Tr

They agree with the results which we obtain by the Replica trick (See My paper!!).

Comments on Result

We defined ***the (Renyi) entanglement entropies of operators*** by the late time values of $\Delta S_A^{(n)}$.

The (Renyi) entanglement entropies of $:\phi^k:$ is given by

$$\boxed{\begin{aligned}\Delta S_A^{(n)f} &= \frac{1}{1-n}\log\left(\frac{1}{2^{nk}}\sum_{j=0}^{k}\left({}_kC_j\right)^n\right). \\ \Delta S_A &= k\cdot\log 2 - \frac{1}{2^k}\sum_{j=0}^{k}{}_kC_j\log{}_kC_j.\end{aligned}}$$

Generalize Results

We defined ***the (Renyi) entanglement entropies of operators*** by the late time values of $\Delta S_A^{(n)}$.

The (Renyi) entanglement entropies of specific operators ($:(\partial^m \phi)^k:$) which are composed of single species operator are given by

$$\boxed{\begin{aligned} \Delta S_A^{(n)f} &= \frac{1}{1-n} \log \left(\frac{1}{2^{nk}} \sum_{j=0}^{k} \left({}_kC_j\right)^n \right). \\ \Delta S_A &= k \cdot \log 2 - \frac{1}{2^k} \sum_{j=0}^{k} {}_kC_j \log {}_kC_j . \end{aligned}}$$

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for ***any dimension*.**

⇨ They characterize the local operators from the viewpoint of quantum entanglement!!

Generalize Results

We defined ***the (Renyi) entanglement entropies of operators*** by the late time values of $\Delta S_A^{(n)}$.

The (Renyi) entanglement entropies of specific operators ($:(\partial^m \phi)^k:$) which are composed of single species operator are given by

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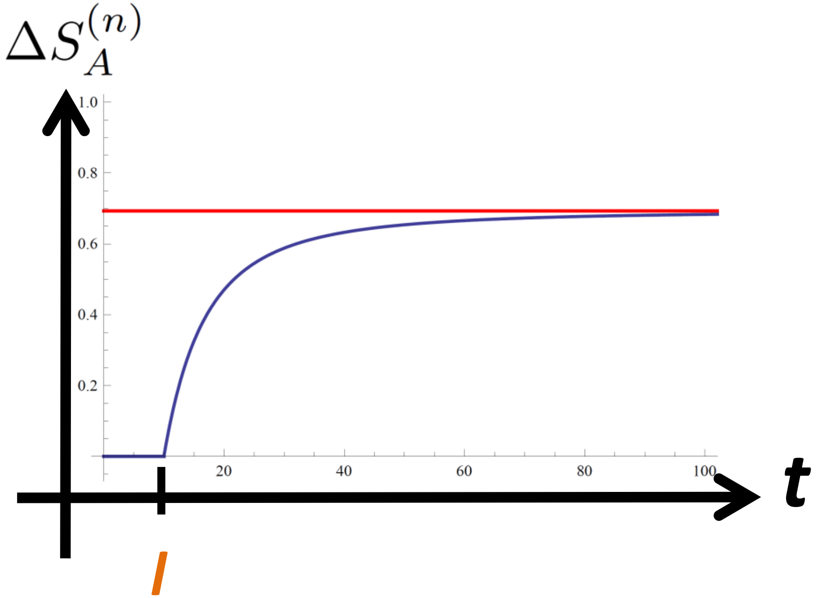
for ***any dimension***.

Large k, $\Delta S_A^{(n)} \sim \frac{1}{2}\log k$

Time Evolution of $\Delta S_A^{(n)}$

$\Delta S_A^{(n)}$ for locally excited states **obey causality**.



$\Delta S_A^{(n)}$ approach constant ((Renyi) entanglement entropy of local operators)

(Renyi) entanglement entropy of local operators

= Entanglement between Quasi-particles

Field Theory

3. Free massless fermionic field theory

   arXiv:1507.04352 [hep-th] MN, Tokiro Numasawa, Shunji Matsuura

   Work in Progress Pawel Caputa, MN, Tokiro Numasawa

4. Charged Renyi Entanglement Entropy (CREE)

   Work in Progress Pawel Caputa, MN, Tokiro Numasawa

Fermionic Field Theory

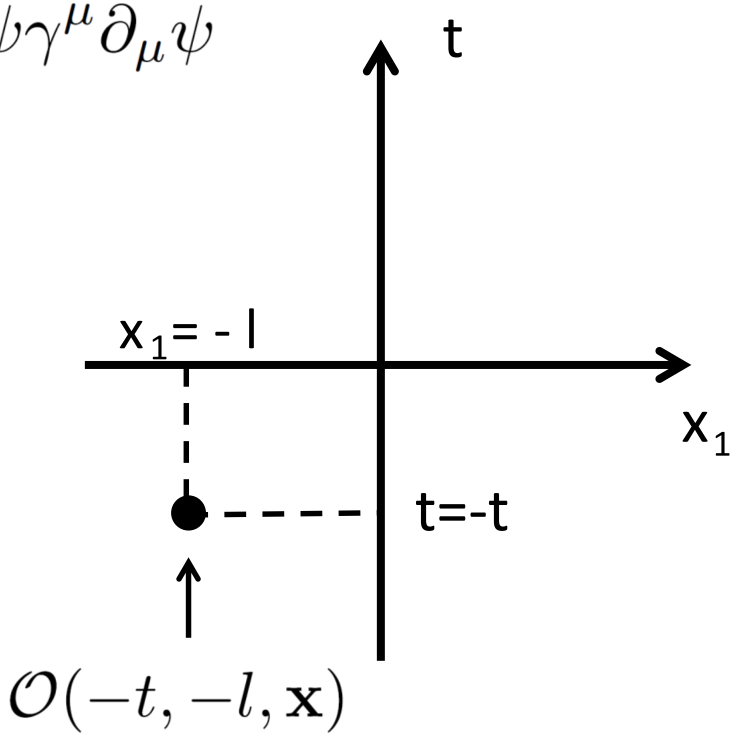
Theory:

4-dimentional free massless fermionic field theory

$$S_{\text{fermion}} = \int d^4x \bar{\psi}\gamma^{\mu}\partial_{\mu}\psi$$

Setup:

$$|\Psi\rangle = \mathcal{N}\mathcal{O}(-t,-l,\mathbf{x})\,|0\rangle$$



Examples

An Excited State: $|\Psi\rangle = \mathcal{N} \bar{\psi}\, \psi\, (-t, -l, \mathbf{x})\, |0\rangle$

SL(2,C) invariant!!

Excess of Renyi Entanglement Entropy (t→∞):

Examples

An Excited State: $|\Psi\rangle = \mathcal{N}\bar{\psi}\,\psi\,(-t,-l,\mathbf{x})\,|0\rangle$

SL(2,C) invariant!!

Excess of Renyi Entanglement Entropy (t→∞):

$$\Delta S_A^{(n)} = \frac{1}{1-n}\log\left[2\left(\frac{12}{64}\right)^n + 8\left(\frac{1}{64}\right)^n \sum_{k\in 2\mathbb{Z}}^{k\leq n} {}_nC_k 4^k 5^{n-k}\right]$$

Examples

An Excited State: $|\Psi\rangle = \mathcal{N}\bar{\psi}\,\psi\,(-t,-l,\mathbf{x})\,|0\rangle$

SL(2,C) invariant!!

