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Entanglement Entropy in String Theory

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Collaboration with

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①Introduction, Motivatoin

- In field theories in d dim, EE has a UV divergence:

$$S_A = s \cdot \frac{V_{d-2}}{\varepsilon^{d-2}} + (\text{subleading})$$

- String theory is a UV finite theory .

  Naturalcutoff : $l_s = \sqrt{\alpha'}$ (string length)

  How about EE ?

  Natural expectation: replace with $\varepsilon = l_s$

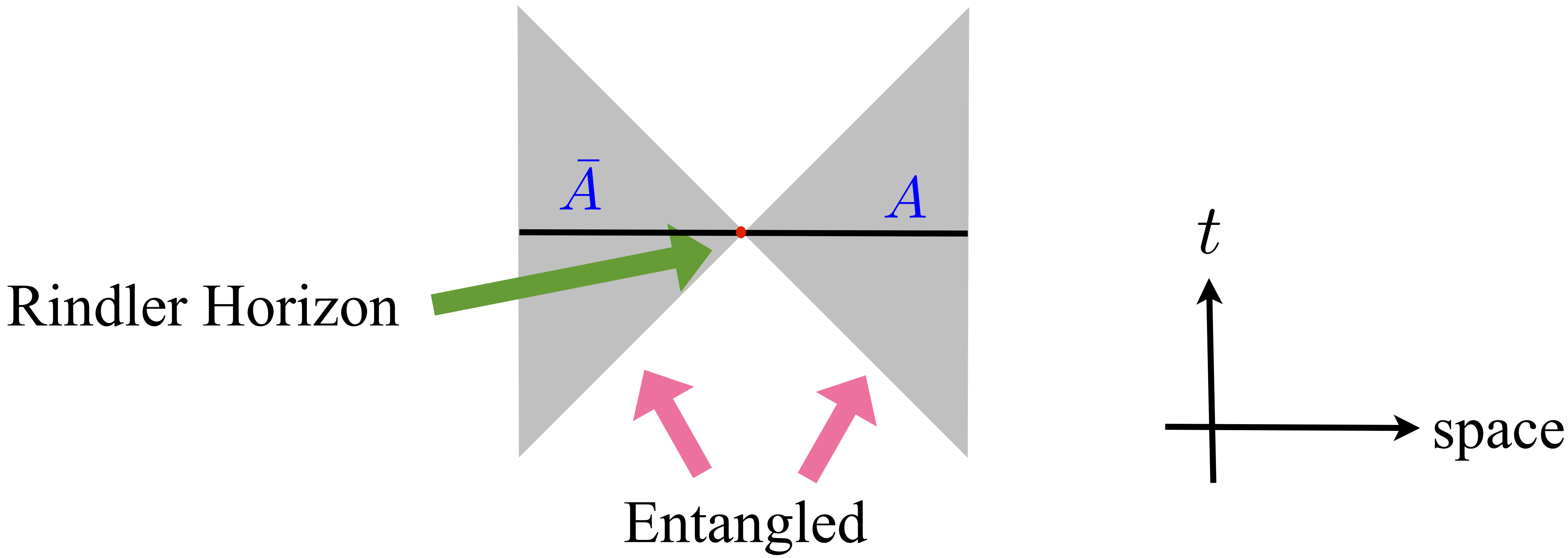
$$S_A = s \cdot \frac{V_{d-2}}{l_s^{d-2}} + (\text{subleading})$$

What is the definition of EE in string theory ?

Another Motivation:

EE as quantum correction of BH entropy

Consider Minkowski vacuum in Rindler space



Entangment across the Rindler Horizon = Quantum correction of BH entropy

Understand EE in sting theory → Understand sting around BH horizon

②Definition

- Field Theory case:

