Perspective in Geometric Analysis

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In this talk, I shall discuss some directions in the subject of geometric analysis. I have written several times about the subject in the past years. Many of my talks can be found in my selected works on survey papers (International Press of Boston, 2014).

In this talk, I shall focus on several problems that may be interesting to students who are new to the subject.

1. General relativity

There are many interesting problems in this classical subject, which we may say to be geometry of Lorentzian manifolds. However, we should always remember that this is also a branch of physics. It has to be compatible with the phenomena that we know about gravity according to the basic principle of General relativity .

a. Within the subject of General relativity, a very fundamental question is to find the right definition of quasi-local mass and quasi-local linear momentum.

Quasi-local mass means that given a two dimensional spacelike closed surface in spacetime, we want to associate a quantity which satisfies several "axioms", namely that

i. It depends only on the metric of the surface, and the second fundamental form of the surface.
ii. It is trivial if the surface is situated within the Minkowski spacetime.
iii. It should be positive if the spacelike surface is convex in a certain sense and the spacetime satisfies the local energy condition.

iv. When a sequence of surfaces converges to the sphere at spatial infinity, the quasi-local mass should converge to the ADM mass as was defined by Arnowit-Deser-Miser in the following manner.

$$M_{ADM} = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S^2(r)} (\partial_j g_{ij} - \partial_j \mathrm{tr}\, g)\nu^j$$

Here ν^j is the outward unit normal of $S^2(r)$.

v. When a sequence of surfaces converges to the sphere at null infinity, the quasi-local mass should converge to the Bondi mass defined by Bondi et al. in (Proc. Roy. Soc. London A 1962).

vi. When the surfaces converge to a point, it should reproduce the matter density when the matter tensor is not zero, otherwise to the quantity related to Bel-Robinson tensor.

vii. When the space-time has global symmetric group given by $SU(2)$, the quasi-local mass of the orbit of $SU(2)$ is equivalent to the Komar mass defined by the method of Emmy Noether.

The important question is whether such a definition is unique up to equivalence or not? Mu-Tao Wang and I (CMP & PRL 2009) succeeded to provide a construction of such a quasi-local mass. Besides the definition given by Bartnik, this is the only known definition of quasi-local mass that satisfy all of the above properties.

- What is the relation between the Wang-Yau quasi-local mass with Bartnik mass?
- The quasi-local mass that was defined Wang-Yau or Bartnik are modeled after Minkowski spacetime. One can model the definition after other static spacetime. Can we find similar properties?

b. Isometric embeddings of surfaces into Minkowski spacetime. In the works of Chen-Wang-Yau, we encountered the question of optimal isometric embedding into Minkowski spacetime. We know that the optimal isometric embedding is unique in many cases. It will be good to find general conditions to establish uniqueness. It is also interesting to know whether a general compact surface with nonzero genus can be isometrically embedded into Minkowski spacetime or not.

c. Classification of complete spacelike hypersurfaces in Schwarzschild spacetime with constant mean curvature. This was done by Andrejs Treibergs in (Invent. Math. 1982). It will be nice to find the detailed structure of its asymptotic structure even if we assume the mean curvature is equal to zero and the spacelike hypersurface is asymptotic to a hyperplane that is boosted. Such structure can be very useful for many discussions of general spacetime which is asymptotically flat.

d. What is the fate of maximal spacelike hypersurfaces with nontrivial second fundamental forms when when we evolve them according to the Einstein equation?

2. Metric geometry

a. Prove that when dimension is greater than 26, every compact manifold with positive sectional curvature is diffeomorphic to a homogeneous space.

b. Formulate and prove the Hamilton Inequality for Ricci flow when the curvature is not necessarily positive.

c. Consider the space of compact manifolds with positive scalar curvature that is the conformal boundary of an Einstein manifold which is asymptotic to hyperbolic space form. Can one do surgery in codimension 3 for this class of manifolds, as was demonstrated by Schoen-Yau within the class of metrics with positive scalar curvature (cf. the paper of Gromov-Lawson). It is not hard to demonstrate that connected sum preserves this class of metrics.

Witten and I proved that the conformal boundary of such Einstein manifold must be connected. It seems to me that there is no new obstruction for such class of metrics than those from positive scalar curvature alone.

d. For a compact manifold with dimension greater than 4, do they always an Einstein metric and in fact , would there be infinite number of such metrics? If we impose extra structure on the metric such as Kählerian, we know that they are parametrized by the space of Kählerian complex structures (due to the uniqueness theorems developed by Calabi and Bando-Mabuchi).

We can weaken the requirement of Kählerness by requiring the metric to be a sum of Riemannian metrics that are locally conformal to some Kähler metrics.

3. Complex geometry

a. On an *n*-dimensional Kähler manifold with nonvanishing holomorphic *n*-form, there exists a Calabi-Yau metric and associated with it, the concept of special Lagrangian submanifolds. Prove that embedded special Lagrangian torus always exists. If there is nontrivial deformation of the complex structure of the Calabi-Yau manifold, there is nontrivial family of special Lagrangian torus that are mutually disjoint.