Excess of Renyi Entanglement Entropy (t→∞):

Reduced Density Matrix

$$\rho_A = \frac{1}{64}\begin{pmatrix} 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 & 0 & 0 \\ 0 & 0 & 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 \end{pmatrix}$$

$$B = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

Examples

An Excited State: $|\Psi\rangle = \mathcal{N}\psi_a(-t,-l,\mathbf{x})\,|0\rangle$

Excess of Renyi Entanglement Entropy

$$\Delta S_A^{(n)} = \frac{1}{1-n}\log\left[A_1 + A_2\right],$$

$$A_1 = \left(\frac{t+l}{4t}\right)^n \left(\left(\frac{t-l}{t}\right)(\gamma^t\gamma^1)_{aa} + 2\right)^n,$$

$$A_2 = \left(\frac{t-l}{4t}\right)^n \left(-\left(\frac{t+l}{t}\right)(\gamma^t\gamma^1)_{aa} + 2\right)^n.$$

Examples

An Excited State: $|\Psi\rangle = \mathcal{N}\psi_a(-t,-l,\mathbf{x})\,|0\rangle$

Excess of Renyi Entanglement Entropy (t→∞):

$$\Delta S_A^{(n)} = \frac{1}{1-n}\log\left[A_1 + A_2\right],$$
$$A_1 = \left(\frac{(\gamma^t\gamma^1)_{aa}+2}{4}\right)^n,$$
$$A_2 = \left(\frac{-(\gamma^t\gamma^1)_{aa}+2}{4}\right)^n.$$

Examples

An Excited State: $|\Psi\rangle = \mathcal{N}\psi_a(-t,-l,\mathbf{x})\,|0\rangle$

Excess of Renyi Entanglement Entropy (t→∞):

$$\Delta S_A^{(n)} = \frac{1}{1-n}\log\left[A_1 + A_2\right],$$

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$$A_2 = \left(\frac{-(\gamma^t\gamma^1)_{aa} + 2}{4}\right)^n.$$

Real

Examples

An Excited State: $|\Psi\rangle = \mathcal{N}\psi_a(-t,-l,\mathbf{x})\,|0\rangle$

Excess of Renyi Entanglement Entropy (t→∞):

Reduced Density Matrix

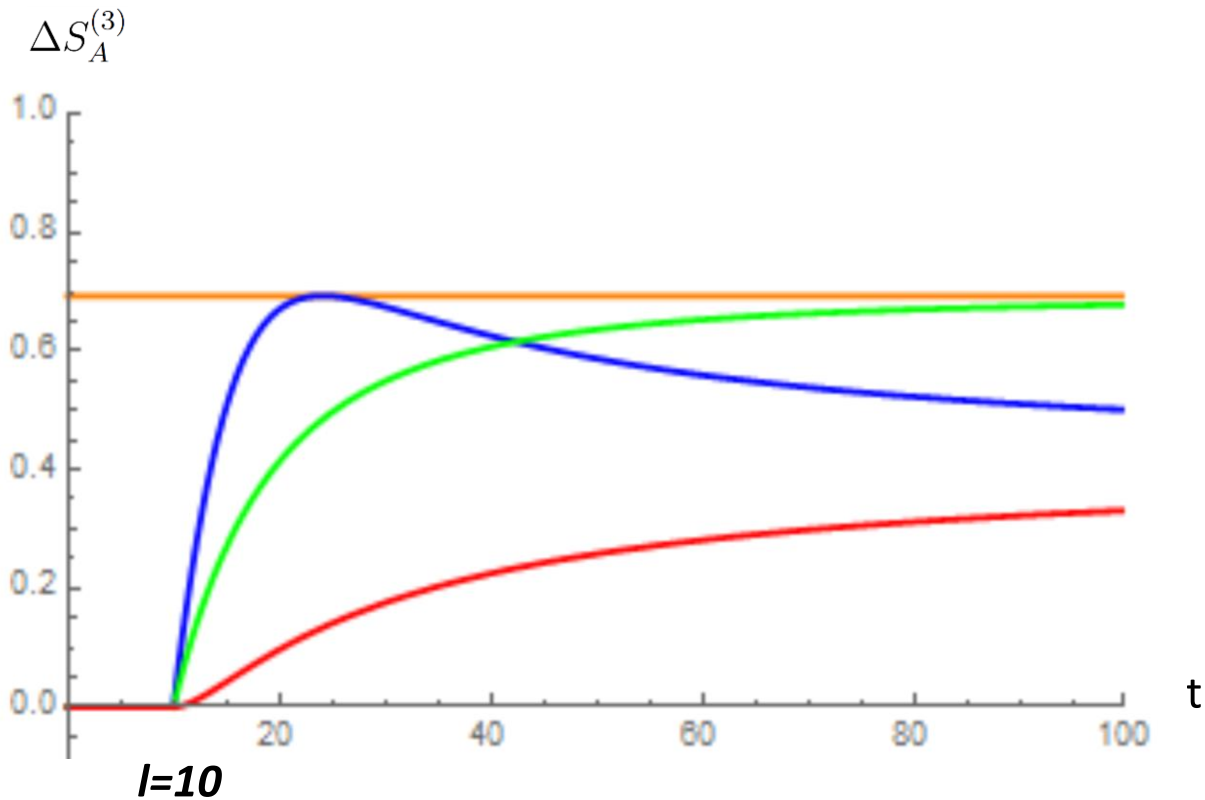
$$\rho_A = \frac{1}{4}\begin{pmatrix} (\gamma^t\gamma^1)_{aa}+2 & 0 \\ 0 & -(\gamma^t\gamma^1)_{aa}+2 \end{pmatrix}$$

Time Evolution of $\Delta S_A^{(n)}$

$$\Delta S_A^{(n)} = \frac{1}{1-n} \log\left[A_1 + A_2\right],$$

$$A_1 = \left(\frac{t+l}{4t}\right)^n \left(\left(\frac{t-l}{t}\right)(\gamma^t\gamma^1)_{aa} + 2\right)^n,$$

$$A_2 = \left(\frac{t-l}{4t}\right)^n \left(-\left(\frac{t+l}{t}\right)(\gamma^t\gamma^1)_{aa} + 2\right)^n.$$