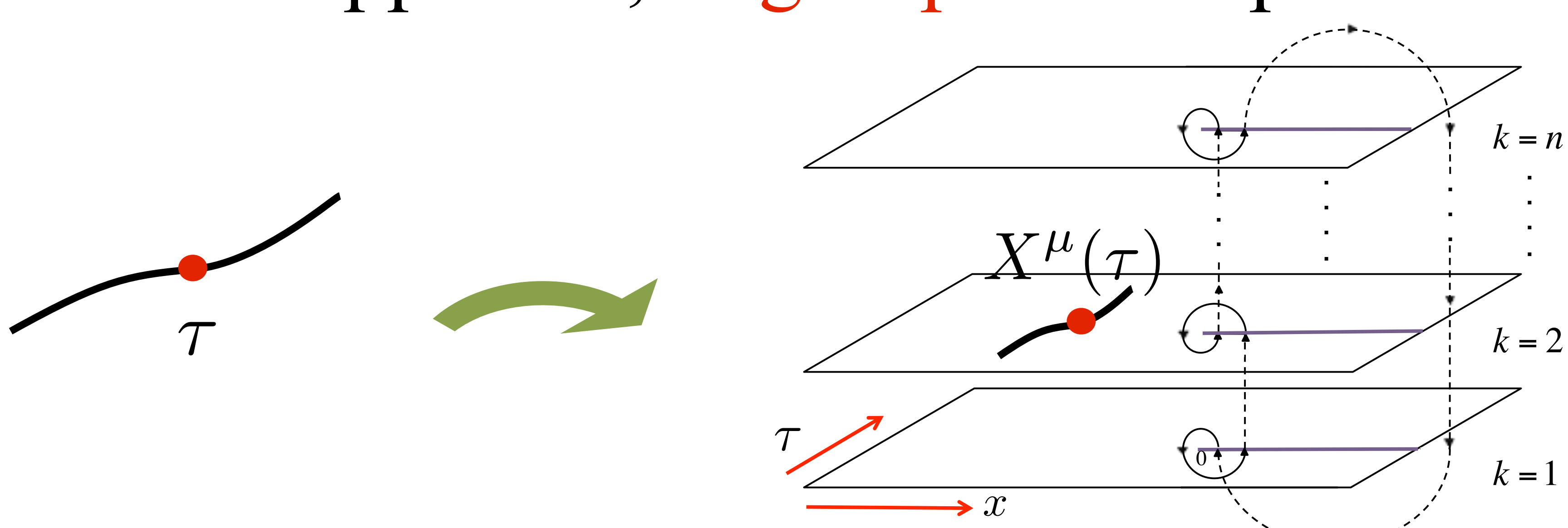
  Consider a free scalar.

  We can compute EE using replica method:

$$S_A = \lim_{n \to 1} \frac{1}{1-n} \log \frac{Z_n}{(Z_1)^n}$$

Z_n :partition function on n-sheeted manifold Σ_n

In world line approach, target space is replicated.



In world line approach, field theory partition function Z_{field} can be computed by world line partition function Z_{line} .

Using diagram, vacuum amplitude is

$$Z_{field} = 1 + \bigcirc + \frac{1}{2!}\,\bigcirc\bigcirc + \frac{1}{3!}\,\bigcirc\bigcirc\bigcirc + \cdots$$

$$= \exp(\bigcirc)$$

On the other hand,

$$Z_{line} = \bigcirc$$

$$\Rightarrow \quad \log Z_{field} = Z_{line}$$

EE can be computed using world line partition function with target Σ_n :

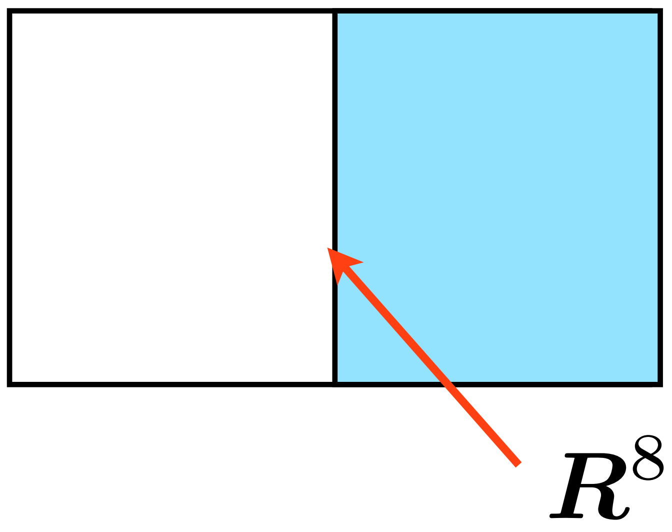
$$S_A = \lim_{n \to 1} \frac{1}{1-n} \left(Z_{line}(\Sigma_n) - n Z_{line}(\Sigma_1) \right)$$

- String Theory case:

  We consider at 1-loop level.

  Entangling surface: $\boldsymbol{R}^8$

  (Total spacetime: $\boldsymbol{R}^{10}$)



- We don't know the decomposition of the Hilbelt space in string theory

- but Z_{line} can be generalised to the world sheet partition function Z_{sheet} .

