Superstring amplitudes in genus 0 and 1

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String Math
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18th June 2018
Overview
Superstring perturbation

- Expansion in series of genus \( g \) world-sheets. Integrate over world-sheet moduli space.
- Expansion in \( \alpha' = \ell_S^2 \)

We only consider \( g = 0, 1 \).
Superstring perturbation

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- Expansion in $\alpha' = \ell_s^2$

We only consider $g = 0, 1$.

Study integrals of the shape (where $g = 0, 1$)

$$\int_{\mathcal{M}_{g,n}(\mathbb{C})} \exp \left( \sum_{i<j} \alpha' s_{ij} G(z_i - z_j) \right) \omega$$
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Strategy: first integrate over configuration space of points on a Riemann surface of genus $g$. Then integrate over the moduli of the Riemann surface.

The last step is redundant in the case $g = 0$. 
Beta function is an integral on $\mathcal{M}_{0,4} = \mathbb{P}^1\backslash\{0, 1, \infty\}$

$$\int_0^1 x^{\alpha's-1}(1-x)^{\alpha't-1}dx = \frac{\Gamma(\alpha's)\Gamma(\alpha't)}{\Gamma(\alpha's + \alpha't)}$$
Beta function is an integral on $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

\[
\int_0^1 x^{\alpha' s - 1} (1 - x)^{\alpha' t - 1} \, dx = \frac{\Gamma(\alpha' s) \Gamma(\alpha' t)}{\Gamma(\alpha' s + \alpha' t)}
\]

\[
= \frac{s + t}{st\alpha'} \exp \left( \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\zeta(n)}{n} \sigma_n \right)
\]

where $\sigma_n = (\alpha')^n((s + t)^n - s^n - t^n)$. Involves all zeta values.
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Closed string gives complex beta function:

$$\int_{\mathbb{P}^1(\mathbb{C})} |x|^{-2\alpha's - 2} |1 - x|^{-2\alpha't - 2} \, d^2 x \quad = \quad \frac{\Gamma(\alpha's)\Gamma(\alpha't)\Gamma(1 - \alpha's - \alpha't)}{\Gamma(s\alpha' + t\alpha')\Gamma(1 - \alpha's)\Gamma(1 - \alpha't)}$$
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\]

\[
= \frac{s + t}{st\alpha'} \exp \left( \sum_{n=1}^{\infty} \frac{2\zeta(2n + 1)}{2n + 1} \sigma_{2n+1} \right)
\]

where $d^2x = (2\pi i)^{-1} dx \wedge \overline{dx}$. Only involves odd zeta values.
Distinct points $z_0, \ldots, z_{n+2}$ on a Riemann sphere. By $\text{PSL}_2(\mathbb{C})$ action, can place $z_0 = 0, z_{n+1} = 1, z_{n+2} = \infty$.

For a permutation $\pi \in \Sigma_{n+3}$, let

$$z_\pi = \prod_{i \in \mathbb{Z}/(n+3)\mathbb{Z}} (z_\pi(i) - z_\pi(i+1))$$

omitting term $z_{n+2} = \infty$. 
Open vs closed amplitudes in genus 0

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Open string amplitudes reduce to \( n! \) integrals:

\[
A_{\text{open}}(\pi) = \int_{0 < z_1 < \ldots < z_n < 1} \prod_{i < j} (z_i - z_j) \alpha' s_{ij} \frac{dz_1 \ldots dz_n}{z_\pi}
\]

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A_{\text{closed}}(\pi, \pi') = \int_{C_n} \prod_{i < j} |z_i - z_j|^2 \alpha' s_{ij} \frac{dz_1 \ldots dz_n}{z_\pi} \wedge dz_1 \ldots dz_n
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Open string amplitudes reduce to $n!$ integrals:

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Closed string amplitudes reduce to complex integrals:

$$A^{\text{closed}}(\pi, \pi') = \int_{\mathbb{C}^n} \prod_{i<j} |z_i - z_j|^{2\alpha'_sij} \frac{dz_1 \ldots dz_n}{z_\pi} \wedge \frac{d\bar{z}_1 \ldots d\bar{z}_n}{\bar{z}_{\pi'}}$$
Kawai-Lewellen-Tye formula (1986)

Expresses closed tree-level ($g = 0$) amplitudes as quadratic expression in open amplitudes: approximately

$$A_{\text{closed}}(\rho, \sigma) = \sum_{\rho, \sigma} A_{\text{open}}(\rho) S(\rho; \sigma) A_{\text{open}}(\sigma)$$

for certain factors $S(\rho; \sigma)$ in the Mandelstam variables $s_{ij}$. 