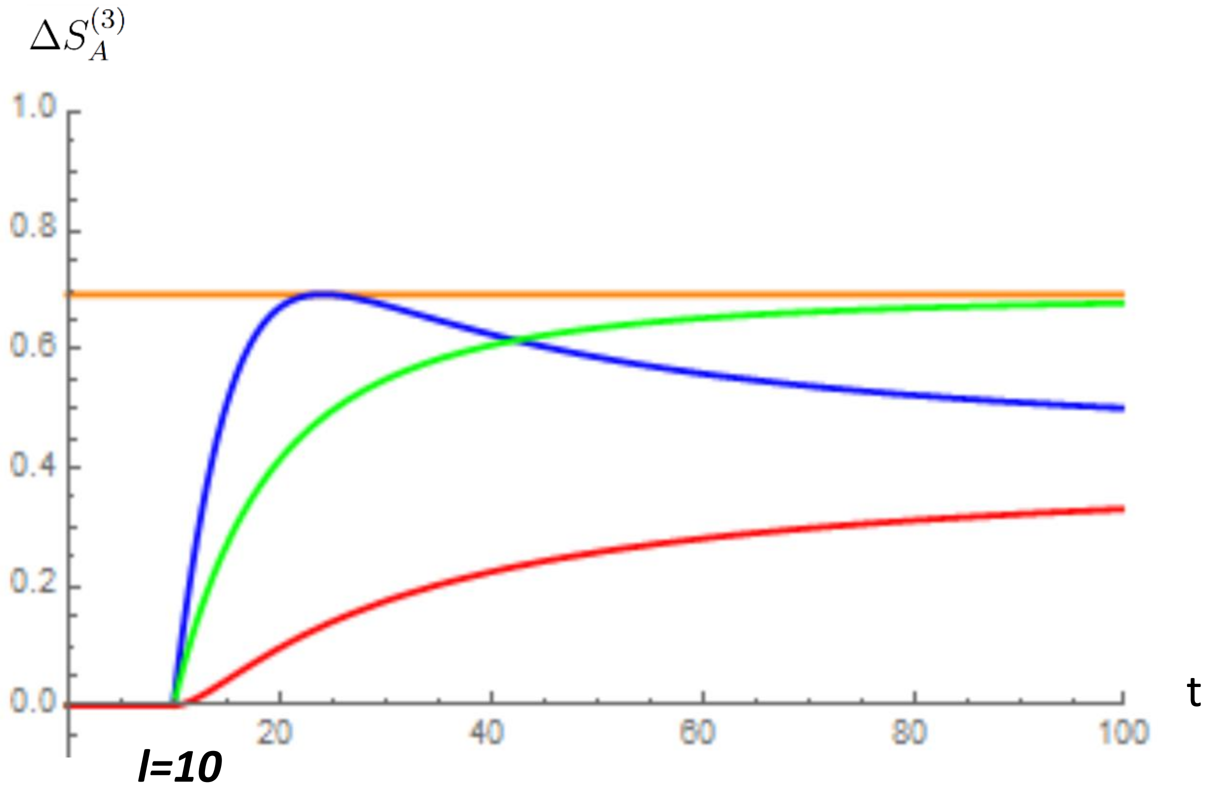


Red: $(\gamma^t\gamma^1)_{aa} = 1$

Green: $(\gamma^t\gamma^1)_{aa} = 0$

Blue: $(\gamma^t\gamma^1)_{aa} = -1$

Orange: Log2



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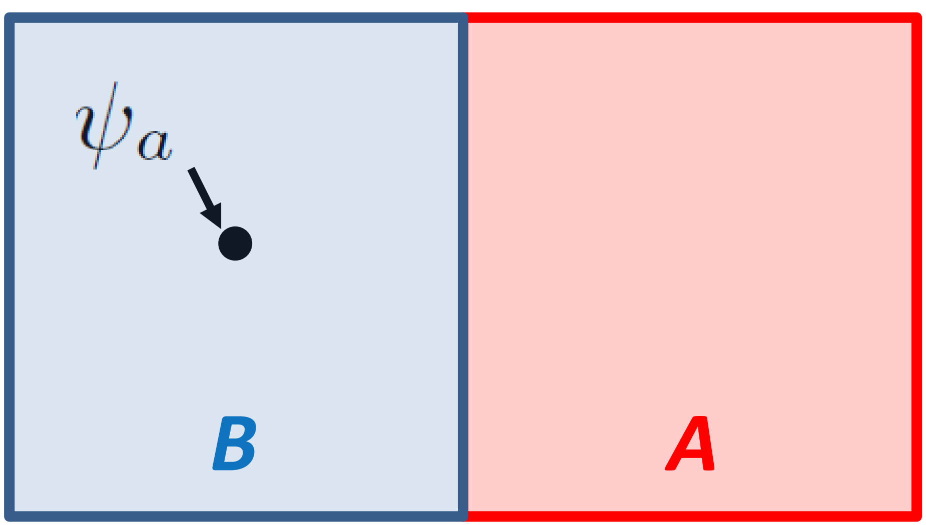
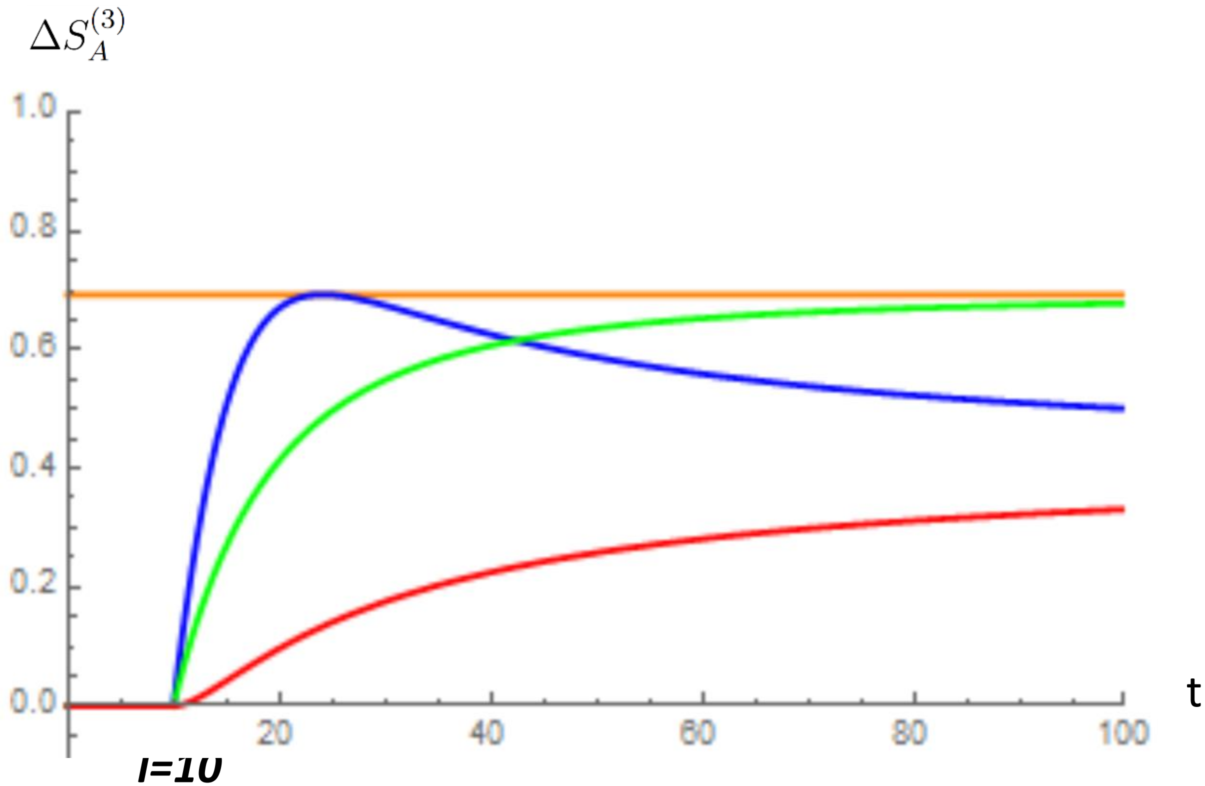
Orange: Log2

Example: $(\gamma^t\gamma^1)_{aa} = -1$

ψ_a Creates anti- particle.





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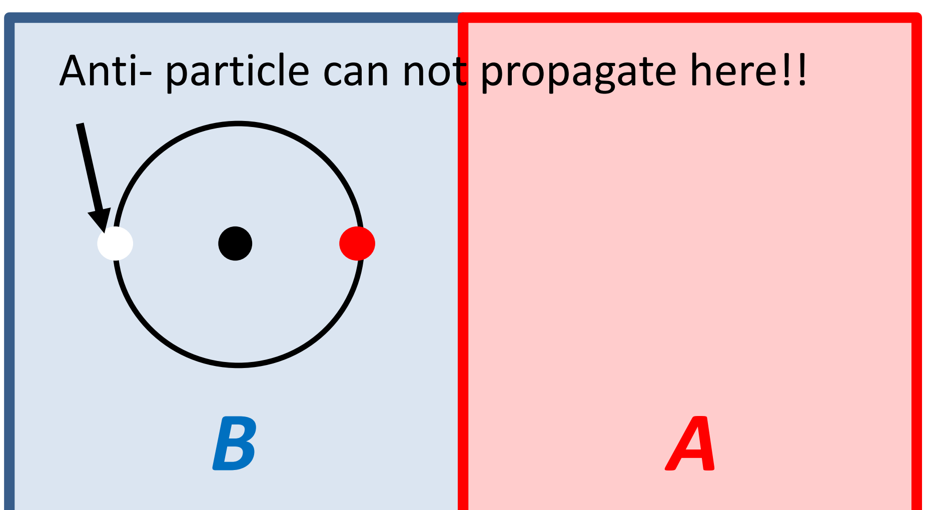
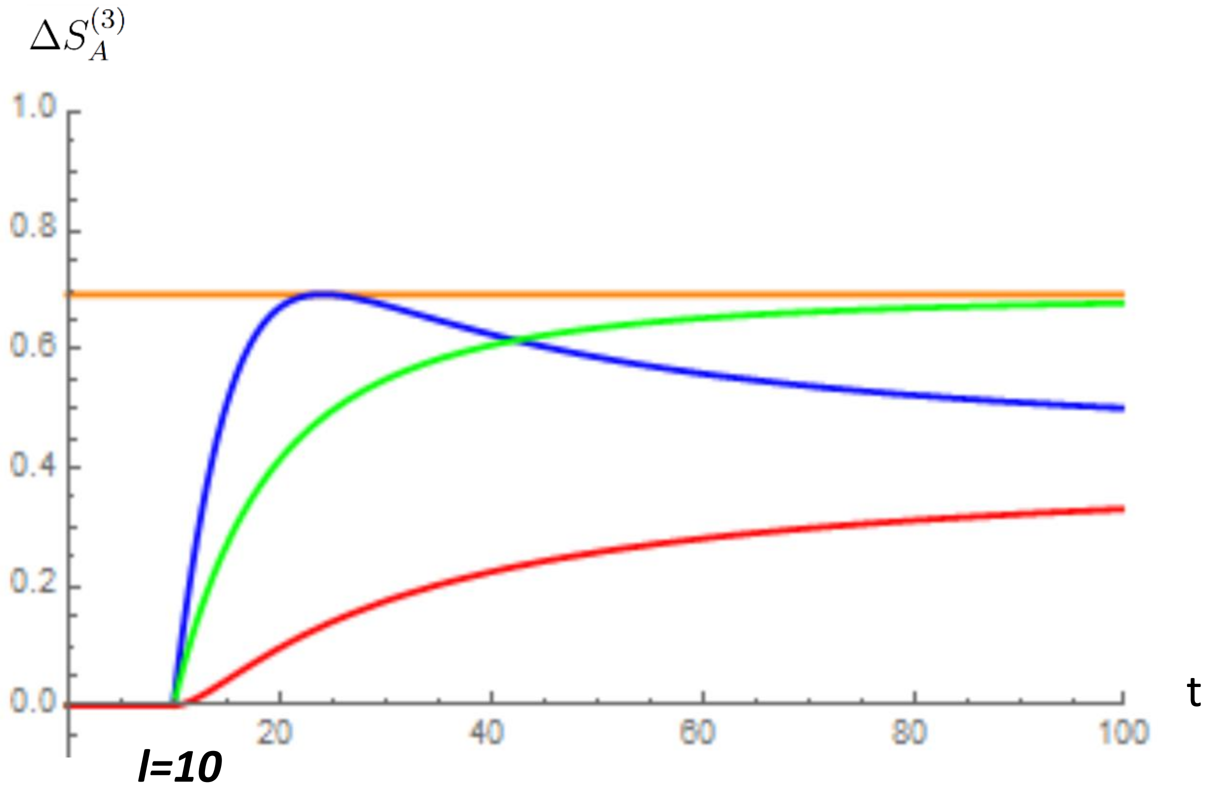
Orange: Log2



Example: $\left(\gamma^t\gamma^1\right)_{aa} = -1$

ψ_a Creates anti- particle.