- We don't know the string theory with target space Σ_n

  ➡ For fractional $n = \dfrac{1}{N}$, Σ_n becomes

  orbifold $\boldsymbol{R}^8 \times \boldsymbol{C}/\mathbb{Z}_N$

Final expression:

$$S_A = \lim_{N \to 1} \frac{1}{1 - 1/N} \left(Z_{sheet}(\boldsymbol{R}^8 \times \boldsymbol{C}/\mathbb{Z}_N) - \frac{1}{N} Z_{sheet}(\boldsymbol{R}^{10}) \right)$$

③EE for free higher spin fields

Sting theory includes many massive higher spin modes.

⇨ EE for higher spin fields is important.

- partition function of higher spin fields on orbifold

$$Z_{line}(\boldsymbol{C}/\mathbb{Z}_N \times \boldsymbol{R}^{D-2}) = (-1)^F \int_{\epsilon^2}^{\infty} \frac{ds}{2s} \mathrm{Tr} \frac{1}{N} \sum_{j=0}^{N-1} g^j e^{-s(\hat{k}^2+m^2)}$$

Projection operator

Complete set: $\{ |\vec{k}, a> \}$

(momentum + spin component)

⇨ $$< \vec{k}, a| g^j |\vec{k}, a >= \frac{V_{D-2}}{(2\pi)^{D-2}} \frac{e^{\frac{2\pi i s_a}{N}}}{4\sin^2 \frac{\pi j}{N}}$$

s_a : spin under $SO(2) \subset SO(D)$

To calculate entropy, we need

$$J(r,N)=\sum_{\beta=1}^{N-1}\frac{\cos\frac{\pi\beta r}{N}}{\sin^2\frac{\pi\beta}{N}}$$

Finally we get

fermion: $J(r,N)|_{r\in odd}=\frac{1}{3}(N^2-1)+2N^2\left[\left\{\frac{r+N}{2N}\right\}^2-\left\{\frac{r+N}{2N}\right\}\right]$

boson: $J(r,N)|_{r\in even}=\frac{1}{3}(N^2-1)+2N^2\left[\left\{\frac{r}{2N}\right\}^2-\left\{\frac{r}{2N}\right\}\right]$

$\{x\}$: fractional part of x

non analytic function except for $r=0,1,2$(scalar, spinor, gauge field)

We need to define the rule of differential

(1) assume N is sufficiently large

fermion: $J(r,N)|_{r\in odd} = -\frac{1}{6}N^2 + \frac{r^2}{2} - \frac{1}{3}$ for $-N \le r \le N$

boson: $J(r,N)|_{r\in even} = \frac{1}{3}N^2 - rN + \frac{r^2}{2} - \frac{1}{3}$ for $0 \le r \le 2N$

$$\Rightarrow \quad \frac{\partial J(r,N)}{\partial N}\Big|_{N=1} = -\frac{1}{3}, (r \in odd)$$

$$\frac{\partial J(r,N)}{\partial N}\Big|_{N=1} = \frac{2}{3} - |r|, (r \in even)$$

Finallly

$$S_A = c_{ent} \cdot V_{d-2} \int_{\varepsilon^2}^{\infty} \frac{ds}{2s(4\pi s)^{\frac{d-2}{2}}} e^{-m^2 s}$$

where

$$c_{ent}^{Fermion} = \frac{1}{12} \cdot [\#\text{Majorana spin components}]$$

$$c_{ent}^{Boson} = \frac{1}{6} N_{dof} - \frac{1}{2} \sum_{a=1}^{N_{dof}} |s_a|$$

generalization of [Furusaev-Miele 96 for s=1,3/2,2](also coinside with [Kabat 95] for s=1)

Note

- bosonic higher spin modes have negative contributes.

➡ possibility of cancelltion (because of susy)

[cf Susskind Uglum 94]

Historically, this called conical entropy

(2)ignore r dipendent part

fermion: $$\frac{\partial J(r,N)}{\partial N}\Big|_{N=1} = -\frac{1}{3}, (r \in odd)$$

boson: $$\frac{\partial J(r,N)}{\partial N}\Big|_{N=1} = \frac{2}{3}, (r \in even)$$

➡ Thermodynamical entropy in Rindler space.

Historically, this called entanglement entropy

We concentrate on conical entropy.

