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Tensor Networks and Holography (Review Talk)

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Contents

Refer also to Matsueda's talk

Our Original Contributions

[1] arXiv:1208.3469 (JHEP 1210 (2012) 193)
with Masahiro Nozaki (YITP, Kyoto)
and Shinsei Ryu (Illinois, Urbana–Champaign)

[2] arXiv:1412.6226 (JHEP 1505 (2015) 152)
with Masamichi Miyaji (YITP, Kyoto),
Shinsei Ryu and Xueda Wen (Illinois, Urbana–Champaign).

[3] arXiv:1503.08161 (PTEP 2015 (2015) 7)
with Masamichi Miyaji (YITP, Kyoto)

[4] arXiv:1506.01353 with Masamichi Miyaji (YITP, Kyoto),
Tokiro Numasawa (YITP, Kyoto), Noburo Shiba (YITP, Kyoto),
and Kento Watanabe (YITP, Kyoto)

① Introduction

The AdS/CFT and more generally holography argues

``Quantum Gravity = Quantum Many-body Systems''.

on $\mathbf{M_{d+2}}$ on $\mathbf{\partial M_{d+1}}$

Then one may ask what are the most *elementary degrees of freedom* in the correspondence ?

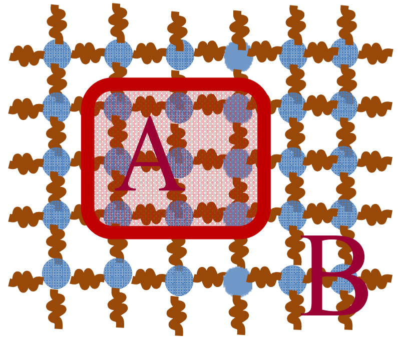
⇒ One attractive possibility is that

quantum entanglement explains the degrees of freedom.

``Emergence of spacetime from Qubits''

⇔ Tensor Networks

Entanglement Entropy (EE)



$$H_{tot} = H_A \otimes H_B, \qquad \rho_{tot} = |\Psi\rangle\langle\Psi|,$$

$$\rho_A \equiv \mathrm{Tr}_B[\rho_{tot}] \rightarrow S_A = -\mathrm{Tr}[\rho_A \log \rho_\mathrm{A}].$$

$\Rightarrow$ The best measure of quantum entanglement

EE measures

(1) How many EPR pairs can be extracted.

$$\left(|\uparrow\rangle|\downarrow\rangle \pm |\downarrow\rangle|\uparrow\rangle\right)/\sqrt{2}$$

(2) Active degrees of freedom. (~central charges)

(3) A quantum order parameter.(~topological order)

(4) a ***geometry*** of quantum many-body system.

(~holography)

(i) Area law of EE [Bombelli-Koul-Lee-Sorkin 86, Srednicki 93]

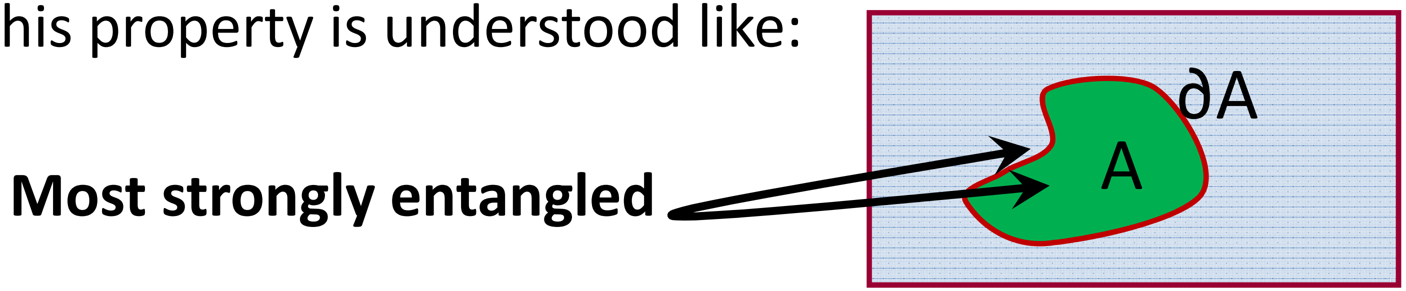
EE in QFTs includes UV divergences.

In a d+1 dim. QFT (d>1) with a UV relativistic fixed point, the leading term of EE at its ground state behaves like

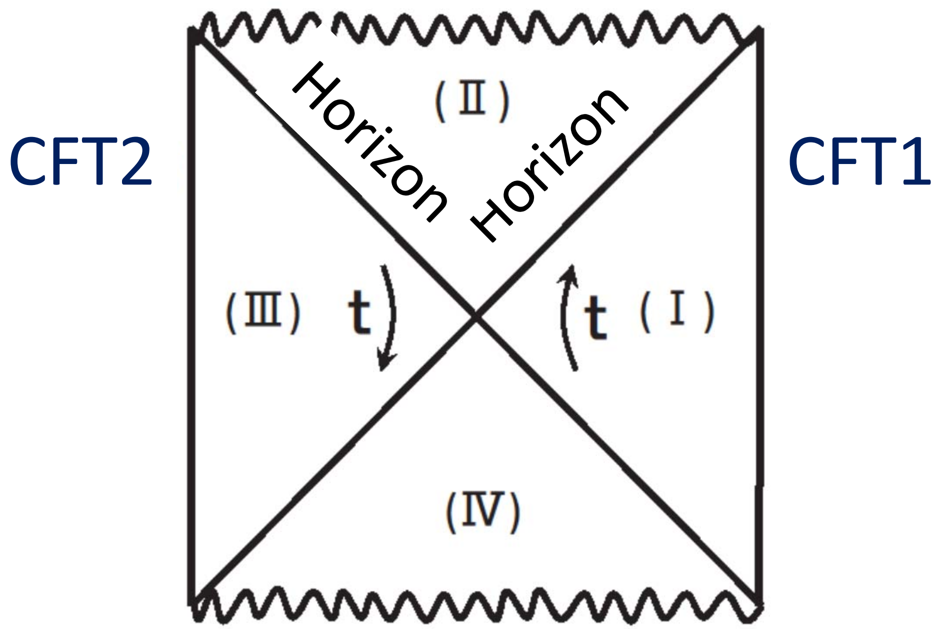
$$S_A \sim \frac{\mathrm{Area}(\partial A)}{\varepsilon^{d-1}} + (\text{subleading terms}),$$

where ε is a UV cutoff (i.e. lattice spacing). [d=1: log div.]

Intuitively, this property is understood like:



(ii) Eternal BH in AdS/CFT [Maldacena 00]



$$|\Psi\rangle = \frac{1}{Z(\beta)}\sum_n e^{-\beta E_n/2}|n\rangle_1|n\rangle_2 \quad , \quad Z(\beta)=\sum_n e^{-\beta E_n},$$

$$\rho_1 = \mathrm{Tr}_2[|\Psi\rangle\langle\Psi|] = \frac{1}{Z(\beta)}\sum_n e^{-\beta E_n}|n\rangle_1\langle n|_1 \quad ,$$

$$S_{ent} = -\mathrm{Tr}[\rho_1 \log \rho_1] = S_{thermal} = S_{BH} = \frac{\mathrm{Area(Horizon)}}{4G_N}.$$

(iii) Holographic Entanglement Entropy (HEE)