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Slogan:

‘Multiply then integrate $=$ integrate then multiply’
Plan

- Mathematics behind generalised KLT formulae
- Single-valued projections
- (Cosmic Galois group)
- New theory of modular forms from genus 1 string amplitudes
Single-valued integration
The usual theory of integration pairs a differential form $\omega$ with a domain of integration $\sigma$

$$I = \int_{\sigma} \omega \in \mathbb{C}$$
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We shall discuss how to pair certain differential forms $\omega$ with a ‘dual differential form’ $\nu$

$$I^{sv} = \oint_{\nu} \omega \in \mathbb{R}$$
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$$I^{sv} = \oint_\nu \omega \in \mathbb{R}$$

It can be interpreted as a ‘$p$-adic period at the infinite prime $p = \infty$’. First some examples.
Examples of single-valued functions

The *logarithm* is a multi-valued function on $\mathbb{C}\{0\}$:

$$\log z = \int_1^z \frac{dx}{x}.$$  

Changing path of integration results in $\log z \mapsto \log z + 2\pi i \mathbb{Z}$.  

The *dilogarithm* (Leibniz) is multi-valued on $\mathbb{C}\{0, 1\}$:

$$\text{Li}_2(z) = \sum_{k \geq 1} \frac{z^k}{k^2}.$$  

It has a single-valued version, the Bloch-Wigner dilogarithm $D(z) = 2i \text{Im}(\text{Li}_2(z) + \log |z| \log(1 - z))$. 
Examples of single-valued functions

The logarithm is a multi-valued function on \( \mathbb{C}\setminus\{0\} \):

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It has a single-valued version which is well-defined:

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2 \text{Re}(\log z) = \log |z|^2
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Multiple zeta values (MZV’s)

Defined by Euler (1730’s),

\[ \zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < \ldots < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}} \]

where \( n_1, \ldots, n_r > 0 \) integers, \( n_r \geq 2 \). They satisfy a plethora of complicated algebraic relations.
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They are values at 1 of multiple polylogarithms (Poincaré, Kummer, Lappo-Danilevskyy):

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These are multi-valued functions on $\mathbb{C} \setminus \{0, 1\}$. Coefficients of the universal KZ equation in genus 0. Iterated integrals on $\mathbb{C} \setminus \{0, 1\}$. 
Theorem (B. 2004)

By taking combinations of products of real and imaginary parts, there is a canonical way to define single-valued versions

$$\mathcal{L}_{n_1,\ldots,n_r}(z)$$ of $$\text{Li}_{n_1,\ldots,n_r}(z)$$

preserving algebraic and differential (with respect to $$\frac{\partial}{\partial z}$$) relations.

The linear combinations involve coefficients which are MZV’s.
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Definition 1

The *single-valued multiple zeta values* are defined by

\[ \zeta_{sv}(n_1, \ldots, n_r) = \mathcal{L}_{n_1, \ldots, n_r}(1) \]
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Example:

\[ \zeta_{sv}(2) = D(1) = 0 \]
Single-valued MZV’s continued

\[ \zeta_{sv}(2n) = 0 \]
\[ \zeta_{sv}(2n + 1) = 2\zeta(2n + 1) \]
The first non-trivial $\zeta_{sv}$ is at weight 11:

$$\zeta_{sv}(3, 5, 3) = 2\zeta(3, 5, 3) - 2\zeta(3)\zeta(5) - 10\zeta(3)\zeta(5)$$

**Theorem (B. 2013)**

The single-valued MZV’s satisfy all ‘motivic’ relations for MZV’s. ('Motivically') there is a ‘single-valued projection’ $\zeta_{sv} : \zeta \mapsto \zeta_{sv}$.

**Warning:** Does not exist for general period integrals.

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\begin{align*}
\zeta_{sv}(2n) &= 0 \\
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\zeta_{sv}(5, 3) &= 14\zeta(3)\zeta(5)
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Theorem

The open superstring amplitudes for $g = 0$ admit a canonical Laurent expansion in $s_{ij}$ whose coefficients are multiple zeta values.