④closed string case

partition function is given by

$$Z_{closed}(C/\mathbb{Z}_N \times R^8) = V_8 \int_F \frac{d\tau^2}{4\tau_2} \cdot (4\pi^2\alpha\tau_2)^{-4} \cdot \sum_{l,m=0}^{N-1} \frac{|\theta_1(\nu_{lm}/2|\tau)|^8}{N|\eta(\tau)|^{18}|\theta_1(\nu_{lm}|\tau)|^2}$$

F:fundamental region of torus moduli

difficulty: we can not do the double summation

[cf: Susskind-Uglum 94, Dabholkar 94,95, Emparan 94, Lowe-Strominger 94]

To discuss more quantitatively, we consider the modified quantity twisted conical entropy

To avoid the double summation, we consider string theory on Melvin (Flux tube) background and consider the orbifold as a limit of this

Melvin background

First we compactify one direction to S^1(radius: R, $R = \dfrac{\alpha'}{N R_{orb}}$)
To get Melvin BG, we includes the twisting around S^1 direction.

Action of $\mathbb{Z}_N$ Melvin background:

$$g : (X, \bar{X}, y) \to (e^{\frac{4\pi i}{N}}, e^{-\frac{4\pi i}{N}}\bar{X}, y + 2\pi R)$$

$$X \in \boldsymbol{C} \qquad y \in S^1$$

when $R \to 0$ and using T-duality, This reduces to ordinary orbifold. [Takayanagi-Uesugi 01]

⇨ leading contribution is not changed

Merits of Melvin BG

- finite summation becomes infinite summation(effect of twist)
- one of infinite summation can be absorbed to the change of integral region

Partition function on Melvin BG

Partition function is given by

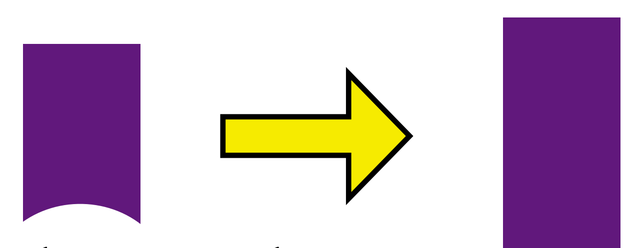
$$Z_{closed}\left[(\boldsymbol{C}\times S^1)/\mathbb{Z}_N\times\boldsymbol{R}^7\right]=$$

$$\frac{V_7 R}{4(2\pi)^7\alpha'^4}\cdot\int_F \frac{d\tau^2}{\tau_2^5}\sum_{w',w=-\infty}^{\infty} e^{-\frac{\pi R^2}{\alpha'\tau_2}|w-w'\tau|^2}\cdot\frac{|\theta_1((w-w'\tau)/N|\tau)|^8}{|\eta(\tau)|^{18}|\theta_1(2(w-w'\tau)/N|\tau)|^2}$$

[Takayanagi-Uesugi 01]

Infinite summation due to twist

One of the summation is absorbed by the change of integral region:



F:fundamentral region

S:strip

$$-\frac{1}{2}\leq\tau_1\leq\frac{1}{2}$$

$$\tau_2>0$$

Using Poisson summation formula, finally we get

$$Z_{closed}\left[(\boldsymbol{C}\times S^1)/\mathbb{Z}_N\times\boldsymbol{R}^7\right]=$$

$$\frac{V_7 R}{4(2\pi)^7\alpha'^4}\int_S\frac{d\tau^2}{\tau_2^5}\frac{\sqrt{\alpha'\tau_2}}{NR}\sum_{\gamma\in\mathbb{Z}}\sum_{\beta=0}^{N-1}e^{-\frac{\pi\alpha'\tau_2}{R^2N^2}\gamma^2}\cdot e^{2\pi i\frac{\beta\gamma}{N}}\frac{|\theta_1(\beta/N|\tau)|^8}{|\eta(\tau)|^{18}\cdot|\theta_1(2\beta/N|\tau)|^2}$$

$$w=N\gamma+\beta$$

To evaluate this integral, we investigate the possiblly divergent regions.

