Quantum Dimension as Entanglement Entropy in 2D RCFTs

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- S.H, Tokiro Numasawa, Tadashi Takayanagi, Kento Watanabe, Phys. Rev. D90, 041701(Rapid Comm.) (2014).
- Wuzhong Guo, S.H. JHEP 1504 (2015) 099 .
- Bin Chen, Wuzhong Guo, S.H., Jie-Qiang Wu, arXiv:1507.01157.

Outline

- Background of Entanglement Entropy (EE).
- Setup of Entanglement Entropy (EE).
- EE and Quantum dimension in CFT.
- EE and Quantum dimension in BCFT.
- EE for Descendent Local Operators.
- Summary.

Basics of EE

- EE is a useful measure of the degrees of freedom in quantum many body systems.
  1. Using EE to detect the central charge (the coefficient of logarithmic divergent term in Odd dimesion)[C. Holzhey, F. Larsen and F. Wilczek, 94][P. Calabrese and J. L. Cardy, 04][S. Ryu and T. Takayanagi,06][...].
  2. Detecting the topological degrees of freedom of topological field theories (finite piece of EE)[A. Kitaev and J. Preskill,05][M. Levin and X.G.Wen,05].
  3. Measuring the degrees of freedom of local operators (Quantum dimension).[M. Nozaki, T. Numasawa and T. Takayanagi,14][S. He, T. Numasawa, T. Takayanagi and K. Watanabe,14][P. Capta, M. Nozaki and T. Takayanagi, 14][M. Nozaki,14][Wu-Zhong Guo, S. He,15][Masahiro Nozaki, Tokiro Numasawa, Shunji Matsuura,15]...
- ...

Basics of EE

- General diagnostic: divide quantum system into two parts (A and B) and use entropy as measure of correlations between subsystems

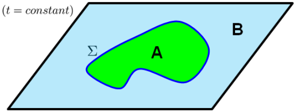


Figure: This figure is learnt from Prof. Rob Myers talk in String conference 2013 and also today.

- Integrate out degrees of freedom in outside region (B). Remaining dof are described by a density matrix ρ_A.

$$S_A = -\mathrm{Tr}_A \rho_A \log \rho_A \tag{1}$$

Replica to calculate EE in QFT

- How to calculate EE in quantum system.
- A basic method of calculating EE in QFTs is so called the replica method.

$$S_A = -\frac{\partial \mathrm{Tr}(\rho_A)^n}{\partial n}|_{n=1} = \lim_{n\to 1} S_A^n$$

- The relation provides a practical way to compute EE in field theory, although it is difficult.

Motivation from 'First Law'

- First law of thermodynamics: $TdS = dE$. In a generic quantum system which are far from the equilibrium, can we find the analogous relation between the EE (information) and energy of A:

$$T_{ent} dS_A = dE_A \quad ?$$

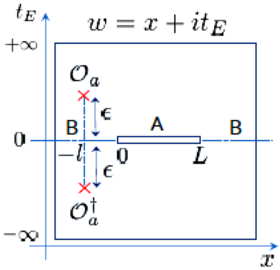
- The first study in field theory in [F. C. Alcaraz, M. I. Berganza, G. Sierra, PRL 106, 201601]
- First holographic studied in [Jyotirmoy Bhattacharya, Masahiro Nozaki, Tadashi Takayanagi, Tomonori Ugajin, PRL 110, 091602]
- More general studies given by [Nozaki,Numasawa,Prudenziati,Tadashi Takayanagi 13],[Wu-zhong Guo, S.H, Jun Tao, 13](Higher derivative gravity)[S.H, Danning Li, Jun-Bao Wu, 13](Non-conformal cases with full backreaction) [Bhattacharya,Tadashi Takayanagi, 13]...

Motivation from 'First Law'

- More further Progresses: [Blanco-Casini-Hung-Myers 13,Wong-Klich-Pando Zayas-Vaman 13, Nozaki-Numasawa-Prudenziati-TT 13, Bhattacharaya-TT 13, Lashkari-McDermott-Raamsdonk 13, Faulkner-Guica-Hartman-Myers-Raamsdonk 13]...
- In this talk, we will focus on EE of the excited states with large size limit and also finite variation of EE.

Our main motivation

- In this talk, our setup in 1+1 dimension space time denoted by w



- Focus on the excess of REE of $\Delta S_A^{(n)}$ is

$$\Delta S_A^{(n)} = S_A^{(n)}[|\Psi\rangle] - S_A^{(n)}[|0\rangle], \tag{2}$$

where $S_A^{(n)}[|\Psi\rangle]$ denote the n-th REE for the state.