- Will this family fill up the Calabi-Yau manifold with some closed exceptional set with codimension not less than two?
- What is the structures of this exceptional set?

b. In the previous problem, if we look at the moduli space of the pairs (special Lagrangian torus vs. a flat complex line bundle over this special Lagrangian torus).

In the paper of Strominger-Yau-Zaslow, this moduli space has a complex Kähler structure and that after some “quantum correction”, it becomes a Calabi-Yau manifold mirror to the original Calabi-Yau manifold. How to justify these claims?

If the special Lagrangian is nonsingular, the moduli space near it is smooth. The major question is to prove that we can complete this part of moduli space to be a complex variety.

c. Special Langragian submanifolds are supposed to be mirror to certain holomorphic bundles defined in the mirror manifold. (The bundle may have singularity. The rank of the bundle is the intersection number of the special Lagrangian submanifolds with the special Lagrangian torus.)

Leung-Yau-Zaslow translated the equation of special Lagrangian to equation on the mirror bundle. The existence theory for this equation was studied by Adam Jacob, Tristan Collins and me. It is not completely understood. It is important to relate such existence and the corresponding stability condition to the bridgeland stability.

d. In studying eigenvalue of Laplacian on a projective manifold, Peter Li and I came up with a balanced condition for its projective embedding. By changing the measure to be the induced measure of the ambient projective space, my former student Huazhang Luo found that existence of such balanced condition implies Chow stability of manifolds.

An important question is to find an effective way to projectively move an embedded algebraic manifold to a balanced position. For algebraic manifolds with ample canonical line bundle K, we know that by taking a power of K, the normalized induced metric at balanced position will converge to the Kähler-Einstein metric when the power of K tends to infinity. Can we compute the Kähler-Einstein metric effectively by this method?

4. Eigenvalues of elliptic operator and minimal surface

Both of these topics have long history. There are several approaches to study eigenvalues of Laplacian which went back to mathematicians in 19th century through variational principle. Important contributors include Kelvin, Rellich, Pólya, Szegö and others.

Then a breakthrough happened in 1911 when Weyl solved a famous problem of Lorentz on asymptotic behavior of eigenvalues of Laplacian. (Hilbert had thought that he would not see a proof in his life time.) The argument used Tauberian theorem which was popular in analytic number theory.

This line of research developed into later works based on heat and wave equation methods. For manifolds whose universal cover is symmetric space, all eigenvalues and eigenfunctions are computed. In this case, the method of Selberg is powerful and gives a lot of information about eigenvalues related to the discrete groups acting on symmetric space.

Pólya conjecture

a. Consider the Dirichlet problem for a bounded domain Ω in Euclidean space $\mathbb{R}^n$, Pólya (1961) conjectured that the kth eigenvalue λ_k has a lower bound which is the same as the Weyl asymtotics of λ_k, i.e.,

$$\lambda_k \geq \frac{4\pi^2 k^{2/n}}{(C_n \operatorname{vol}(\Omega))^{2/n}}$$

Here C_n is the volume of the unit ball in $\mathbb{R}^n$.

Peter Li and I (CMP 1983) proved Pólya conjecture in the average. But the conjecture is still open.

b. For a compact surface M, Peter Li and I (Invent. Math. 1982) proved that the first eigenvalue times the area of the surface is bounded above by twice the conformal area.

- Compute the conformal area as a function defined on the moduli space of the Riemann surface.
- Is the Li-Yau upper bound optimal? For a fixed conformal structure, what is the optimal metric to optimize the Li-Yau inequality?
- For some special surface such as those defined by arithmetic group, the lower bound also holds qualitatively as was demonstrated by Selberg's famous $3/16$ lower estimate. What is the conformal area of such surfaces that are defined by arithmetic group quotient?

If we fix the conformal structure, and maximize the product of first eigenvalue with area for metrics within that conformal class, what is the result compared with the conformal area defined by Li-Yau? If one does not fix the conformal structure, the problem of extremal metric was studied by Nadirashvili (GAFA 1996) and in recent works of Schoen and Fraser.

For higher dimensional Kähler manifold, we can ask similar question of optimizing the first eigenvalue among Kähler metrics within the same Kähler class. Similar questions can be asked for higher eigenvalues.

Suppose we fix the Kähler class as a cohomology class, and we optimize eigenvalues of the Laplacian, we obtain infinite number of numbers that are invariants of the complex structure and the Kähler class. Can these numbers determine the complex structure and the Kähler class?

Generalize my work with Peter Li and Bourguignon to Sasakian manifolds.