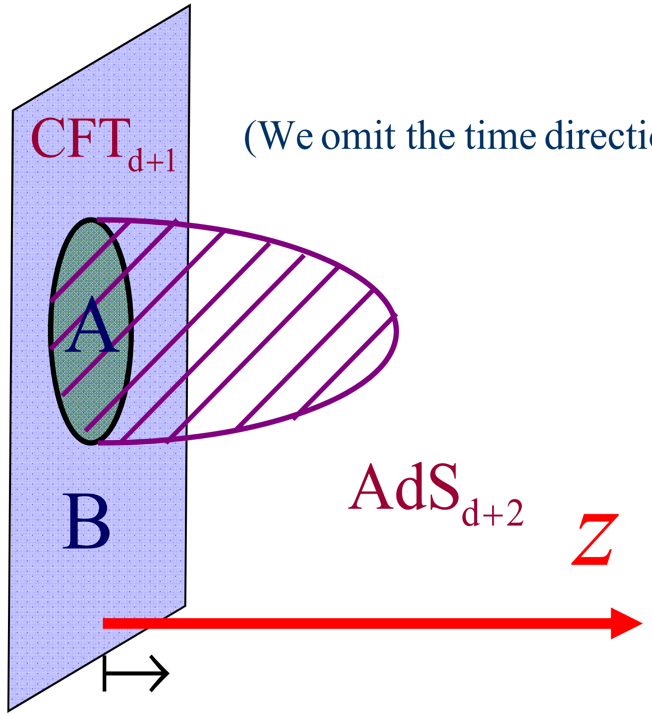
[Ryu-TT 06; a derivation: Casini-Huerta-Myers 11, Lewkowycz-Maldacena 13]

$$\boxed{S_A = \underset{\substack{\partial\gamma_A = \partial A \\ \gamma_A \approx A}}{\mathrm{Min}} \left[\frac{\mathrm{Area}(\gamma_A)}{4G_N} \right]}$$

γ_A is the minimal area surface (codim.=2) such that

$$\partial A = \partial \gamma_A \ \text{ and } \ A \sim \gamma_A \ .$$

homologous



$$z > \varepsilon \quad \text{(UV cut off)}$$

$$ds^2 = R^2 \cdot \frac{dz^2 - dt^2 + \sum_{i=1}^{d} dx_i^2}{z^2}$$

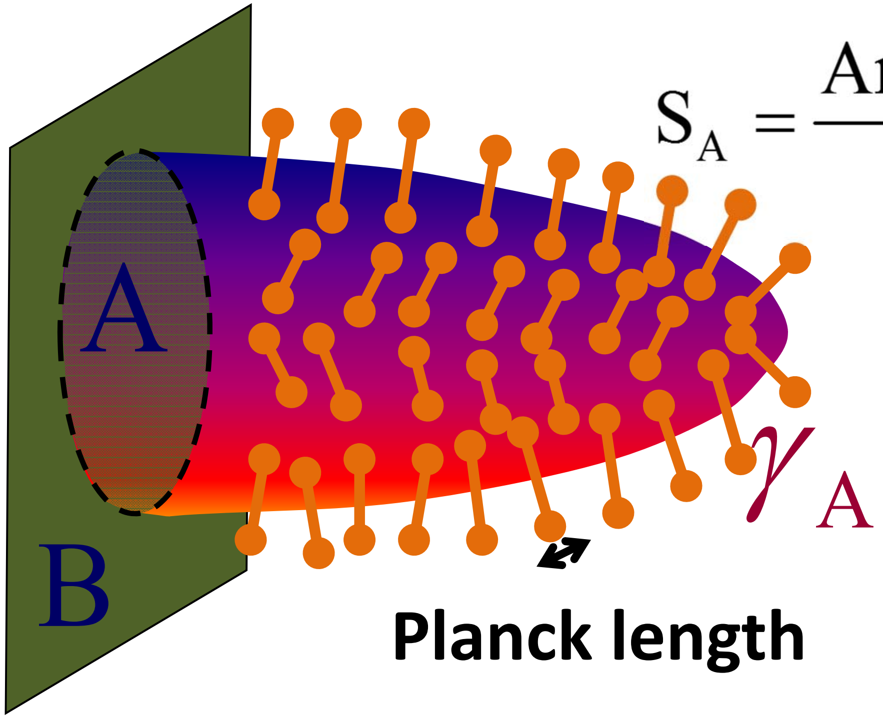
Note: In time-dependent spacetimes, we need to take extremal surfaces.

[Hubeny-Rangamani-TT 07]

The HEE suggests the following novel interpretation:

``A spacetime in gravity

= Collections of bits of quantum entanglement''



$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N} \approx \frac{\mathrm{Area}(\gamma_A)}{l_{pl}^2}.$$

This entanglement/geometry duality may work for any surfaces [Bianchi-Myers 12]

⇒ Manifestly realized in the proposed connection between AdS/CFT and tensor networks ! [Swingle 09]

② Review of Tensor Networks

(2-1) Tensor Network [See e.g. Cirac-Verstraete 09(review)]

Tensor network states

= Efficient variational ansatz for the ground state wave functions in quantum many-body systems.

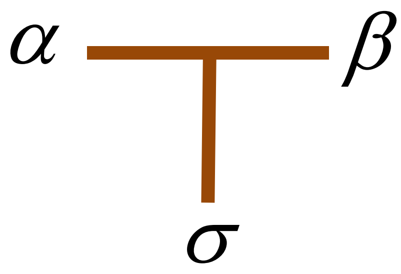
[A tensor network diagram = A wave function]

⇒ An ansatz should respect the correct **quantum entanglement** of ground state.

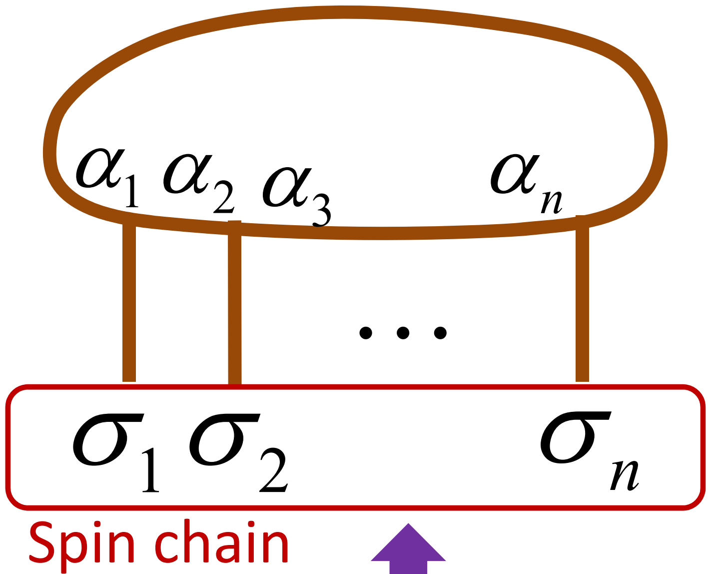
~Geometry of Tensor Network

Ex. Matrix Product State (MPS)

[DMRG: White 92,...,
Rommer-Ostlund 95,..]