It can not propagate parallel to the left!



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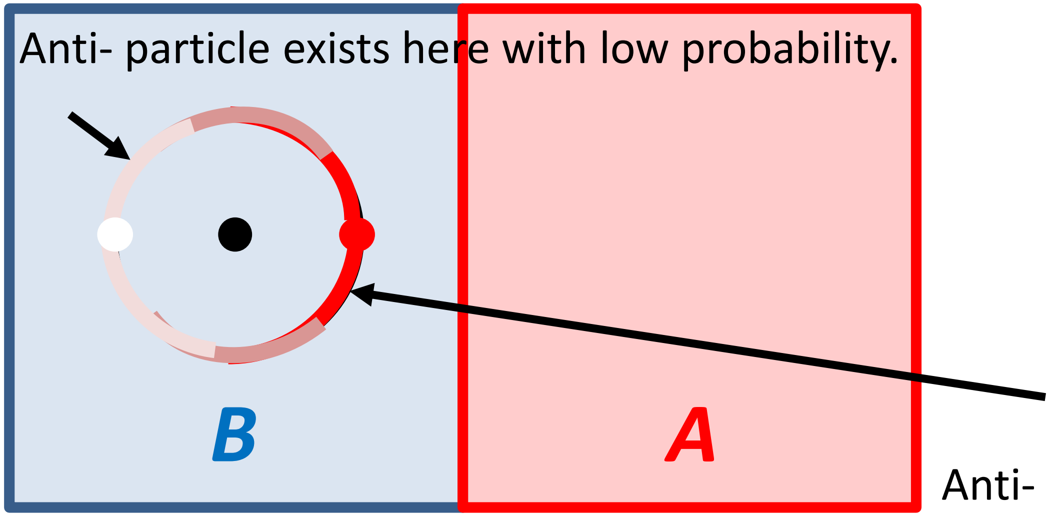
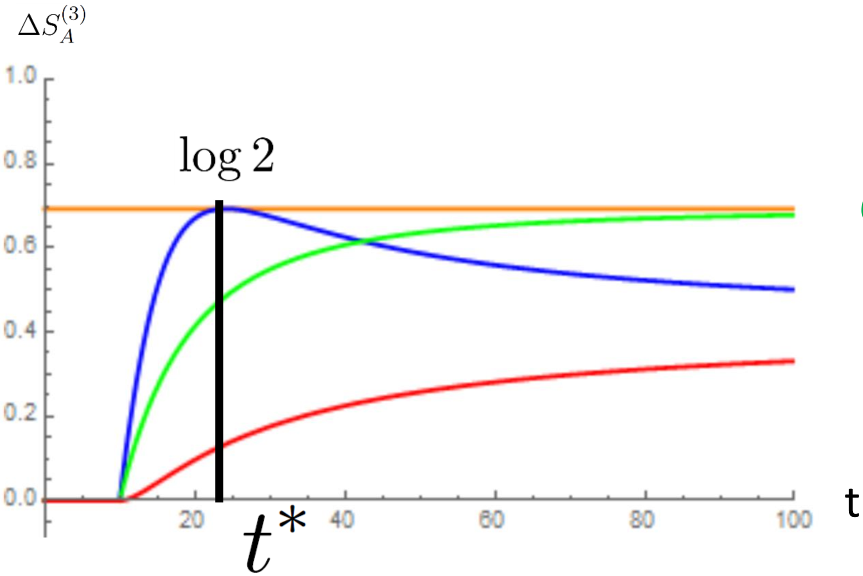
Orange: Log2



Example: $\left(\gamma^t\gamma^1\right)_{aa}=-1$

The existence probability of anti-particle is continuous.

Anti- particle exists here with high probability.



Red: $\left(\gamma^t\gamma^1\right)_{aa} = 1$

Green: $\left(\gamma^t\gamma^1\right)_{aa} = 0$

Blue: $\left(\gamma^t\gamma^1\right)_{aa} = -1$

Orange: Log2

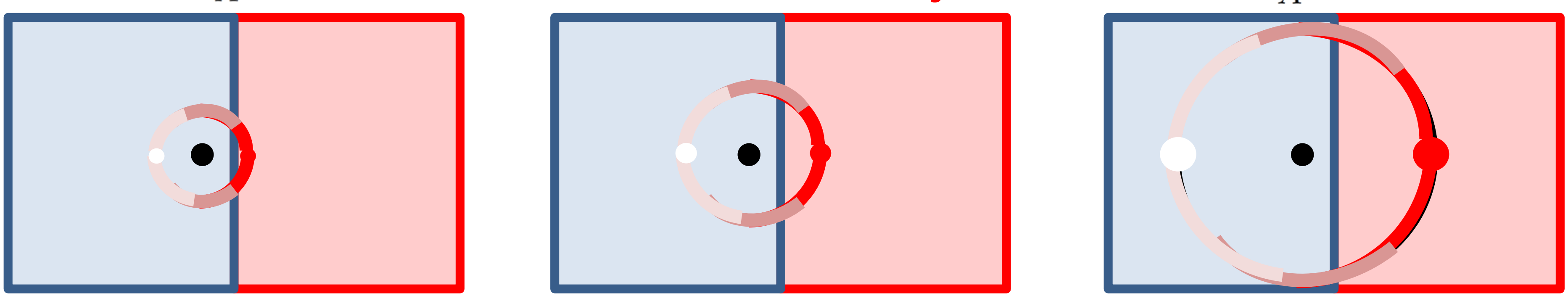
$$\rho_A \sim \begin{pmatrix} P_A & \cdot \\ \cdot & P_B \end{pmatrix}$$

Example: $\left(\gamma^t\gamma^1\right)_{aa} = -1$

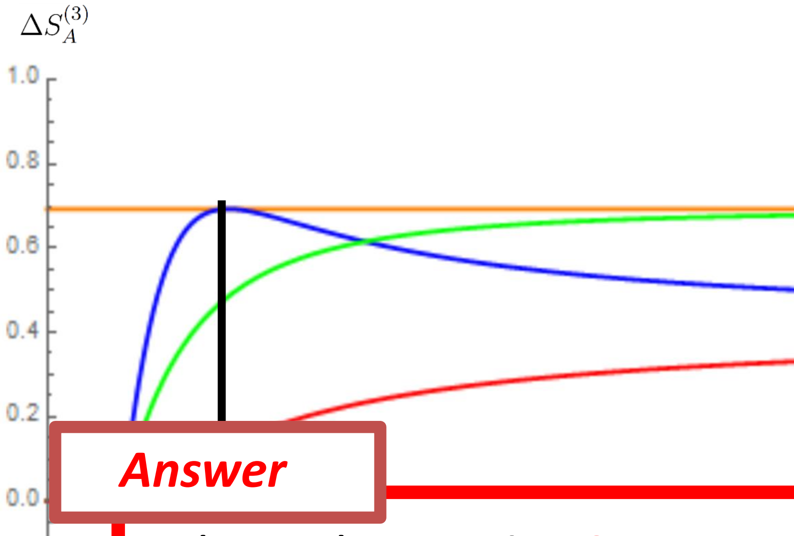
P_A,P_B :Existence probability in A, B.