Follows from conjecture of Goncharov-Manin (B. 2006), Terasoma, Schlotterer, Stieberger, Broödel, ...
Stieberger’s conjecture

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Laurent expansion in $s_{ij}$ whose coefficients are multiple zeta values

Follows from conjecture of Goncharov-Manin (B. 2006), Terasoma, Schlotterer, Stieberger, Broël, . . .

Stieberger and Stieberger-Taylor made the following conjecture:

Theorem (B. and Dupont ’18)

$$sv A^{\text{open}}(\pi) = A^{\text{closed}}(\pi; \text{id})$$

Apply $sv$ term by term in the Laurent expansion in $s_{ij}$. 
X a smooth algebraic variety over \( \mathbb{Q} \). A period integral on \( X \)

\[
I = \int_{\sigma} \omega
\]

where \( \omega \in \Omega^n(X; \mathbb{Q}) \) algebraic differential form. Chain \( \sigma \subset X(\mathbb{C}) \) has boundary \( \partial \sigma \subset D(\mathbb{C}) \) where \( D \subset X \) a divisor.
$X$ a smooth algebraic variety over $\mathbb{Q}$. A *period integral* on $X$

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where $\omega \in \Omega^n(X; \mathbb{Q})$ algebraic differential form. Chain $\sigma \subset X(\mathbb{C})$ has boundary $\partial \sigma \subset D(\mathbb{C})$ where $D \subset X$ a divisor.

$$[\omega] \in H^n_{dR}(X, D)$$
$$[\sigma] \in H_n(X(\mathbb{C}), D(\mathbb{C})) = H^n_B(X, D)^\vee$$

Integration is a pairing

$$H^n_{dR} \otimes H_n \longrightarrow \mathbb{C}$$
$X$ a smooth algebraic variety over $\mathbb{Q}$. A \textit{period integral} on $X$

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Integration is a pairing

$$H^n_{dR} \otimes H_n \longrightarrow \mathbb{C}$$

It defines a canonical isomorphism (Grothendieck 1964):

$$H^n_{dR}(X, D) \otimes \mathbb{C} \sim H^n_B(X, D) \otimes \mathbb{C}$$
Complex conjugation is continuous

\[(X(\mathbb{C}), D(\mathbb{C})) \xrightarrow{\sim} (X(\mathbb{C}), D(\mathbb{C}))\]

It induces the *real Frobenius*

\[F_{\infty} : H^*_B(X, D) \xrightarrow{\sim} H^*_B(X, D)\]
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\[F_{\infty} : H^n_B(X, D) \sim H^n_B(X, D)\]

We get

\[H^n_{dR}(X, D) \otimes \mathbb{C} \sim H^n_B(X, D) \otimes \mathbb{C} \xrightarrow{F_{\infty}} H^n_B(X, D) \otimes \mathbb{C} \sim H^n_{dR}(X, D) \otimes \mathbb{C}\]
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Complex conjugation is continuous

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It induces the real Frobenius

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We get

$$H^n_{dR}(X, D) \otimes \mathbb{C} \sim H^B_{\infty}(X, D) \otimes \mathbb{C} \xrightarrow{F_{\infty}} H^B_{\infty}(X, D) \otimes \mathbb{C} \xleftarrow{\sim} H^n_{dR}(X, D) \otimes \mathbb{C}$$

It defines a real comparison isomorphism

$$sv : H^n_{dR}(X, D) \otimes \mathbb{R} \sim H^n_{dR}(X, D) \otimes \mathbb{R}$$
This gives a way to pair forms with ‘dual forms’:

\[ [\omega] \in H^n_{dR}(X, D) \]
\[ [\nu] \in H^n_{dR}(X, D)^\vee \]

to get a real number, which we denote by

\[ \int_{\nu} \omega = \langle [\nu], sv [\omega] \rangle \in \mathbb{R} \]

It satisfies the usual rules of integration (bilinearity, change of variables, etc). How to make sense of a ‘dual form’?
Duality

Suppose $X$ smooth projective of dimension $n$, $A \cup B \subset X$ normal crossing divisor. Then Poincaré-Verdier:

$$H^k_{dR}(X \setminus A, B)^\vee \cong H^{2n-k}_{dR}(X \setminus B, A)(n)$$

Use to replace ‘dual forms’ with actual forms.
Suppose $X$ smooth projective of dimension $n$, $A \cup B \subset X$ normal crossing divisor. Then Poincaré-Verdier:

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Use to replace ‘dual forms’ with actual forms.