$$M_{\alpha\beta}(\sigma)$$



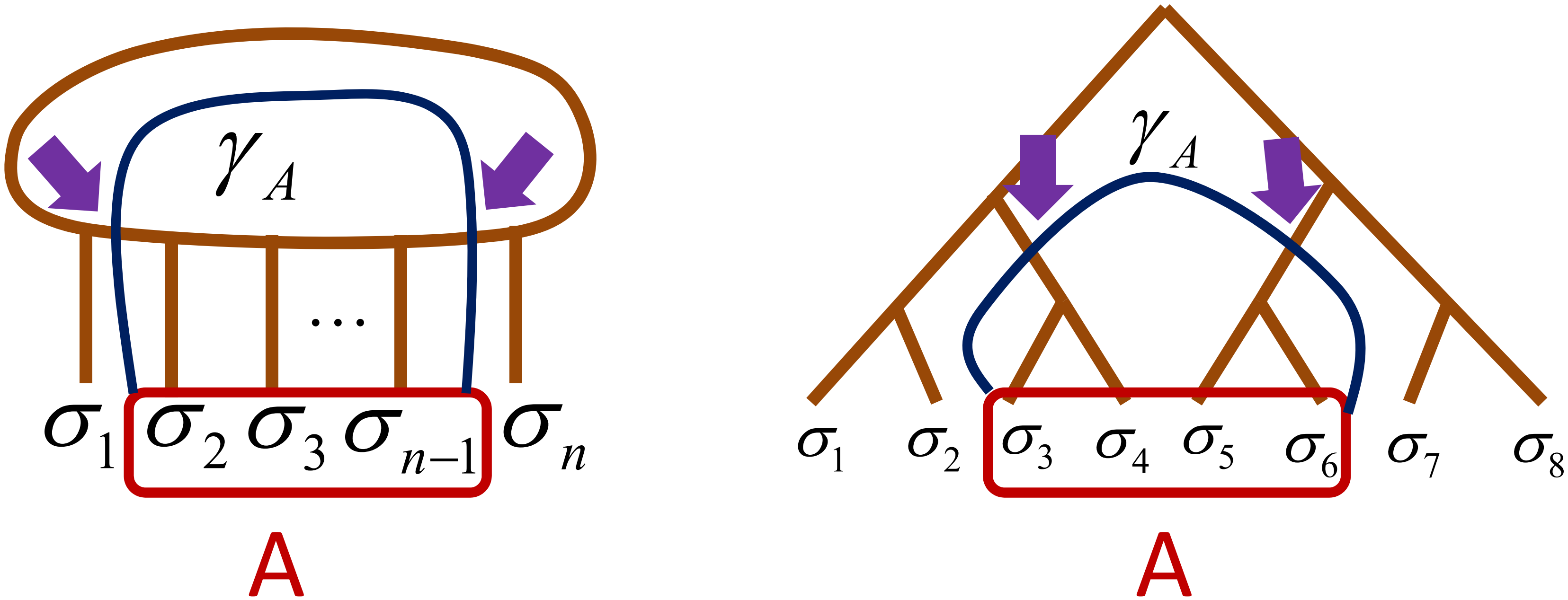
$$\alpha_i = 1, 2, \ldots, \chi,$$

$$\sigma_i = \uparrow \text{ or } \downarrow .$$

$$\left|\Psi\right\rangle = \sum_{\sigma_1, \sigma_2, \cdots, \sigma_n} \mathrm{Tr}[M(\sigma_1) M(\sigma_2) \cdots M(\sigma_n)] \, \underbrace{\left|\sigma_1, \sigma_2, \cdots, \sigma_n\right\rangle}_{n \;\; Spins}$$

MPS with finite χ does not have enough EE to describe 1d quantum critical points (2d CFTs) :

$$S_A \leq 2\log\chi \quad (<< \log L \sim S_A^{CFT}).$$



In general,

$$\boxed{\begin{aligned} &S_A \leq N_{\text{int}} \cdot \log\chi, \\ &N_{\text{int}} \equiv \min[\#\text{Intersections of } \gamma_A]. \end{aligned}}$$

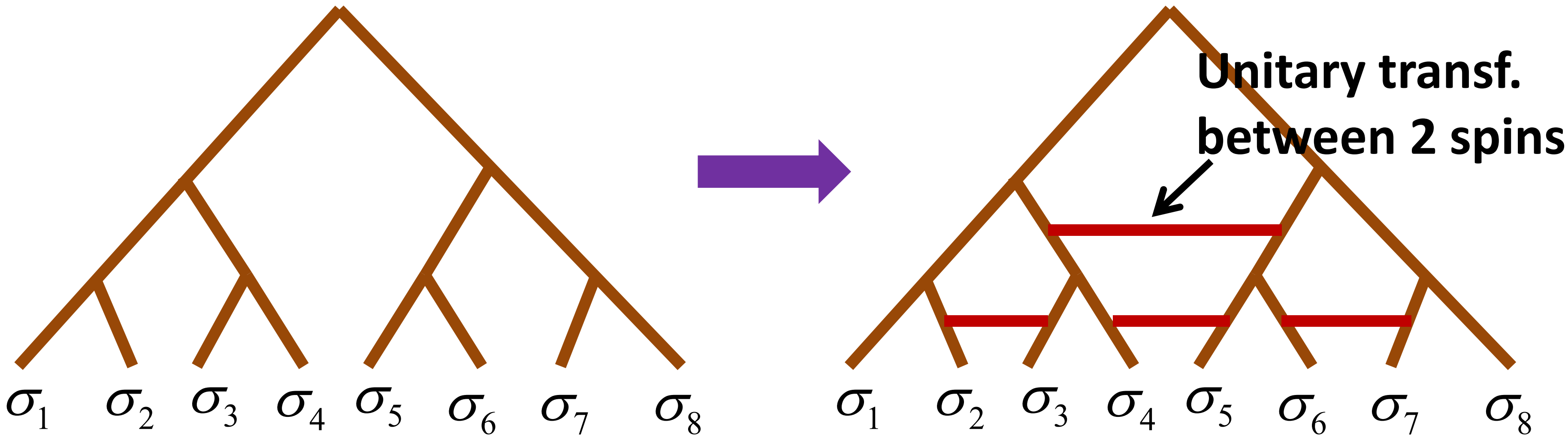
(2-2) MERA

<u>MERA</u> (Multiscale Entanglement Renormalization Ansatz):

$\Rightarrow$ An efficient variational ansatz for CFT ground states.

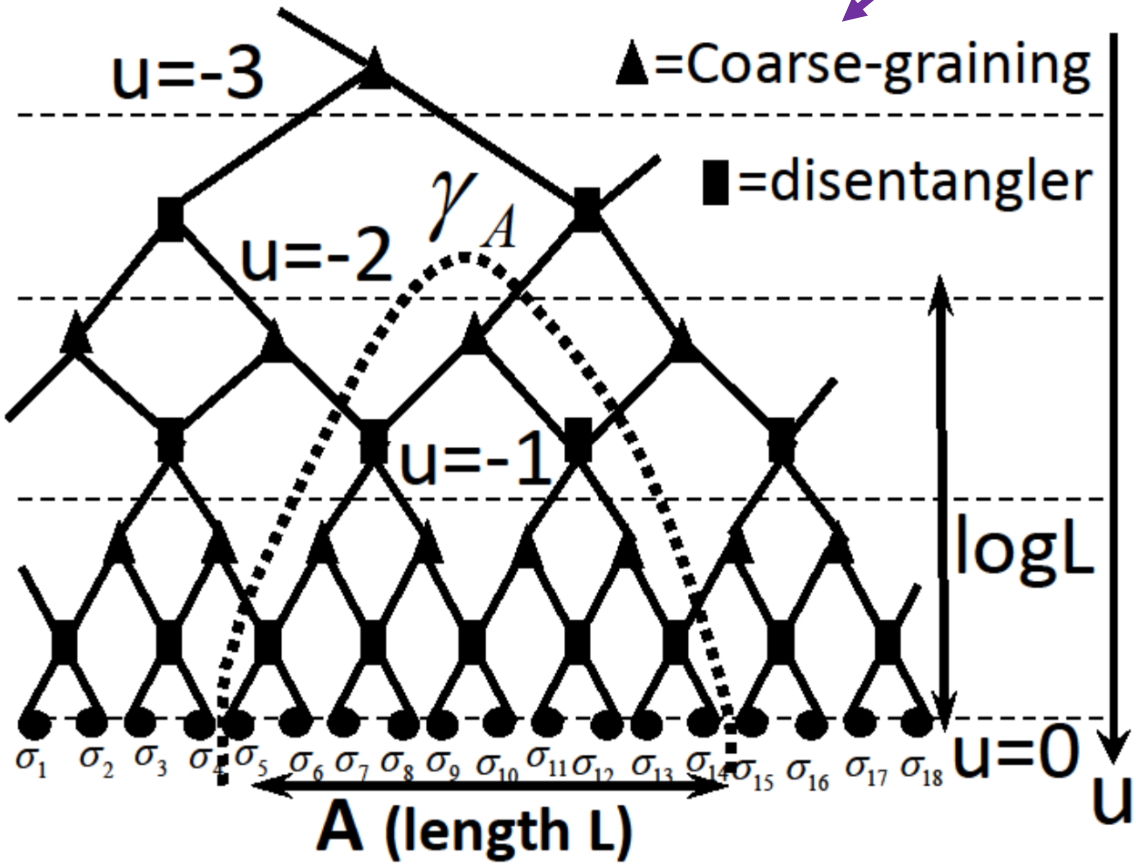
[Vidal 05]

To increase entanglement in a CFT, we add (dis)entanglers.