- Can we map a Sasakian manifold to odd dimensional sphere preserving transverse holomorphic structure?
- Can we generalize the above discussions to balanced metrics?
- Is any of these discussions still valid for Laplacian acting on differential forms?

c. Can we estimate lower bound of the first eigenvalue of Laplacian acting on differential forms in terms of isometric inequalities for subvarieties that are homologous to zero. Hence the volume of the subvariety compared with the volume of the area minimizing subvarieties that it bounds.

d. In previous problem, we propose a sequence of numbers within a fixed a Kähler class and complex structure. It may be called Type B eigenvalues of the Laplacian. Is there a mirror picture of these numbers according to mirror principle for Calabi-Yau manifolds? In that case, we are looking for eigenvalues related to the symplectic structure.

- Is there natural operator associate to the symplectic structure?
- If we fixed a symplectic form, then for each almost complex structure that is compatible with the symplectic form, we have Laplacian which gives rise to eigenvalues. Can we find some inequality associated to these eigenvalues?

5. Submanifolds

a. For which Calabi-Yau manifold and which integral homology class of (k, k)-type, can it be represented as positive sum of holomorphic cycles? This assertion refines the famous Hodge conjecture and is not understood even when the manifold has dimension three. Strictly speaking, there is counterexample due to Wolfson for some K3 surfaces.

The general idea was to minimize area and attempt to prove the minimial variety is in fact a sum of holomorphic cycles. This idea was effective when the cycle is 2-dimensional and has genus zero as was demonstrated by Siu and Yau in the proof of the Frenkel conjecture, by studying the second variational formula. The idea of Siu-Yau was used later by Micallef-Moore and Brendle to treat surfaces of high codimension in Riemannian manifolds.

b. Classify embedded minimial surfaces in the three sphere. Several mathematicians including Lawson, Meeks, Rosenberg, Schoen et al had basically accomplished the classifications of completed embedded surfaces in Euclidean space by reducing most problems to those minimal surfaces with finite total curvature where algebraic geometry method can be brought in. Perhaps we can hope to classify such surfaces in 3-sphere by algebro-geometric means?

c. Consider closed minimal hypersurfaces $M \subset S^{n+1}$ with constant scalar curvature κ. Chern conjectured that for each n the set of all possible values for κ is discrete. It has not been completely solved despite much progress was made.

One way of generalizing the Chern conjecture as originally proposed is the following: Assume furthermore that some scalar invariant of the second fundamental form h, or its covariant derivatives, is constant on M ($|h|^2$ in the case of the Chern conjecture, other examples would be $\det h$ and $|\nabla^k h|^2$). Then one can ask what values the constant can assume, in particular whether the set of these values is discrete.

It is also possible to generalize questions of this type to submanifolds of higher codimension.

One can of course ask similar question for compact minimal submanifolds of constant scalar curvature in other compact Riemannian homogeneous spaces. What kind of values are their possible scalar curvatures? One can also ask the same question for submanifolds where some symmetric polynomials of the second fundamental form is constant.

d. Given a 3-dimensional Riemannian manifold M, can we find a graph of it in $M \times \mathbb{R}$ so that this graph can be conformally embedded into $\mathbb{R}^4$. This is a question that arises from my discussion with Mu-Tao Wang and Po-Ning Chen on quasi-local mass.

6. Special structure on manifolds

a. Understand which compact connected Lie group can be holonomy group of some Riemannina manifold. The discrete part is more complicated and is related to fundamental group of the manifold. It will be nice to find a systematic way to understand this problem.

The question is reasonably understood for manifolds with constant positive or zero curvature. They are covered by spheres or torus. In the first case, it is called Vincent problem and solved by J. A. Wolf. The second case was started by Bieberbach, but not completely understood.

The negative curvature case is much more difficult. They are torsion-free discrete cocompact subgroups of $SO(n, 1)$. A great progress was accomplished by Thurston's geometrization theorem which says that every compact 3-dimensional irreducible aspherical and atoroidal 3-manifold must admit a metric with constant curvature. Which finitely presented torsion free atoroidal group can appear as fundamental group of compact manifold with negative curvature? Of course, we know there are various kind of hyperbolicity that can attach to it as was initiated by the famous paper of Milnor.

The same question can be asked for other compact locally symmetric space, positive curvature or not. Can we decide which smooth manifold admits such a structure?

b. Berger has classified all manifolds with compact connected Lie group as holonomy group. They can also be realized by examples. However those with holonomy group $\mathrm{Spin}(7)$ and G_2 are not understood as their moduli spaces have not been understood well and we cannot tell which 7 or 8-dimensional manifolds can support such structures.

It is an important question to reduce the problem of existence of such structures based on simple criterion. Donaldson and Thomas introduced special bundles over such manifolds. It should be important to understand their moduli space also.

c. Classify those Lorenzian manifolds whose holonomic groups are proper subgroups of $SO(n, 1)$. What kind of group can occur as holonomy group?

d. Let M be a compact complex manifold that admits a balanced metric such that the tangent bundle is slope stable with respect to the balanced metric. Supposed the Chern number inequality becomes equality, what can we say about such balanced manifolds?

Thank you!