$P_B > P_A$ $\quad$ $P_B = P_A$ $\quad$ $P_B < P_A$

$\Delta S_A^{(3)} < \log 2$ $\quad$ $\Delta S_A^{(3)} = \log 2$ ***Maximally Entangled*** $\quad$ $\Delta S_A^{(3)} < \log 2$



$l < t < t^*$ $\quad$ $t = t^*$ $\quad$ $t > t^*$



Red: $\left(\gamma^t\gamma^1\right)_{aa} = 1$

Green: $\left(\gamma^t\gamma^1\right)_{aa} = 0$

Blue: $\left(\gamma^t\gamma^1\right)_{aa} = -1$

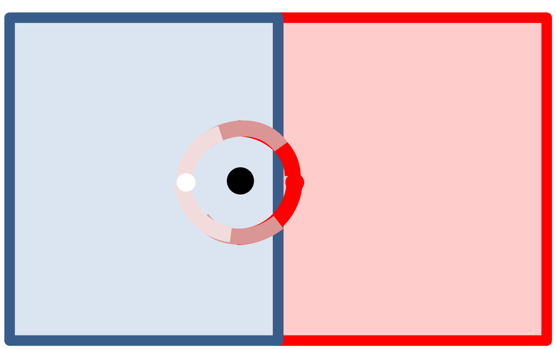
Orange: Log2

Answer

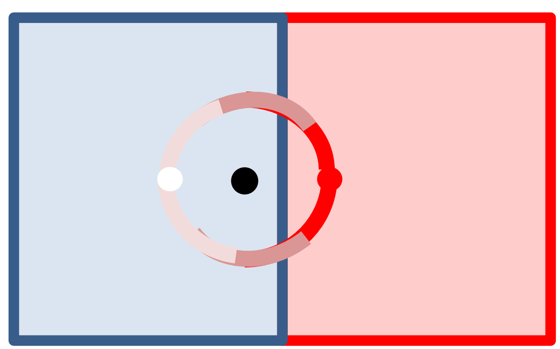
(Anti-)particle ***does not*** propagate with equivalent probability.

Example: $\left(\gamma^t\gamma^1\right)_{aa} =$ $\begin{pmatrix} P_A & \cdot \\ \cdot & P_B \end{pmatrix}$

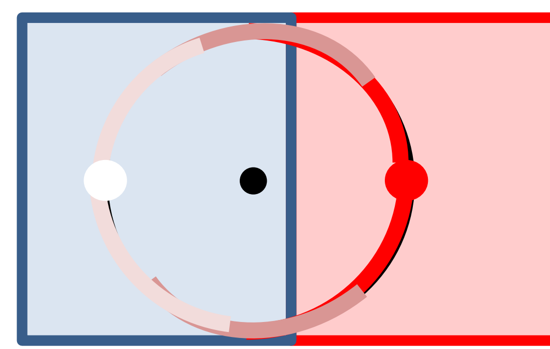
$P_B > P_A$	$P_B = P_A$	$P_B < P_A$
$\Delta S_A^{(3)} < \log 2$	$\Delta S_A^{(3)} = \log 2$ ***Maximally Entangled***	$\Delta S_A^{(3)} < \log 2$



$l < t < t^*$



$t = t^*$



$t > t^*$

Quasi-Particle Interpretation

- Decomposition into ***left movers*** and ***right movers***

$$\psi_a = \psi_a^{L\dagger} + \psi_a^{R\dagger} + \phi_a^L + \phi_a^R, \quad \bar{\psi}_a^{L\dagger} \equiv i\left(\phi^{L\dagger}\gamma^t\right)_a,$$

$$\psi_a^\dagger = \psi_a^L + \psi_a^R + \phi_a^{L\dagger} + \phi_a^{R\dagger}, \quad \bar{\psi}_a^{R\dagger} \equiv i\left(\phi^{R\dagger}\gamma^t\right)_a.$$

$$\bar{\psi}_a = \bar{\psi}_a^{L\dagger} + \bar{\psi}_a^{R\dagger} + i\left(\psi^L\gamma^t\right)_a + i\left(\psi^R\gamma^t\right)_a,$$

Quasi-Particle Interpretation

- Decomposition into ***left movers*** and ***right movers***

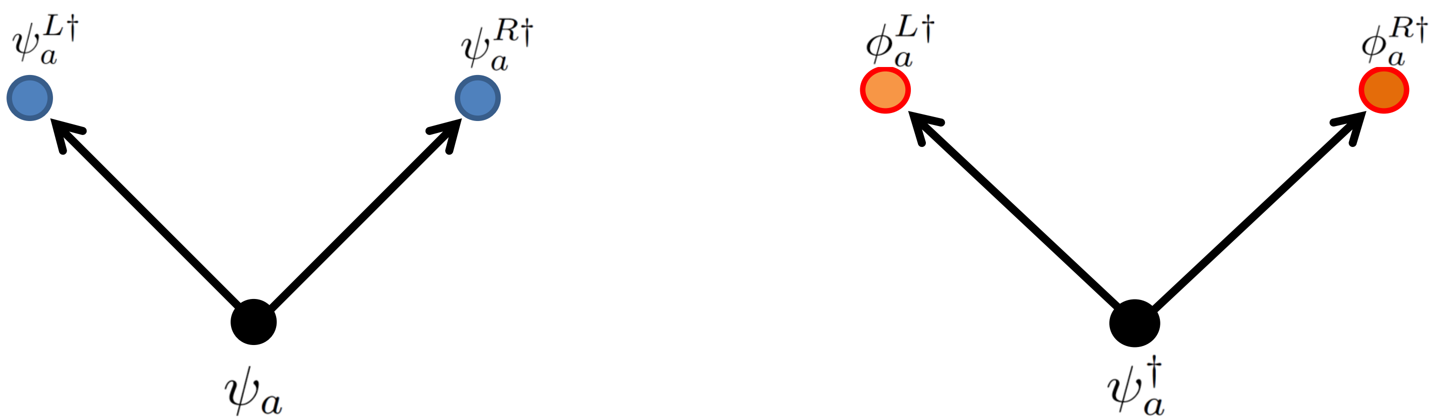
Anti- Particle

$$\psi_a = \psi_a^{L\dagger} + \psi_a^{R\dagger} + \phi_a^L + \phi_a^R,$$

Particle

$$\psi_a^\dagger = \psi_a^L + \psi_a^R + \phi_a^{L\dagger} + \phi_a^{R\dagger},$$

$$\bar{\psi}_a = \bar{\psi}_a^{L\dagger} + \bar{\psi}_a^{R\dagger} + i\left(\psi^L\gamma^t\right)_a + i\left(\psi^R\gamma^t\right)_a,$$



Exotic Quantization in 4d

Exotic Quantization

$$\{\phi^R_a, \phi^{R\dagger}_b\} = \left(\delta_{ab} - \frac{1}{2}\left(\gamma^t\gamma^1\right)_{ab}\right), \{\psi^{L\dagger}_a, \psi^L_b\} = \left(\delta_{ab} + \frac{1}{2}\left(\gamma^t\gamma^1\right)_{ab}\right),$$

$$\{\phi^L_a, \phi^{L\dagger}_b\} = \left(\delta_{ab} + \frac{1}{2}\left(\gamma^t\gamma^1\right)_{ab}\right), \{\bar{\psi}^R_a, \bar{\psi}^{R\dagger}_b\} = \left(\delta_{ab} - \frac{1}{2}\left(\gamma^1\gamma^t\right)_{ab}\right),$$

$$\{\psi^{R\dagger}_a, \psi^R_b\} = \left(\delta_{ab} - \frac{1}{2}\left(\gamma^t\gamma^1\right)_{ab}\right), \{\bar{\psi}^L_a, \bar{\psi}^{L\dagger}_b\} = \left(\delta_{ab} + \frac{1}{2}\left(\gamma^1\gamma^t\right)_{ab}\right).$$

(Anti-)particles do not propagate with equivalent probability.

$\Downarrow$

We need to impose exotic quantization condition.