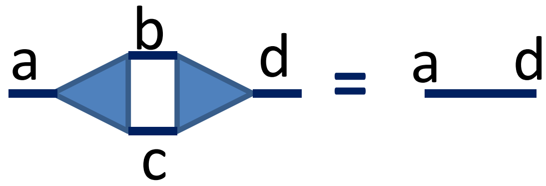


MERA in a ``more official'' diagram

Coarse-graining = Isometry



$$[T]^{\dagger}_{abc}[T]_{bcd} = \delta_{ad}$$



$$S_A \propto \mathrm{Min}[\#\mathrm{links}]$$
$$\propto \log L$$

$\Rightarrow$ agrees with results in 2d CFT!

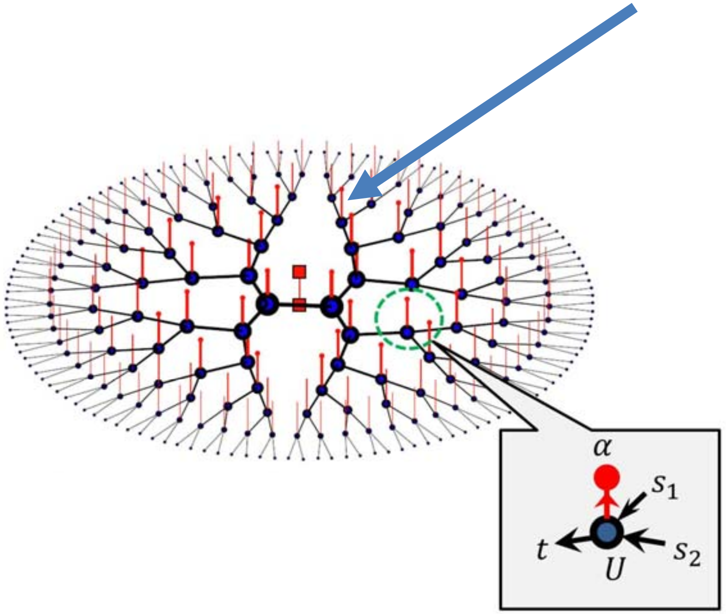
Cf. Bulk State construction

⇒ Low energy bulk excitations (e.g. bulk scalars)

[Exact holographic mapping: Xiao-liang Qi 13]

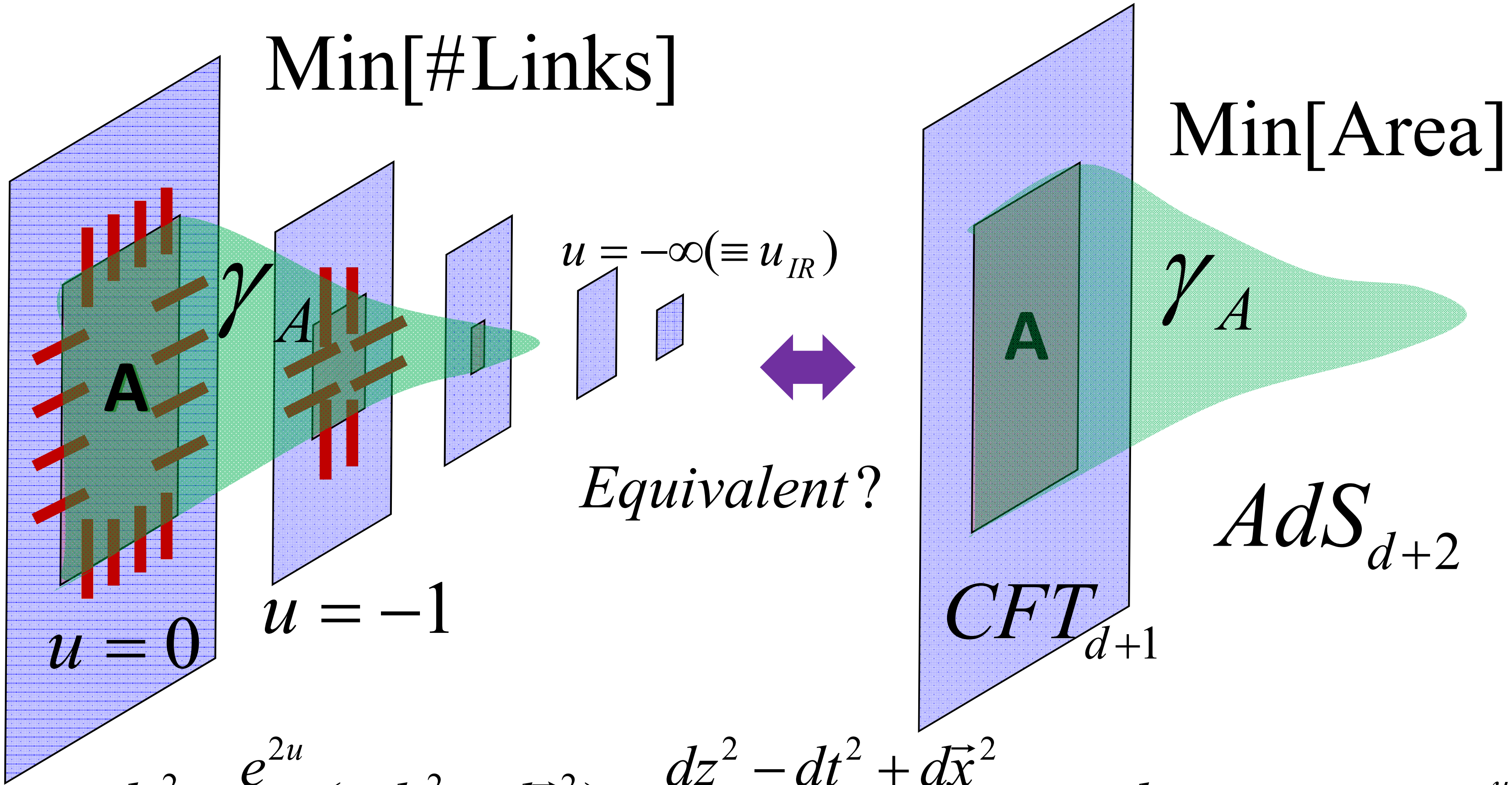
Idea: Keep the network and introduce bulk states

[tips of red vertical bonds]



③ Tensor Networks and AdS/CFT

(3-1) AdS/MERA proposal [Swingle 09]



$$ds^2 + \frac{e^{2u}}{\varepsilon^2}(-dt^2 + d\vec{x}^2) = \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2}, \quad where \quad z = \varepsilon \cdot e^{-u} \quad .$$

The idea: Tensor Network of MERA (∃ scale inv.)
= a time slice of AdS space

Qualitative evidences

(i) Real space RG = radial evolution in AdS
→ something we usually expect in AdS/CFT.

(ii) The bound of EE in MERA: $S_A \leq \mathrm{Min}[N_{Int}] \cdot \log\chi.$

If this is saturated, it seems to agree with the HEE

$$S_A = \frac{1}{4G_N} \underset{\gamma_A}{\mathrm{Min}} \left[\mathrm{Area}(\gamma_A) \right].$$

In this way, AdS/MERA seems to work qualitatively.

However, if we look more closely, several questions arise

[see e.g. Bao-Cao-Carroll-Chatwin-Davies-Hunter-Jones-Pollack-Remmen 15]

(a) **Locality of bulk AdS**: The disentangler carries $O(N^2)$ Qubits.

$\Rightarrow$ $\exists$ Locality only at the AdS radius (not Planck scale)....

(b) **Conformal invariance is not clear** $\Rightarrow$ Lattice artifact ?

(c) **Why the EE bound is saturated ?**

$\Rightarrow$ If disentanglers ~ large N Random tensors, that can be true. But this is not clear.

(e) **$\exists$ RG causal structure in MERA**

$\Rightarrow$ No causal structure on the time slice of AdS ?

(3-2) Refinement 1: Integral geometry

[Czech, Lamprou, McCandlish, Sully 15]

Focus on AdS3/CFT2. Keep MERA network as it is.