**Theorem (B.-Dupont 2018)**

Let $\omega \in \Omega^n_X(\log A)$, $\nu \in \Omega^n_X(\log B)$ meromorphic, log. sings. Then

$$\oint_\nu \omega = \frac{1}{(2\pi i)^n} \int_{X(\mathbb{C})} \omega \wedge \overline{\nu},$$

which is Lebesgue integrable.
Duality

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$$\oint_{\nu} \omega = \frac{1}{(2\pi i)^n} \int_{X(\mathbb{C})} \omega \wedge \overline{\nu},$$

which is Lebesgue integrable.

Warning. LHS defined in general. RHS badly defined for non-logarithmic $\omega, \nu$. Kazhdan-Felder. Very subtle issues.
Theorem (B.-Dupont 2018)

In the same setting,

$$\int_{X(\mathbb{C})} \omega \wedge \nu = \sum_{\sigma, \tau} \langle \sigma, \tau \rangle \int_{\sigma} \omega \int_{\bar{\tau}} \nu$$

sum over $\sigma$ a relative homology basis of $H_n(X \setminus A, B)$ and $\tau$ a relative homology basis of $H_n(X \setminus B, A)$.
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‘KLT’ slogan revisited:

“Multiply then integrate = integrate then multiply”
Recall

\[ \log x = \int_1^x \frac{dz}{z} \]

period of

\[ H^1(\mathbb{P}^1 \setminus \{0, \infty\}, \{1, x\}) . \]
Example: logarithm

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$$\log x = \int_1^\infty \frac{dz}{z}$$

period of

$$H^1(\mathbb{P}^1 \setminus \{0, \infty\}, \{1, x\}) .$$

Its single-valued version served two ways:

$$\log |x|^2 = \frac{1}{2\pi i} \int_{\mathbb{P}^1(\mathbb{C})} \frac{dz}{z} \wedge \frac{(\overline{x} - 1)d\overline{z}}{(\overline{z} - 1)(\overline{z} - \overline{x})}$$

$$= \int_1^\infty \frac{dz}{z} . 1 + 1 . \int_1^\infty \frac{dz}{z}$$
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period of

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$$= \int_{1}^{x} \frac{dz}{z} \cdot 1 + 1 \cdot \int_{1}^{\bar{x}} \frac{dz}{z}$$
Moduli spaces $\mathcal{M}_{0,n}$
Moduli of Riemann spheres with \( n \) ordered marked points

Let \( n \geq 4 \).

\[
\mathcal{M}_{0,n} = \{(z_1, \ldots, z_n) \in \mathbb{P}^1 : z_i \neq z_j\}/\text{PGL}_2
\]

Place \( z_1 = 0, z_{n-1} = 1, z_n = \infty \). Then

\[
\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}
\]
Dihedral coordinates

Let $\mathcal{M}_{0,S}$ be moduli of R.S. with marked points indexed by a set $S$. Suppose that $S$ has a dihedral (= cyclic up to reversal) order.

Every chord $c$ in the polygon $S$ determines two consecutive pairs

$$c = \{z_i, z_{i+1}, z_j, z_{j+1}\}$$

Forgetting all other marked points defines a dihedral coordinate

$$u_c : \mathcal{M}_{0,S} \longrightarrow \mathcal{M}_{0,4} \subset \mathbb{P}^1$$
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Example: On \( \mathcal{M}_{0,5} \) there are five such

\[
1 - z_1, \quad \frac{z_1}{z_2}, \quad \frac{z_2 - z_1}{z_2(1 - z_1)}, \quad \frac{1 - z_2}{1 - z_1}, \quad z_2
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Let $\mathcal{M}_{0,S}$ be moduli of R.S. with marked points indexed by a set $S$. Suppose that $S$ has a dihedral (=: cyclic up to reversal) order.

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General dihedral coordinates are *not* cluster coordinates.
Every open string amplitude can be written in dihedral coordinates:

\[
I^{\text{open}} = \int_{X} \Omega \quad \text{where} \quad \Omega = \left( \prod_{c} u_{c}^{s_{c}} \right) \omega
\]

where \( X = \{ 0 < u_{c} < 1 \} \), for some \( \omega \in \Omega^{n}(\log(\partial M_{0,S})) \).
Renormalised amplitudes

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**Theorem (canonical renormalisation) (B.-Dupont '18)**

\[ I^{\text{open}} = \sum_J \frac{1}{s_J} \int_{X_J} \Omega_J^{\text{ren}} \quad \text{where} \quad s_J = \prod_{c \in J} s_c \]

\( J \) are sets of non-crossing chords in \( S \)-gon. Each integrand \( \Omega_J^{\text{ren}} \) is convergent for \( \Re(s_c) > -1 \), so has Taylor expansion.