Possible divergent region

(1)IR limit $\tau_2\to\infty$

(2)UV limit $\tau_2\to 0$

⇨ Check these regions

(1)IR limit $\tau_2 \to \infty$

summation is localized to $\gamma = 0$ and

$$Z_{closed}[(\boldsymbol{C} \times S^1)/\mathbb{Z}_N \times \boldsymbol{R}^7] \simeq 64 \cdot Z_0 \int_S \frac{d\tau^2}{\tau_2^5} \frac{\sqrt{\alpha' \tau_2}}{NR} \cdot \sum_{\beta=0}^{N-1} \frac{\sin^8(\frac{\pi\beta}{N})}{\sin^2(\frac{2\pi\beta}{N})}$$

$$\left(Z_0 = \frac{V_7 R}{4(2\pi)^7 \alpha'^4} \right)$$

This match with type II supergravity contribution $\sum_{\text{Sugra}} J(s_a, N)$

Entropy is given by

$$S_A(R_{orb}) \simeq 2Z_0 \cdot \frac{R_{orb}}{\sqrt{\alpha'}} \int_S \frac{d\tau^2}{\tau_2^9} \sim \frac{V_7}{\alpha'^{\frac{7}{2}}}$$

(2)UV limit $\tau_2 \to 0$

$$Z_{closed}\left[(\boldsymbol{C}\times S^1)/\mathbb{Z}_N\times \boldsymbol{R}^7\right]=$$

$$\frac{V_7 R}{4(2\pi)^7\alpha'^4}\int_S \frac{d\tau^2}{\tau_2^5}\frac{\sqrt{\alpha'\tau_2}}{NR}\sum_{\gamma\in\mathbb{Z}}\sum_{\beta=0}^{N-1} e^{-\frac{\pi\alpha'\tau_2}{R^2N^2}\gamma^2}\cdot e^{2\pi i\frac{\beta\gamma}{N}}\frac{|\theta_1(\beta/N|\tau)|^8}{|\eta(\tau)|^{18}\cdot|\theta_1(2\beta/N|\tau)|^2}$$

We evaluate $f(\tau)=\dfrac{\theta_1(\frac{\beta}{N}|\tau)^4}{\eta(\tau)^9\theta_1(\frac{2\beta}{N}|\tau)}=\displaystyle\sum_{n=0}^{\infty} d_n e^{2\pi i\tau n}$

like Cardy formula, we find $d_n \sim (\dfrac{\beta}{N})^{\frac{7}{4}} n^{-\frac{9}{4}} e^{\sqrt{4\pi\frac{\beta n}{N}}}$
$(n >> 1, 0 < \beta/N < 1/2)$

Then, $\displaystyle\int_{-1/2}^{1/2} d\tau_1 |f(\tau)|^2=\sum_n (d_n)^2 e^{-4\pi\tau_2 n}\sim\sqrt{\frac{N}{\beta}}(\tau_2)^{\frac{15}{2}}\cdot e^{4\pi\frac{\beta}{N\tau_2}}$

- β summation

$$\frac{\partial}{\partial N}\sum_{\beta=1}^{N-1}\left(\sqrt{\frac{N}{\beta}}e^{2\pi i\frac{\beta\gamma}{N}}e^{\frac{4\pi\beta}{N\tau_2}}\right)\Big|_{N=1}\simeq\frac{32\pi}{3}\tau_2^{-3/2}$$

$$\tilde{S}_A \sim \frac{V_7}{\alpha'^{\frac{7}{2}}} \int_0^{\tau_{max}} d\tau_2 (\tau_2)^{\frac{3}{2}} \sum_{\gamma \in \mathbb{Z}} e^{-\frac{\pi R_{orb}^2 \tau_2 \gamma^2}{\alpha'}}$$

Finally UV contribution is evaluated as

$$[\tilde{S}_A]_{UV} \simeq \frac{V_7}{\alpha'^{\frac{7}{2}}} (s_1 + s_2 \frac{\alpha'^{\frac{5}{2}}}{R_{orb}^5})$$

<u>Combine UV + IR</u>

Total twisted conical entropy is estimated as

$$\tilde{S}_A(R_{orb}) = \frac{V_7}{\alpha'^{\frac{7}{2}}} \cdot \tilde{S}(\frac{R_{orb}}{\sqrt{\alpha'}})$$

taking orbifold limit $R_{orb} \to \infty$

$$\tilde{S}_A(R_{orb}) \simeq \tilde{s} \cdot \frac{V_7}{\alpha'^{\frac{7}{2}}}$$

no area law term $V_7 R_{orb}$

suggest actual conical entropy vanish

[cf Susskind Uglum]

Conclusion

- Orbifold calculation enables to calculate the EE for general Higher spin fields

- Conical entropy seems to be finite and more strongly becomes 0

- Our results suggest that actual conical entropy vanish and support Susskind Uglum conjecture

  (non renormalization of Newton constant)

Outlook

- explicit evaluation of orbifold part.func.
- Relation with EE in 2d string ?[Hartnoll-Mazenc 15]