A metric for MERA: EE for the interval A=[x-L,x+L]

$$S_A(x-L, x+L) = \frac{c}{3}\log 2L,$$

$$\Rightarrow \quad ds^2 = \frac{\partial^2 S_A(u,v)}{\partial u \partial v} du dv = \frac{c}{12}\cdot\frac{-dL^2 + dx^2}{L^2}$$

$$\rightarrow 2d \ \ \text{deSitter space}$$

(i.e. MERA=dS≠AdS)

[⇒Rob Myers's talk on Thu.]

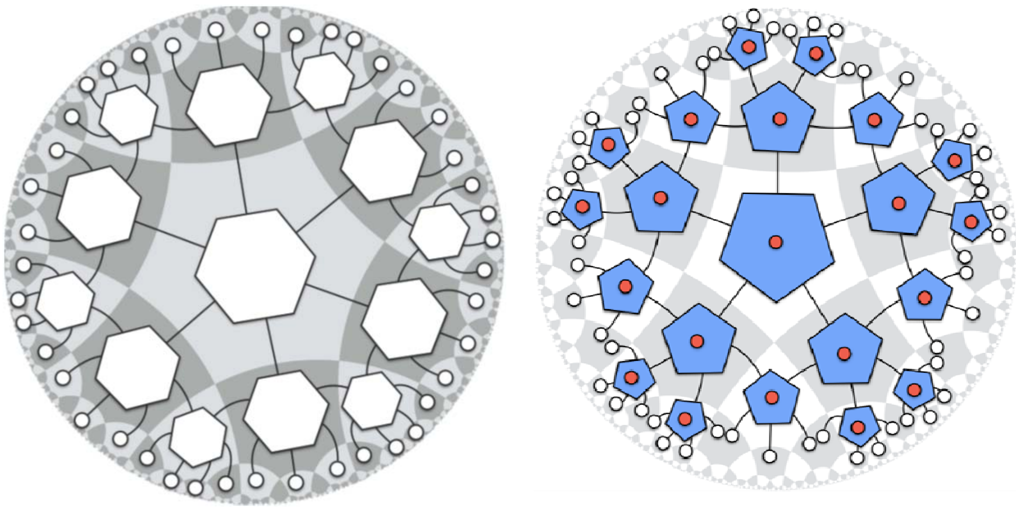
(3-3) Refinement 2: Perfect Tensor Network

[Pastawski-Yoshida-Harlow-Preskill 15]

Consider tensors which are unitary or isometry w.r.t any legs, called *perfect tensors*.

[Bulk reconstruction ⇔ quantum error corrections]

[Almheiri-Dong-Harlow 14]



[i.e. PTN=AdS]

⇒   The EE bound is saturated for single intervals.

④ cMERA and AdS/CFT [Refinement 3]

[Haegeman-Osborne-Verschelde-Verstraete 11, see also Nozaki-Ryu-TT 12]

(4-1) cMERA formulation

To remove lattice artifacts, take a continuum limit of MERA:

$$\underbrace{|\Phi(u)\rangle}_{\text{State at scale u}} = P\cdot\exp\left(-i\int_{u_{IR}}^{u} ds\,\hat{K}(s)\right)\cdot \underbrace{|\Omega\rangle}_{\text{IR state}}.$$

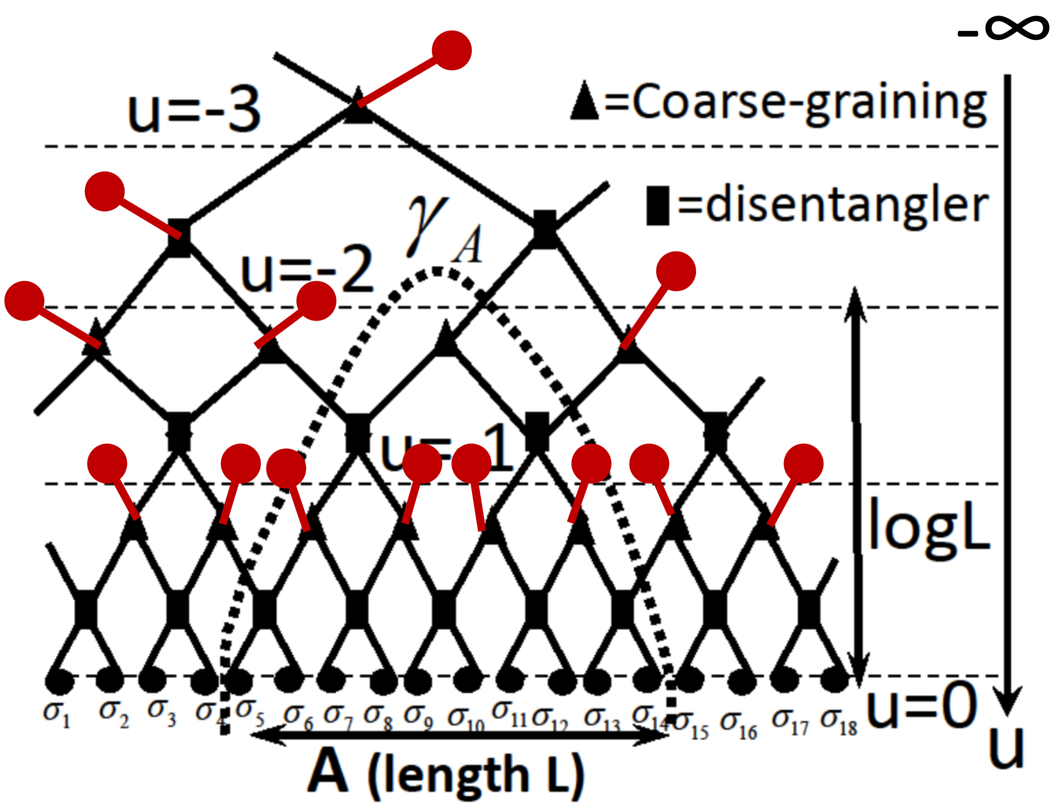
$$u_{IR} = -\infty$$

$\hat{K}(u)$: (dis)entangler at length scale $\sim \varepsilon \cdot e^{-u}$

$|\Omega\rangle$: unentangled state in real space

$\rightarrow S_A = 0$ for any A. ➡ What is this state ?
We will come back later.

Relation to (discrete) MERA



By adding dummy states |0>  , we keep the dimension of Hilbert space for any u to be the same.

⇒ We can formally describe the real space RG by a **unitary transformation**.

Example: cMERA for a (d+1) dim. Free Scalar Theory

Hamiltonian: $H = \frac{1}{2}\int dk^d [\pi(k)\pi(-k) + (k^2 + m^2)\phi(k)\phi(-k)].$

Ground state $\left|0\right\rangle$ **:** $a_k\left|0\right\rangle = 0.$

IR state: $a_x\left|\Omega\right\rangle = 0, \qquad \left(a_x \equiv \sqrt{M}\phi(x) + \frac{i}{\sqrt{M}}\pi(x) \right),$

i.e. $\left|\Omega\right\rangle = \prod_x \left|0\right\rangle_x \qquad \Rightarrow \quad S_A = 0.$

cMERA: $\hat{K}(u) = \frac{i}{2}\int dk^d \left[\chi(u)\Gamma\left(ke^{-u}/M\right)a_k^+ a_{-k}^+ + (h.c.)\right],$

where $\Gamma(\mathrm{x})$ is a cut off function : $\Gamma(\mathrm{x}) = \theta(1 - |\mathrm{x}|).$

$$\chi(s) = \frac{1}{2}\cdot\frac{e^{2u}}{e^{2u} + m^2/M^2}, \quad (\text{for } m = 0,\ \chi(\mathrm{u}) = 1/2.)$$

(4-2) Boundary State as Gravity Dual of Point-like Space

[Miyaji-Ryu-Wen-TT 14]

Q. A general construction of the IR states |Ω> in CFTs ?

Argument 1

We can realize disentangled states (IR states |Ω>)

⇔ Trivial (Point-like) spaces

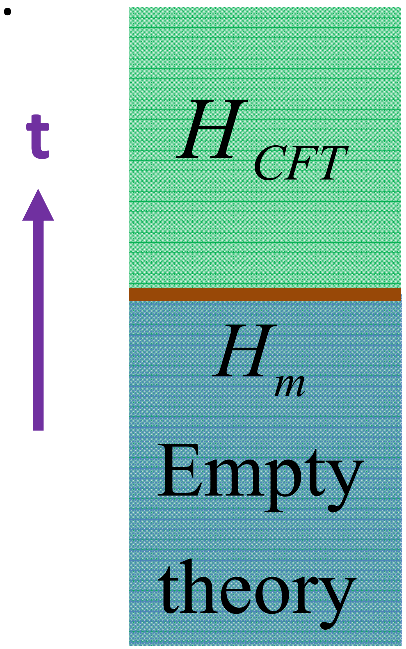
by performing a (infinitely) massive deformation:

$$H_m = H_{CFT} + m^{d+1-\Delta_O} \int dx^d O(x),$$

$$\underset{m\to\infty}{\Longrightarrow} \left|\Omega\right\rangle = \text{the ground state of } H_m .$$

Now we apply the idea of ***quantum quenches***.