EE for Excited State

- For $|\Psi(t)\rangle = e^{-itH-\epsilon H}O(-l)|0\rangle$,

$$S^{(n)}[|\Psi(t)\rangle] = \frac{1}{1-n}\log\left[\frac{\int \phi O^{+}(x_1)O(x_2)...O^{+}(x_{2n-1})O(x_{2n})e^{-S}}{(\int \phi O^{+}(x_1)O(x_2)e^{-S})^n}\right] \tag{3}$$

- The excess of REE $\Delta S_A^{(n)}$ between excited state and vacuum

$$\begin{aligned}\Delta S_A^{(n)} &= \frac{1}{1-n}\Big[\log\left\langle \mathcal{O}_a^{\dagger}(w_l,\bar{w}_1)\mathcal{O}_a(w_2,\bar{w}_2)..\mathcal{O}_a(w_{2n},\bar{w}_{2n})\right\rangle_{\Sigma_n} \\ &\quad -n\log\left\langle \mathcal{O}_a^{\dagger}(w_l,\bar{w}_1)\mathcal{O}_a(w_2,\bar{w}_2)\right\rangle_{\Sigma_1}\Big],\end{aligned} \tag{4}$$

- Especially, we are interested in two different time evolution region, $t < l$ and $t > L$ (earlier time) and $L > t \gg l$ (late time).

Earlier time and Late time

- Using conformal map $z^n = \frac{\omega+l}{\omega-l}$ to obtain n copies of original manifold.
- The behavior of $(z, \bar{z})$ in the limit $\epsilon \to 0$. When $0 < t < l$ or $t > L + l$, we find $(z, \bar{z}) \to (0, 0)$:
- In the late time limit $l < t < L + l$, we find $(z, \bar{z}) \to (1, 0)$:

$$z \simeq 1 - \frac{L^2\epsilon^2}{4(l-t)^2(L+l-t)^2}, \quad \bar{z} \simeq \frac{L^2\epsilon^2}{4(l+t)^2(L+l+t)^2}.$$

Though this limit $(z, \bar{z}) \to (1, 0)$ does not seem to respect the complex conjugate due to analytical continuation of t.

EE in Ising model

- Spin operator $O_{2,2}$ in Ising model whose conformal dimension is

$$\Delta_{2,2} = \frac{3}{4m(m+1)}\Big|_{m=3} = \frac{1}{16}. \tag{5}$$

- For Ising model, the Green function

$$G(z,\bar{z}) = \frac{1}{\sqrt{2}}\sqrt{\sqrt{\frac{|z|}{|1-z|}} + \frac{1}{\sqrt{|z||1-z|}} + \sqrt{\frac{|1-z|}{|z|}}}. \tag{6}$$

In late time limit, (4)

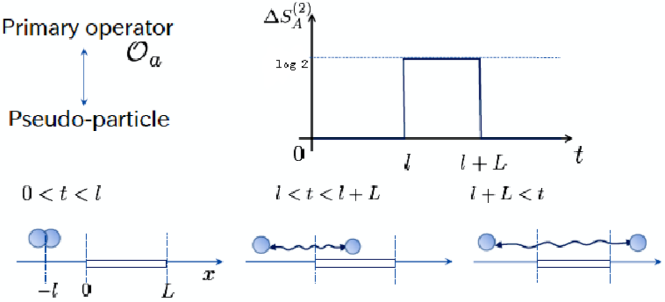
$$\Delta S_A^{(2)} = \log\sqrt{2}. \tag{7}$$

EE in Ising model

- Through very very highly nontrivial calculation, we can show that $\Delta S_A^{(2)} = \Delta S_A^{(3)} = \Delta S_A^{(4)} = ... = \log \sqrt{2}$.
- So it is nature to ask What is the meaning of $\sqrt{2}$.
- An: The $\sqrt{2}$ is exact quantum dimension of spin operator σ in Ising model.
- In 2D RCFTs, the quantum dimension is associated with fusion matrix.