Exotic Quantization in 4d

Exotic Quantization

Boost

$$\{\phi_a^R, \phi_b^{R\dagger}\} = \left(\delta_{ab} - \frac{1}{2}\left(\gamma^t\gamma^1\right)_{ab}\right), \{\psi_a^{L\dagger}, \psi_b^L\} = \left(\delta_{ab} + \frac{1}{2}\left(\gamma^t\gamma^1\right)_{ab}\right),$$

$$\{\phi_a^L, \phi_b^{L\dagger}\} = \left(\delta_{ab} + \frac{1}{2}\left(\gamma^t\gamma^1\right)_{ab}\right), \{\bar{\psi}_a^R, \bar{\psi}_b^{R\dagger}\} = \left(\delta_{ab} - \frac{1}{2}\left(\gamma^1\gamma^t\right)_{ab}\right),$$

$$\{\psi_a^{R\dagger}, \psi_b^R\} = \left(\delta_{ab} - \frac{1}{2}\left(\gamma^t\gamma^1\right)_{ab}\right), \{\bar{\psi}_a^L, \bar{\psi}_b^{L\dagger}\} = \left(\delta_{ab} + \frac{1}{2}\left(\gamma^1\gamma^t\right)_{ab}\right).$$

Exotic Quantization in 4d

Deformed Quantization

$$\{\phi_a^R, \phi_b^{R\dagger}\} = \left(\delta_{ab} - \frac{1}{2}\left(\gamma^t\gamma^1\right)_{ab}\right), \{\psi_a^{L\dagger}, \psi_b^L\} = \left(\delta_{ab} + \frac{1}{2}\left(\gamma^t\gamma^1\right)_{ab}\right),$$

$$\{\phi_a^L, \phi_b^{L\dagger}\} = \left(\delta_{ab} + \frac{1}{2}\left(\gamma^t\gamma^1\right)_{ab}\right), \{\bar{\psi}_a^R, \bar{\psi}_b^{R\dagger}\} = \left(\delta_{ab} - \frac{1}{2}\left(\gamma^1\gamma^t\right)_{ab}\right),$$

$$\{\psi_a^{R\dagger}, \psi_b^R\} = \left(\delta_{ab} - \frac{1}{2}\left(\gamma^t\gamma^1\right)_{ab}\right), \{\bar{\psi}_a^L, \bar{\psi}_b^{L\dagger}\} = \left(\delta_{ab} + \frac{1}{2}\left(\gamma^1\gamma^t\right)_{ab}\right).$$

Example $\quad \psi_a = \psi_a^{L\dagger} + \psi_a^{R\dagger} + \phi_a^L + \phi_a^R$

$$|\Psi\rangle = \mathcal{N}\psi_a |0\rangle \quad \Rightarrow \quad |\Psi\rangle = \frac{1}{\sqrt{2}}\left[\psi_a^{L\dagger}|0\rangle_L \otimes |0\rangle_R + |0\rangle_L \otimes \psi_a^{R\dagger}|0\rangle_R\right]$$

Exotic Quantization in 4d

Deformed Quantization

$$\{\phi_a^R, \phi_b^{R\dagger}\} = \left(\delta_{ab} - \frac{i}{2}\left(\gamma^0\gamma^1\right)_{ab}\right) \quad \{\psi_a^{L\dagger}, \psi_b^L\} = \left(\delta_{ab} + \frac{i}{2}\left(\gamma^0\gamma^1\right)_{ab}\right)$$

$$\{\phi_a^L, \phi_b^{L\dagger}\} = \left(\delta_{ab} + \frac{i}{2}\left(\gamma^0\gamma^1\right)_{ab}\right) \quad \{\bar{\psi}_a^R, \bar{\psi}_b^{R\dagger}\} = \left(\delta_{ab} - \frac{i}{2}\left(\gamma^1\gamma^0\right)_{ab}\right)$$

$$\{\psi_a^{R\dagger}, \psi_b^R\} = \left(\delta_{ab} - \frac{i}{2}\left(\gamma^0\gamma^1\right)_{ab}\right) \quad \{\bar{\psi}_a^L, \bar{\psi}_b^{L\dagger}\} = \left(\delta_{ab} + \frac{i}{2}\left(\gamma^1\gamma^0\right)_{ab}\right)$$

Example $\quad \psi_a = \psi_a^{L\dagger} + \psi_a^{R\dagger} + \phi_a^L + \phi_a^R$

$$|\Psi\rangle = \mathcal{N}\psi_a\,|0\rangle \quad \Rightarrow \quad |\Psi\rangle = \frac{1}{\sqrt{2}}\left[\psi_a^{L\dagger}\,|0\rangle_L \otimes |0\rangle_R + |0\rangle_L \otimes \psi_a^{R\dagger}\,|0\rangle_R\right]$$

$$\Rightarrow \quad \rho_A = \frac{1}{4}\left[\left(2 + \left(\gamma^t\gamma^1\right)_{aa}\right)|0\rangle_R\,\langle 0|_R + \left(2 - \left(\gamma^t\gamma^1\right)_{aa}\right)\left|\psi_a^R\right\rangle_R\left\langle\psi_a^R\right|_R\right]$$

Exotic Quantization in 4d

Deformed Quantization

$$\{\phi_a^R, \phi_b^{R\dagger}\} = \left(\delta_{ab} - \frac{i}{2}\left(\gamma^0\gamma^1\right)_{ab}\right) \quad \{\psi_a^{L\dagger}, \psi_b^L\} = \left(\delta_{ab} + \frac{i}{2}\left(\gamma^0\gamma^1\right)_{ab}\right)$$

$$\{\phi_a^L, \phi_b^{L\dagger}\} = \left(\delta_{ab} + \frac{i}{2}\left(\gamma^0\gamma^1\right)_{ab}\right) \quad \{\bar{\psi}_a^R, \bar{\psi}_b^{R\dagger}\} = \left(\delta_{ab} - \frac{i}{2}\left(\gamma^1\gamma^0\right)_{ab}\right)$$

$$\{\psi_a^{R\dagger}, \psi_b^R\} = \left(\delta_{ab} - \frac{i}{2}\left(\gamma^0\gamma^1\right)_{ab}\right) \quad \{\bar{\psi}_a^L, \bar{\psi}_b^{L\dagger}\} = \left(\delta_{ab} + \frac{i}{2}\left(\gamma^1\gamma^0\right)_{ab}\right)$$

Example $\quad \psi_a = \psi_a^{L\dagger} + \psi_a^{R\dagger} + \phi_a^L + \phi_a^R$

$$|\Psi\rangle = \mathcal{N}\psi_a|0\rangle \quad \Rightarrow \quad |\Psi\rangle = \frac{1}{\sqrt{2}}\left[\psi_a^{L\dagger}|0\rangle_L \otimes |0\rangle_R + |0\rangle_L \otimes \psi_a^{R\dagger}|0\rangle_R\right]$$

$$\rho_A = \frac{1}{4}\left[(2 + (\quad \cdots \quad - \left(\gamma^t\gamma^1\right)_{aa}\right) \left|\psi_a^R\right\rangle_R \left\langle\psi_a^R\right|_R\right]$$

Consistent!!

Exotic Quantization in General dim. (Work in Progress)

Exotic Quantization

$$\{\phi_a^R, \phi_b^{R\dagger}\} = \left(\delta_{ab} - c_g \left(\gamma^t\gamma^1\right)_{ab}\right), \{\psi_a^{L\dagger}, \psi_b^L\} = \left(\delta_{ab} + c_g \left(\gamma^t\gamma^1\right)_{ab}\right),$$
$$\{\phi_a^L, \phi_b^{L\dagger}\} = \left(\delta_{ab} + c_g \left(\gamma^t\gamma^1\right)_{ab}\right), \{\bar{\psi}_a^R, \bar{\psi}_b^{R\dagger}\} = \left(\delta_{ab} - c_g \left(\gamma^1\gamma^t\right)_{ab}\right),$$
$$\{\psi_a^{R\dagger}, \psi_b^R\} = \left(\delta_{ab} - c_g \left(\gamma^t\gamma^1\right)_{ab}\right), \{\bar{\psi}_a^L, \bar{\psi}_b^{L\dagger}\} = \left(\delta_{ab} + c_g \left(\gamma^1\gamma^t\right)_{ab}\right).$$

D:Spacetime Dimension

$$D = 2, c_g = 1$$
$$D = 4, c_g = \frac{1}{2},$$
$$D = 6, c_g = \frac{3}{8},$$
$$D = 8, c_g = \frac{5}{16}.$$

Exotic Quantization in General dim. (Work in Progress)

Exotic Quantization

$$\{\phi_a^R, \phi_b^{R\dagger}\} = \left(\delta_{ab} - c_g \left(\gamma^t\gamma^1\right)_{ab}\right), \{\psi_a^{L\dagger}, \psi_b^L\} = \left(\delta_{ab} + c_g \left(\gamma^t\gamma^1\right)_{ab}\right),$$
$$\{\phi_a^L, \phi_b^{L\dagger}\} = \left(\delta_{ab} + c_g \left(\gamma^t\gamma^1\right)_{ab}\right), \{\bar{\psi}_a^R, \bar{\psi}_b^{R\dagger}\} = \left(\delta_{ab} - c_g \left(\gamma^1\gamma^t\right)_{ab}\right),$$
$$\{\psi_a^{R\dagger}, \psi_b^R\} = \left(\delta_{ab} - c_g \left(\gamma^t\gamma^1\right)_{ab}\right), \{\bar{\psi}_a^L, \bar{\psi}_b^{L\dagger}\} = \left(\delta_{ab} + c_g \left(\gamma^1\gamma^t\right)_{ab}\right).$$