⇒ For t<0, we assume the ground state of the massive Hamiltonian H_m. Then at t=0, we suddenly change the Hamiltonian into H_{CFT} as in [Calabrese-Cardy 05].



In this setup, the state at t=0 is identified with the boundary state:

$$\left|\Psi_m(t=0)\right\rangle = \left|\Omega\right\rangle = \left|B\right\rangle.$$

We may introduce the UV cut off like

$$\left|\Omega_m\right\rangle \propto e^{-H/m}\cdot\left|B\right\rangle \ .$$

Boundary states in CFTs (assume 2d CFT)

A **boundary state** (Ishibashi state) : |B>

= A state which gives a conformally invariant boundary condition:

$$\left[L_n - \widetilde{L}_{-n}\right]\left|B\right\rangle = 0.$$

In terms of the Virasoro algebra: $\left|B\right\rangle = \sum_{\vec{k}} \left|\vec{k}\right\rangle_L \left|\vec{k}\right\rangle_R$

where $\vec{k} = (k_1, k_2, \ldots\ldots)$ represent

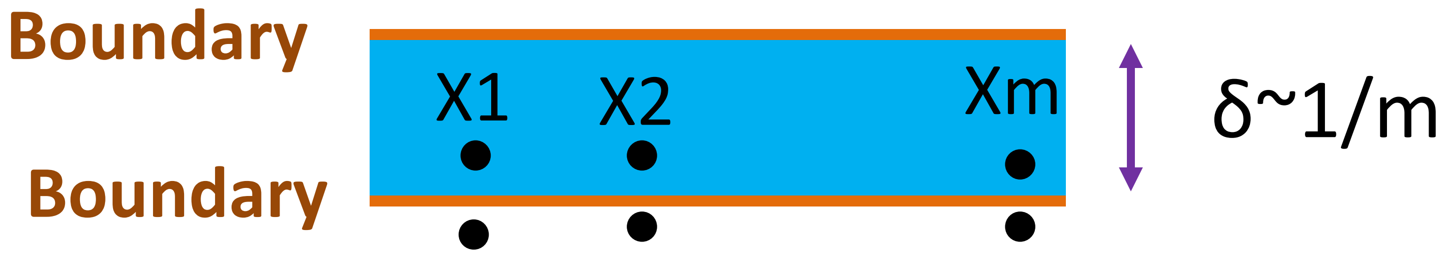
$$\left|\vec{k}\right\rangle = \sum (L_{-1})^{k_1} \cdot (L_{-2})^{k_2} \cdots \left|\Delta\right\rangle.$$

⇒ A maximally entangled state

between left and right moving sectors !

⇒ But, the real space entanglement is quite suppressed !

Argument 2: Correlation functions of local operators



$$\frac{\langle \Omega | O(x_1) O(x_2) \cdots O(x_n) | \Omega \rangle}{\langle \Omega | \Omega \rangle} \approx \prod_{i=1}^{n} \langle O(x_i) \rangle.$$

⇒ When (xi-xj)>>δ, there is no correlations !

⇒ Disentangled !

Argument 3: Direct calculation of EE

For the regularized IR state $\left|\Omega\right\rangle = e^{-H\delta}\left|B\right\rangle$,
we can compute the EE explicitly in free fermion CFTs:

[Ugajin-TT 10]

$$S_A \approx \frac{c}{3}\log\frac{\delta}{\varepsilon} + [\mathrm{Finite}], \qquad (\delta \to 0).$$

Thus we can set $S_A \approx 0 \quad \text{when} \quad \delta \approx \varepsilon.$

Note: Boundary states can still have non-zero finite *topological entanglement.*

(4-3) Aspects of cMERA

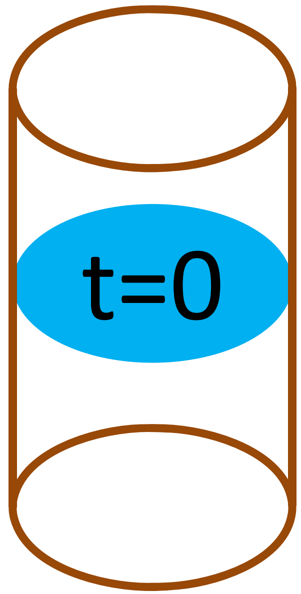
- No lattice artifacts
- Choice of K(u) is flexible → In principle, we can realize not only the continuum limit of original MERA but also those of perfect tensor networks.

  But, we do not know K(u) precisely except free CFTs.
- We can compute information metrics or $\left|\langle \Phi(u) | \Phi(u+du) \rangle\right|$.

  , though the evaluation of EE is not straightforward.
- We can treat excited states in a straightforward way.

Conformal Transformation in cMERA for 2d CFT

The generators which preserve a time slice of AdS3 are given by $l_n = \widetilde{L}_{-n} - L_n$.



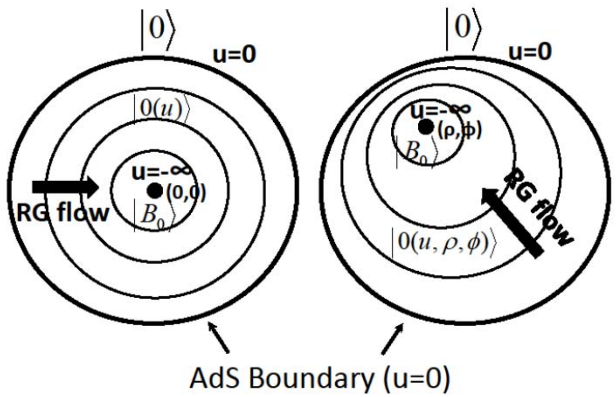
The SL(2,R) action which maps ρ=0 to the point (ρ,ϕ) is given by $g(\rho,\phi) = e^{i\phi l_0} e^{\frac{\rho}{2}(l_{-1}-l_1)}$.

The cMERA flow

$$|0\rangle = P\exp\left(-i\int_{-\infty}^{0} \hat{K}(u)du\right)|B_0\rangle.$$ is transformed by SL(2,R)

$$|0\rangle = P\exp\left(-i\int_{-\infty}^{0} \hat{K}_{(\rho,\phi)}(u)du\right)|B_0\rangle,$$

where $\hat{K}_{(\rho,\phi)}(u) = g(\rho,\phi)\cdot\hat{K}(u)\cdot g(\rho,\phi)$



CFT dual of excited states by bulk local field

[Miyaji-Numasawa-Shiba-Watanabe-TT 15 (see also Nakayama-Ooguri 15)]

We can show that the CFT state dual to an bulk excited state $\varphi_\alpha(\rho,\phi)|0\rangle_{bulk}$ obtained from the bulk scalar field φ_α is given by (in the large N limit) :

$$|\Psi_\alpha(\rho,\phi)\rangle_{CFT} \approx g(\rho,\phi)\cdot e^{\frac{\pi}{2}i(L_0+\tilde{L}_0)} \cdot \underbrace{e^{-\delta H}}_{\substack{\text{some UV cut off} \\ s.t.\ \delta>>1/c}} \cdot \underbrace{|J_\alpha\rangle}_{\substack{SL(2,R) \\ \text{Ishibashi State}}} .$$

Note: this agrees with Kabat-Lifschytz-Lowe's prescription.