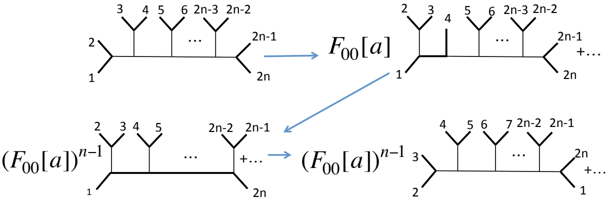
EE in Ising model

Causality argument

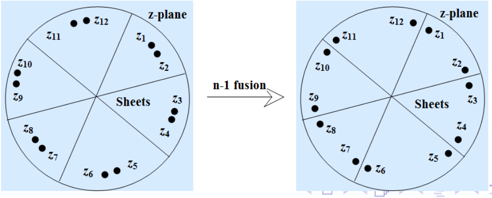


Graph proof

- One can generalize this procedure to arbitrary n to show $\Delta S_{EE}^{(n)}(A) = \log d_Q$. The following procedure can be described by the above cartoon.



- One alternative way to understand this procedure.



Graph proof

- If we just repeat $n-1$ times the fusion transformation. Thus we obtain

$$\langle \mathcal{O}_a(z_1,\bar{z}_1)\mathcal{O}_a(z_2,\bar{z}_2)\ldots\mathcal{O}_a(z_{2n},\bar{z}_{2n})\rangle_{\Sigma_1}$$
$$\simeq (F_{00}[a])^{n-1}\cdot\left[\prod_{k=0}^{n-1}(z_{2k+1}-z_{2k})(\bar{z}_{2k+1}-\bar{z}_{2k+2})\right]^{-2\Delta_a}.$$

Finally, the ratio at late time limit is computed to be $(F_{00}[a])^{n-1}=(d_a)^{1-n}$ And $\Delta S^{(n)}(A)=\frac{1}{1-n}\log(d_a)^{1-n}$.

BCFT

- Two different boundary. $\frac{\partial\phi}{\partial n}|_B = 0$ called Neumann boundary condition and the other is $\phi|_B = 0$ called Dirichlet boundary condition.
- The boundary conditions will affect the Bulk to Bulk Green function in terms of image method.

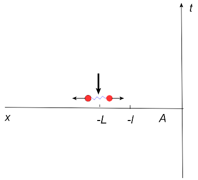


Figure: This figure is to show our setup in two dimensional left half plane $\omega = x + it$ with a boundary $x = 0$. The system will be triggered at $x = -L$ and there are left- and right-moving quasi-particle at $t = 0$.

BCFT

- Two important map. n copies

$$z^n = \frac{\omega + l}{\omega - l} \tag{8}$$

- Map disc to the upper half plane (UHP) $t \geqslant 0$

$$\xi = -i\frac{z+1}{z-1}. \tag{9}$$

- Using the image method in Ising model

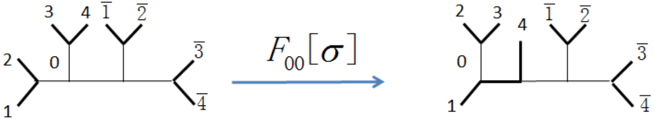


Figure: Where i and $\bar{i}$ presented in the figure stands for the position z_i and $\bar{z}_i$ respectively. $F_{00}[\sigma]$ denotes the fusion transformation and 0 corresponds to the identity operator.

BCFT

- The time evolution in BCFT

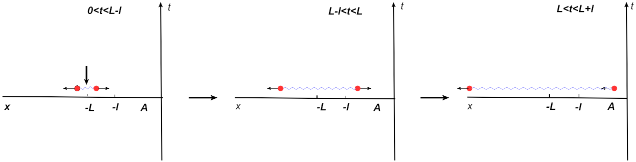


Figure: Where i and $\bar{i}$ presented in the figure stands for the position z_i and $\bar{z}_i$ respectively. $F_{00}[\sigma]$ denotes the fusion transformation and 0 corresponds to the identity operator.

Decendent operators

- For generic descendent operator

$$V = \sum_{m,j,r,k} d_{m,j;r,k} (\partial^m L^{(-,j)})(\bar{\partial}^r \bar{L}^{(-,k)}) O_a(w, \bar{w}) \tag{10}$$

where $L^{(-,j)}$ ($\bar{L}^{(-,k)}$) is a product of Virasoro algebra.

$$[L_0, L^{(-,j)}] = p_j L^{(-,j)}, \quad m + r + p_j + \bar{p}_k + 2h_a = \Delta \tag{11}$$

!! Survive.