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$$\Rightarrow \quad c_g = \frac{1}{2^{D-2}} \sum_{j=0}^{\frac{D-2}{2}} \left({}_{\frac{D-2}{2}}C_j\right)^2$$

Exotic Quantization in General dim. (Work in Progress)

Exotic Quantization

$$\{\phi_a^R, \phi_b^{R\dagger}\} = \left(\delta_{ab} - c_g \left(\gamma^t \gamma^1\right)_{ab}\right), \{\psi_a^{L\dagger}, \psi_b^L\} = \left(\delta_{ab} + c_g \left(\gamma^t \gamma^1\right)_{ab}\right),$$
$$\{\phi_a^L, \phi_b^{L\dagger}\} = \left(\delta_{ab} + c_g \left(\gamma^t \gamma^1\right)_{ab}\right), \{\bar{\psi}_a^R, \bar{\psi}_b^{R\dagger}\} = \left(\delta_{ab} - c_g \left(\gamma^1 \gamma^t\right)_{ab}\right),$$
$$\{\psi_a^{R\dagger}, \psi_b^R\} = \left(\delta_{ab} - c_g \left(\gamma^t \gamma^1\right)_{ab}\right), \{\bar{\psi}_a^L, \bar{\psi}_b^{L\dagger}\} = \left(\delta_{ab} + c_g \left(\gamma^1 \gamma^t\right)_{ab}\right).$$

We are studying physical meaning of this term.

D:Space

D

$$D = 4, c_g = \frac{1}{2},$$
$$D = 6, c_g = \frac{3}{8},$$
$$D = 8, c_g = \frac{5}{16}.$$

$$\Rightarrow \quad c_g = \frac{1}{2^{D-2}} \sum_{j=0}^{\frac{D-2}{2}} \left({}_{\frac{D-2}{2}}C_j\right)^2$$

CREE in 2d CFT

- Statistical mechanics

$$\rho = e^{-\beta H} \rightarrow \rho' = e^{-\beta(H+\mu Q)}$$

$$[H, Q] = 0$$

- C(R)EE

JHEP 1312 (2013) 059
Alexandre Belin, Ling-Yan Hung, Alexander Maloney, Shunji Matsuura, Robert C. Myers, Todd Sierens

$$\rho_A = e^{-2\pi H_{mod}} \rightarrow \rho'_A = e^{-2\pi(H_{mod}+\mu Q_A)}$$

$$[Q, H] = 0, \quad [Q_A, H_{mod}] = 0$$

We might study quantum entanglement in more detail.

CREE in 2d CFT

Theory: 2d Massless Fermionic Field Theory

Excited State: $\left|\Psi\right\rangle = \mathcal{N}\bar{\psi}\psi\left|0\right\rangle$

At the late time($t\to\infty$),

$$\boxed{\Delta S_A^{(n)} = \frac{1}{1-n}\log\frac{\cosh n\mu q}{2^{n-1}\left(\cosh\mu q\right)^n}}$$

Physical Interpretation

Under quasi-particle decomposition

Reduced Density Matrix:

$$\rho_A = \frac{1}{2}\left|q\right\rangle_A \left\langle q\right|_A + \frac{1}{2}\left|-q\right\rangle_A \left\langle -q\right|_A$$

$$\Big\Downarrow \; e^{\mu Q_A}$$

$$\rho_A^c = \frac{1}{2\cosh \mu q}\left[e^{\mu q}\left|q\right\rangle_A \left\langle q\right|_A + e^{-\mu q}\left|-q\right\rangle_A \left\langle -q\right|_A\right]$$

$$Q_A\left|q\right\rangle_A = -Q_A\left|-q\right\rangle_A = q\left|q\right\rangle_A = q\left|-q\right\rangle_A$$

Physical Interpretation

Under quasi-particle decomposition

Reduced Density Matrix:

$$\rho_A = \frac{1}{2}\left|q\right\rangle_A \left\langle q\right|_A + \frac{1}{2}\left|-q\right\rangle_A \left\langle -q\right|_A$$

$$\Big\Downarrow \; e^{\mu Q_A}$$

$$\rho_A^c = \frac{1}{2\cosh\mu q}\left[e^{\mu q}\left|q\right\rangle_A \left\langle q\right|_A + e^{-\mu q}\left|-q\right\rangle_A \left\langle -q\right|_A\right]$$

$$\boxed{\Delta S_A^{(n)} = \frac{1}{1-n}\log\frac{\cosh n\mu q}{2^{n-1}\left(\cosh\mu q\right)^n}}$$

Physical Interpretation

Under quasi-particle decomposition

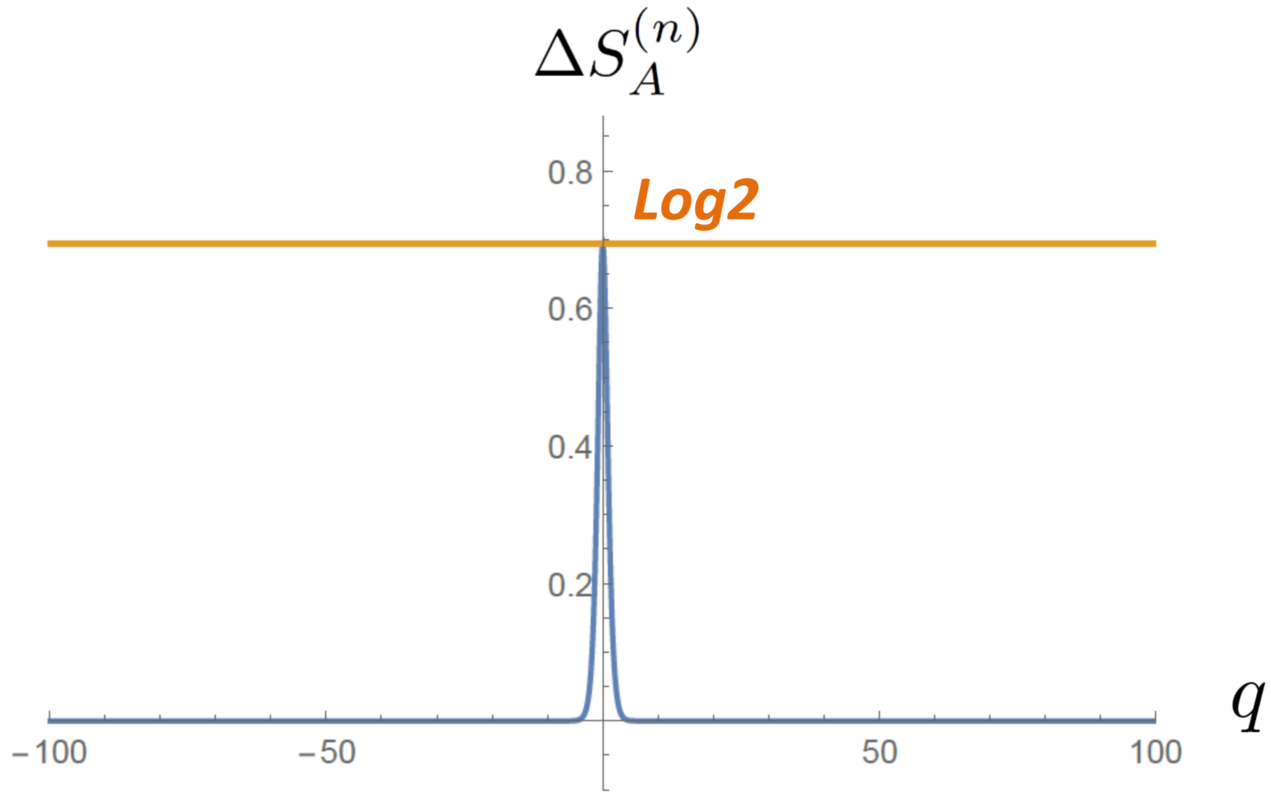
Reduced Density Matrix:

Consistent!!

$$\rho_A^c = \frac{1}{2\cosh \mu q}\left[e^{\mu q}\left|q\right\rangle_A \left\langle q\right|_A + e^{-\mu q}\left|-q\right\rangle_A \left\langle -q\right|_A\right]$$

$$\boxed{\Delta S_A^{(n)} = \frac{1}{1-n}\log\frac{\cosh n\mu q}{2^{n-1}\left(\cosh \mu q\right)^n}}$$

Charge dependence of $\Delta S_A^{(n)}$



$$q \to \pm\infty$$

$$\rho_A^c \to |q\rangle_A \langle q|_A \text{ or } |-q\rangle_A \langle -q|_A$$

Product State

Consistent!!