Why ?

Planck scale cut off

Similar to cMETA IR state

(Full Virasoro Ishibashi state)

[cf. Verlinde 15]

→cMERA flow ?

⑤ Surface/State Correspondence [Miyaji-TT 15]

(5-1) Basic Principle

Consider Einstein gravity on a d+2 dim. spacetime **M**.

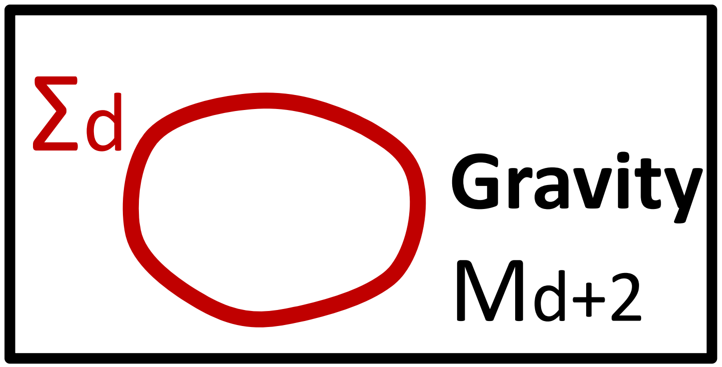
We conjecture the following correspondence:

Σ : an d dim. convex space-like surface in M which is closed and homologically trivial

⇕

$$\boxed{\left|\Phi(\Sigma)\right\rangle \in H_M}$$

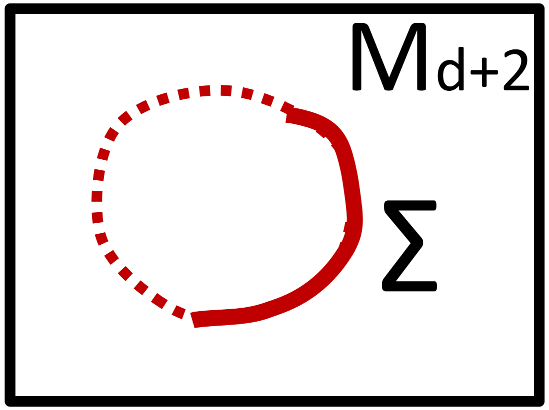
A pure state

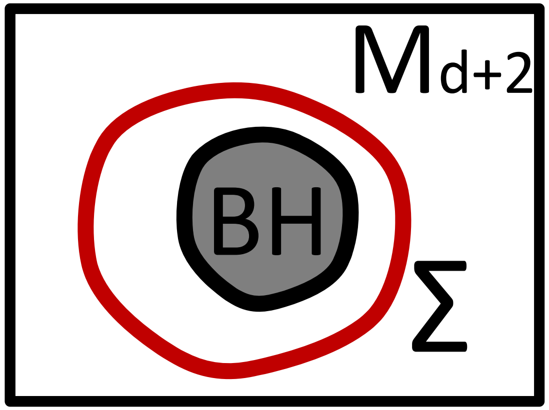


More generally,
the quantum state dual to a convex surface Σ is

a **mixed state** $\rho(\Sigma)$

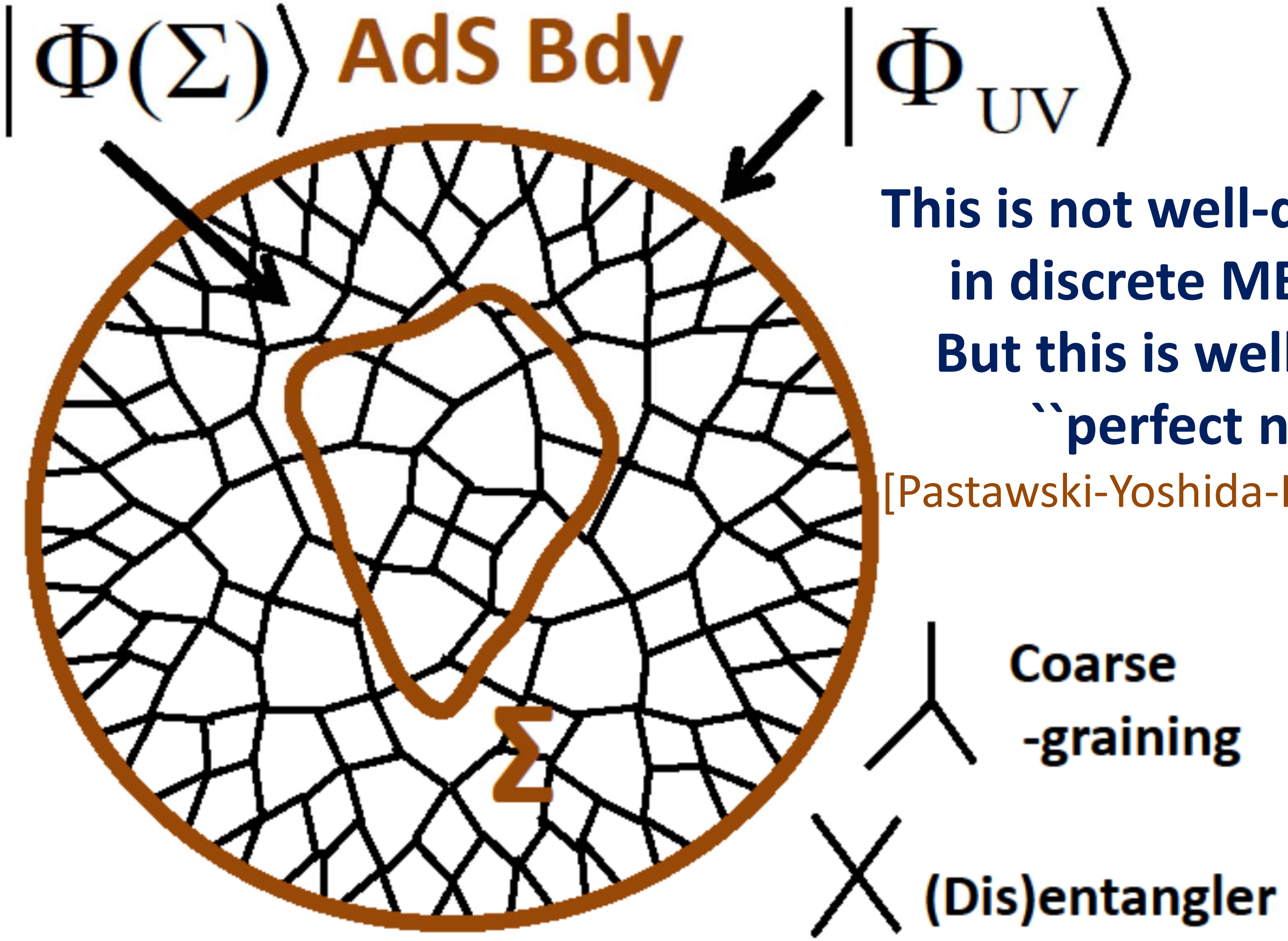
if Σ is open or topologically non-trivial.





On the other hand, the **zero size limit of Σ** corresponds to the **trivial state |Ω>** with no real space entanglement.

This surface/state correspondence is realized in the ``perfect'' tensor network description of holography.



This is not well-defined in discrete MERA.