- Then

$$\Delta S_n = \log d_a - \frac{1}{n-1} \log tr\rho_0^n = \Delta S_n^{\text{primary}} - \frac{1}{n-1} \log tr\rho_0^n, \tag{12}$$

with the normalized density matrix $\rho_0 = \frac{\rho}{tr\rho}$

Descendent operators

- Then

$$\Delta S_n = \log d_a - \frac{1}{n-1} \log tr\rho_0^n = \Delta S_n^{\text{primary}} - \frac{1}{n-1} \log tr\rho_0^n, \tag{13}$$

with the normalized density matrix $\rho_0 = \frac{\rho}{tr\rho}$

- The density matrix

$$\rho = BMB^\dagger M^\dagger. \tag{14}$$

With

$$B_{\{m,j\},\{r,k\}} = d^*_{m,j,r,k}, \tag{15}$$

$$M_{\{m,j\},\{r,k\}} = \langle h \mid L^{(-,j)\dagger} L^{(-,j)} \mid h\rangle \delta_{j,k} i^{r-m}, \tag{16}$$

Comments and Checking

- For
$$V(w,\bar{w}) = \mathcal{L}^{(-)}\bar{\mathcal{L}}^{(-)}O(w,\bar{w}), \tag{17}$$
There is NO additional contribution.
- $\sum d_{\{n_i\}\{n_j\}}(\prod_i L_{-n_i} \prod_j \bar{L}_{-n_j})$, There is additional contribution.
- For example,
$$V = (\partial + \bar{\partial})O(w,\bar{w}) \tag{18}$$
additional contribution $\log 2$ Which is consistent with free field theory (quantum dimension of free field=1).

Summary

1. Both REE ($n \geq 2$) and EE ($n = 1$) for primary operator excitations are log of quantum dimension of primary operator in late time limit.
2. A product of primary operators $\prod_a (\mathcal{O}_a)^{n_a}$, we obtain $\Delta S_A^{(n)} = \sum_a n_a \log d_a$, using the sum rule in [Nozaki,14]. The quantum dimension d_a satisfies $d_a d_b = \sum_c N_{ab}^c d_c$.
3. The maximal value of EE in BCFT is also quantum dimension.
4. For generic Descendent states, the maximal value of EE is quantum dimension plus Normalization contributions.

Thanks for your attention!

What is quantum dimension

- Here we just list the standard alternative definition of quantum dimension in Minimal model.

$\boxed{\text{Quantum Dimension } d_a \ ??}$

2D CFT

$d_a \longrightarrow$ Maximal eigenvalue of N_{ab}^c

$$\mathcal{O}_a \cdot \mathcal{O}_b = \sum_c N_{ab}^c \, \mathcal{O}_c \quad \text{: Fusion rule}$$

\# of the primary fields in $\overbrace{\mathcal{O}_a \cdots \mathcal{O}_a}^{N} = \sum_c (N_a \cdots N_a)_a^c \, \mathcal{O}_c \quad \sim (d_a)^N$

$\longrightarrow M_N \quad (N \to \infty)$

$$\longrightarrow \log d_a = \lim_{N\to\infty} \frac{\log M_N}{N}$$

$\Rightarrow$ $\boxed{\text{Quantum Dimension } d_a = \text{“ The effective d.o.f. of } \mathcal{O}_a \text{”}}$

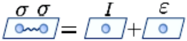
What is quantum dimension

- Especially in Ising model, the quantum dimension of spin operator σ.

$$\boxed{\text{Quantum Dimension } d_a = \text{“ The effective d.o.f. of } \mathcal{O}_a \text{”}}$$

Ising model

$$\mathcal{O}_a = \{I, \sigma, \varepsilon\} \quad \begin{cases} \varepsilon \cdot \varepsilon = I \\ \sigma \cdot \sigma = I + \varepsilon \\ \varepsilon \cdot \sigma = \sigma \end{cases}$$



$$\sigma^{2N} = (I+\varepsilon)^N = 2^{N-1} I + 2^{N-1} \varepsilon \longrightarrow \log d_\sigma = \lim_{N\to\infty} \frac{\log 2^N}{2N} = \log\sqrt{2}$$

$$\longrightarrow d_\sigma = \sqrt{2} \qquad \text{Similarly} \quad d_I = d_\varepsilon = 1$$

- Comment: In the Ising model, the identity I, the spin σ and the energy operator ψ.
  1. $\Delta S_A^{(n)}$ is always vanishing for I and ψ, due to quantum dimension 1.
  2. $\Delta S_A^{(n)} = \log\sqrt{2}$ for any n as $d_\sigma = \sqrt{2}$