Field Theory

2. *U(N)* or *SU(N)* free massless scalar field theory in Large N limit

PTEP 2014 (2014) 093B06 Pawel Caputa, MN, Tadashi Takayanagi

5. Holographic field theory

PTEP 2014 (2014) 093B06 Pawel Caputa, MN, Tadashi Takayanagi

Field Theory Side

We consider large N free massless U(N) scalar field theory.

ϕ_1, ϕ_2 : Adjoint scalar

We act $Tr(\mathcal{Z}^J) = Tr(\phi_1 + i\phi_2)^J$ on the ground state.

$J = 2,$

$$\Delta S_R^{(n)} = \frac{1}{1-n} \log\left(2^{1-2n} + \frac{1}{2^n N^{2(n-1)}}\right).$$

Field Theory Side

We consider large N free massless U(N) scalar field theory.

ϕ_1, ϕ_2 : Adjoint scalar

We act $Tr(\mathcal{Z}^J) = Tr(\phi_1 + i\phi_2)^J$ on the ground state.

$$J = 2, \qquad \Delta S_R^{(n)} = \frac{1}{1-n}\log\left(2^{1-2n} + \frac{1}{2^n N^{2(n-1)}}\right).$$

$$\Big\Downarrow \; N \to \infty$$

$$n \to 1$$

$$\Delta S_R^{(n\geq 2)} \simeq \frac{2n-1}{n-1}\cdot\log 2 \;\Rightarrow\; \textit{Diverge!!}$$

Field Theory Side

We consider large N free massless U(N) scalar field theory.

ϕ_1, ϕ_2 : Adjoint scalar

We act $Tr(\mathcal{Z}^J) = Tr(\phi_1 + i\phi_2)^J$ on the ground state.

$$J = 2, \qquad \Delta S_R^{(n)} = \frac{1}{1-n} \log\left(2^{1-2n} + \frac{1}{2^n N^{2(n-1)}}\right).$$

$$\Downarrow \; n \to 1$$

$$\Delta S_R^{(1)} = \log\left(2\sqrt{2}N\right) \xRightarrow{N \to \infty} \textit{Finite!!}$$

Field Theory Side

We consider large N free massless U(N) scalar field theory

ϕ_1, ϕ_2 : Adjoint scalar

We act $Tr(\mathcal{Z}^J) = Tr(\phi_1 + i\phi_2)^J$ on the ground state.

$N \to \infty$ ⇒ $n \to 1$: ***Incorrect Limit***

$n \to 1$ ⇒ $N \to \infty$: ***Correct limit***

⇓ $n \to 1$

$N \to \infty$

$$\Delta S_R^{(1)} = \log\left(2\sqrt{2}N\right)$$ ⇒ ***Finite!!***

Field Theory Side

In large N limit, for J=2

$$\boxed{\begin{array}{ll} n \geq 2 & \Delta S_A^{(n\geq 2)} \simeq \dfrac{2n-1}{n-1} \cdot \log 2 \\ n = 1 & \Delta S_A = \log\left(2\sqrt{2}N\right) \end{array}}$$

If we think 1/n as an effective temperature ,

n=1 : Deconfinement Phase

n≠1: Confinement Phase

AdS/CFT

At the late time (t>>l) ($1 \ll \Delta \ll c$)

- $n \geq 2$

Conformal dim. of inserted operator

$$\boxed{\Delta S_A^{(n)} \sim \frac{4n\Delta}{d(n-1)} \log t.}$$

⇩

We can not take the von Neumann limit (n→1).

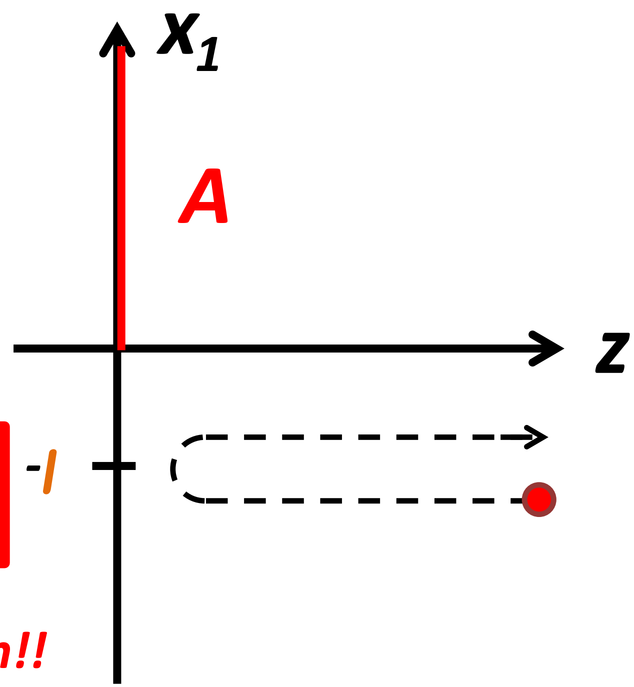
AdS/CFT

At the late time (t>>l) ($\Delta \simeq c$)

- $n = 1$, (2dCFT)

Central charge

$$\boxed{\Delta S_A \simeq \frac{c}{6} \log t}$$



$\Delta S_A^{(n)}$ do not approach constant.

Quasi-particle interpretation breaks down!!

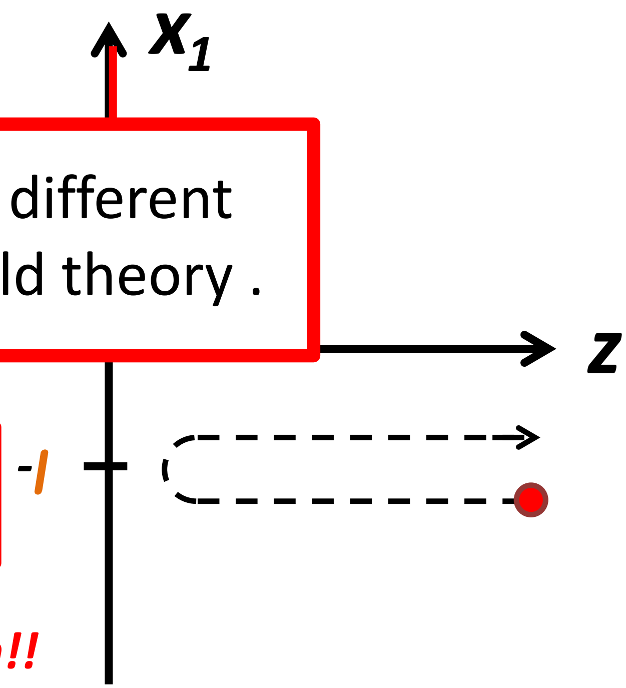
AdS/CFT

At the late time (t>>l) ($\Delta \simeq c$)

- $n=1$, (2dCFT)

Central charge

c



$\Delta S_A^{(n)}$ in holographic field theory is totally different from those in free field theory .

$\Delta S_A^{(n)}$ do not approach constant.

Quasi-particle interpretation breaks down!!

Summary

Free Scalar Field Theory

We defined the (Renyi) entanglement entropies of local operators.

-They characterize local operators from the viewpoint of quantum entanglement.

- These entropies of the operators (constructed of single-species operator) are given by the those of binomial distribution.

-The results we obtain in terms of entangled pair agree with the results we obtain by replica method.

Summary

In Fermionic Field Theory

- We compute $\Delta S_A^{(n)}$ for various locally excited state.
  - They are given by constant values in the late time.
    Some of them depend on the representation(spin).

- When we interpret results in terms quasi-particle picture,
  we should impose exotic quantization condition.
  (Because (anti-)particle does not propagate in any directions
  with equivalent probability.)
- We compute CREE for locally excited state.

Summary

AdS/CFT correspondence

- To take n → 1 limit does not commute with taking N → ∞ limit.

  - After taking large N limit, we can not take n → 1 (EE for excited state diverge.).

- In AdS/CFT correspondence, $\Delta S_A^{(n)}$ does not approach some constants.

  - In large N expansion, the leading term of $\Delta S_A^{(n)}$ for $n \neq 1$ is proportional to the conformal dim. of operators which act on the ground state.

  - In large N expansion, the leading term of $\Delta S_A^{(n)}$ for n=1 is proportional to central charge.

Future Problems

- In non-relativistic case, the time evolution of $\Delta S_A^{(n)}$
- In gauge field theory, $\Delta S_A^{(n)}$ (collaborate with Naoki Watamura, Pawel Caputa)
- The (Renyi) entanglement entropies of operators in the interacting field theory .
- Physical meaning of the coefficient of gamma matrices.
- Excesses of CREE in AdS/CFT