But this is well-defined in the ``perfect network'' in

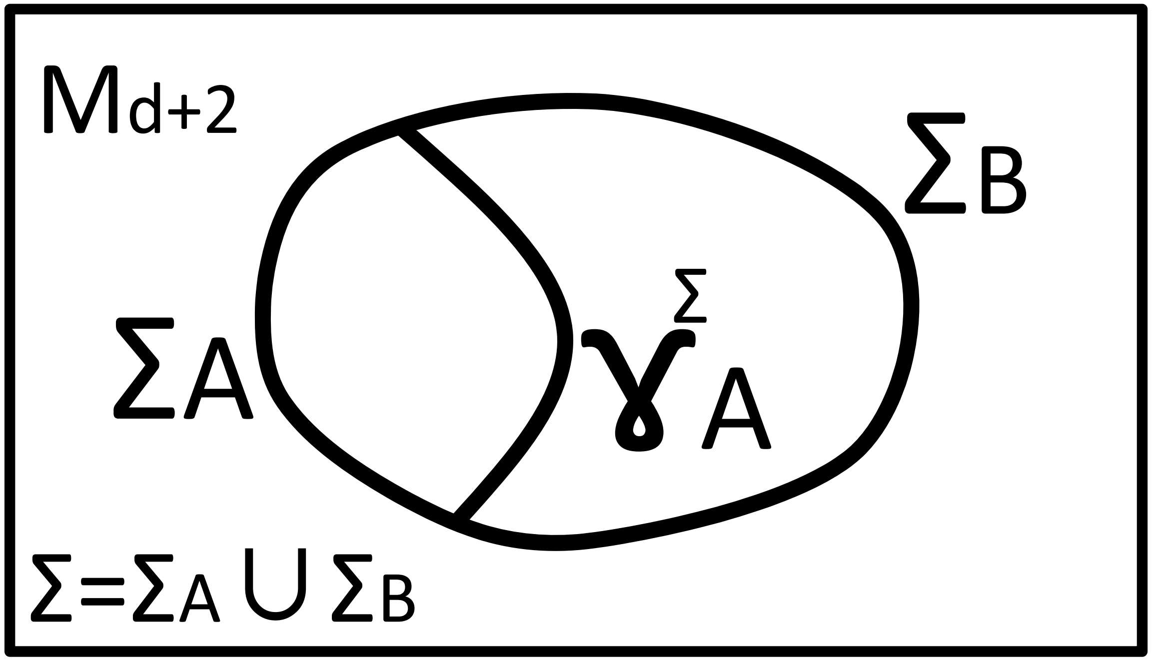
[Pastawski-Yoshida-Harlow-Preskill 15]

(5-2) Entanglement Entropy

We can naturally generalize HEE for our setup :

$$H_{\Sigma} = H_{A} \otimes H_{B}, \qquad \rho_A^{\Sigma} = \mathrm{Tr_B}[\rho(\Sigma)],$$

$$\Rightarrow \qquad S_A^{\Sigma} = \frac{\mathrm{Area}(\gamma_{\mathrm{A}}^{\Sigma})}{4G_N}.$$



(5-3) Effective Dimension

By dividing the surface Σ into infinitesimally small pieces $\Sigma = \cup A_i$, we easily find:

$$\sum_i S_{A_i}^{\Sigma} = \frac{\mathrm{Area}(\Sigma)}{4G_N}.$$

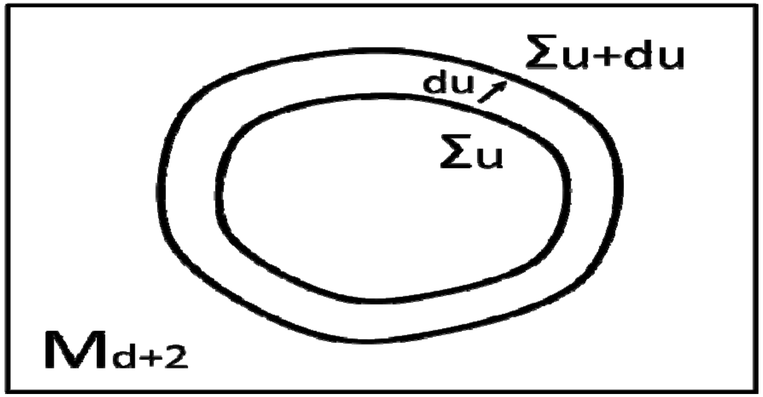
We interpret this as the log of effective dim. for Σ

$$\log[\dim H_{\Sigma}^{eff}]$$

This is because $\rho_{A_i}^{\Sigma}$ is expected to be maximally entangled (except the dummy states).

(5-4) Inner Products and Information Metric

Another intriguing physical quantity is an inner product $\langle \Sigma | \Sigma' \rangle$ between two surfaces.



$$ds^2 = R^2 du^2 + g_{\mu\nu}(x,u) dx^\mu dx^\nu .$$

Here focus on the two surfaces separated infinitesimally.

⇒ Consider an information distance between them

The information metric is defined as

$$1 - \left| \langle \Phi(u) | \Phi(u + du) \rangle \right| = (du)^2 \cdot G_{uu}^{(B)}$$

If the metric is x-independent, we have

$$G_{uu}^{(B)} \sim \frac{1}{G_N} \int_{\Sigma u} dx^d \sqrt{g(x)} \left(K_u\right)^2 .$$

Extrinsic curvature

→ Vanishes on extremal surfaces

Example 1: a flat spacetime $\Rightarrow$ $G_{uu}^{(B)} = 0.$

[u-Translational inv. $\Rightarrow$ $|\Phi(u+du)\rangle = |\Phi(u)\rangle$.]

Example 2: an AdS spacetime [Nozaki-Ryu-TT 12]:

$$G_{uu}^{(B)} = N_{\deg} \cdot \frac{V_d}{\varepsilon^d} \cdot e^{du} \Rightarrow \text{Agrees with cMERA for CFT}_{\text{d+1}}$$

(5-5) Diffeomorphism in cMERA and SS-correspondence

We define $l_n = \widetilde{L}_{-n} - L_n, \quad (\,|n| = 2,3,..).$

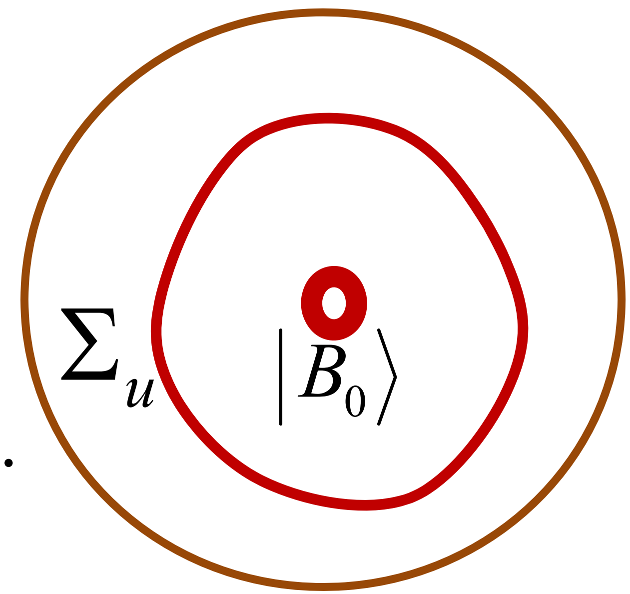
$$|0\rangle = P\exp\left(-i\int_{-\infty}^{0} \hat{K}_g(u)du\right)|B_0\rangle,$$

$$\hat{K}_g(u) = \hat{g}(u)\hat{K}(u)\hat{g}(u)^{-1} + \partial_u g(u)\cdot g(u)^{-1},$$

where $g(u) = \exp\left[\sum_n \xi_u(u)\, l_n\right]$ with $\xi_n(0) = 0.$

The dual state of a surface Σ_u expected in SS-correspondence is given by the form:

$$|\Phi(\Sigma_u)\rangle = P\exp\left(-i\int_{-\infty}^{u} \hat{K}_g(s)ds\right)|B_0\rangle.$$



⑥ Conclusions

- Quantum entanglement represents a *geometry of quantum state* in many-body systems.

Tensor network ⇒ Entanglement = geometry

Boundary states ⇔ trivial (point-like) space

- AdS/MERA duality looks like a zero-th order approximation. ⇒ We need some refinement.

(1) Integral geometry ?

(2) Perfect tensor network ? ⇒ Surface/State corresp.

(3) continuum limit (cMERA) ?

So many future problems

- Planck scale locality in Tensor Networks ?
- Tensor networks for gauge theories ?
- Derivation of Einstein eq. (how to describe strongly coupled gauge theories ?)
- Determinations of K(u) in cMERA ?
- Perfect Tensor models for real CFTs ?
- Holography for more general spacetimes and TNs ?
- Applications to black hole information problem ?

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