Poles in \( s_c \) \( \longleftrightarrow \) poles of \( \omega \) along boundary strata of \( \overline{\mathcal{M}}_{0,S} \)
The identical formalism works for closed string amplitudes:

\[ I_{\text{closed}} = \int_{\mathcal{M}_{0,n}(\mathbb{C})} \Omega \quad \text{where} \quad \Omega = \left( \prod_{c} |u_c|^{2s_c} \right)\omega \wedge \bar{\nu}_X \]

for some \( \nu_X \) ‘dual to’ \( X \). We have

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Each term on the right is the single-valued projection of the corresponding term in the renormalisation of the open amplitude, and proves the formula conjectured by Stieberger:

\[ \text{sv } I_{\text{open}} = I_{\text{closed}} \]
Different point of view: $s_{ij}$ as complex numbers, not formal variables.
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\[ \nabla_s = d - \sum_{i<j} s_{ij} \frac{d(z_i - z_j)}{z_i - z_j} \]

Horizontal sections form a local system

\[ \mathcal{L}_s \cong \mathbb{C} \prod_{i<j} (z_i - z_j)^{s_{ij}}. \]
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Apply sv formalism to the self-dual object

$$H_{dR} = H^n_{dR}(\mathcal{M}_{0,S}, \nabla_s \oplus \nabla_{-s}) \quad , \quad H_B = H^n(\mathcal{M}_{0,S}, \mathcal{L}_s \oplus \mathcal{L}_{-s})$$
KLT and SV for cohomology with coefficients

Different point of view: $s_{ij}$ as complex numbers, not formal variables. Koba-Nielsen rank one connection on $\mathcal{M}_{0,S}$:

$$\nabla_\Sigma = d - \sum_{i<j} s_{ij} \frac{d(z_i - z_j)}{z_i - z_j}$$

Horizontal sections form a local system

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Apply $sv$ formalism to the self-dual object

$$H_{dR} = H^n_{dR}(\mathcal{M}_{0,S}, \nabla_\Sigma \oplus \nabla_{-\Sigma}) \quad , \quad H_B = H^n(\mathcal{M}_{0,S}, \mathcal{L}_\Sigma \oplus \mathcal{L}_{-\Sigma})$$

Immediately deduce KLT-type formula involving intersection numbers on $H_B$. The latter were computed by K. Matsumoto, Mimachi-Yoshida, . . . , Mizera, and implies KLT formula.
Genus 1
Consider now the *closed* string amplitudes

\[
\int_{\mathcal{M}_{1,n}(\mathbb{C})} \exp \left( \sum_{i<j} \alpha' s_{ij} G(z_i - z_j) \right)
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The Greens functions involve logarithms of theta functions.
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Strategy (Green, Russo, d’Hoker, Vanhove,...) integrate first in the fiber, to obtain functions on

$$\mathcal{M}_{1,1}(\mathbb{C}) \cong \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$$

Obtain $\text{SL}_2(\mathbb{Z})$-invariant functions of the modulus $\tau \in \mathbb{H}$. 
Modular graph functions

The fiber integrals can be computed explicitly. To every graph $G$, associate a nested lattice sum $I_G(\tau)$. It is a modular-invariant

$$I_G\left(\frac{a\tau + b}{c\tau + d}\right) = I_G(\tau)$$

where the sum is over $(m_1, n_1), (m_2, n_2) \in \mathbb{Z}^2$ such that $(m_1, n_1) \neq (0, 0), (m_2, n_2) \neq (0, 0), (m_1 + m_2, n_1 + n_2) \neq (0, 0)$. 
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Example:

$G =$

![Graph Diagram](image-url)
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Example:

$$I_G = \pi^{-3} \sum_{m_1,n_1,m_2,n_2} \frac{\text{Im}(\tau)^3}{|m_1\tau + n_1|^2|m_2\tau + n_2|^2|(m_1 + m_2)\tau + n_1 + n_2|^2}$$

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- Fourier-type expansion in $q = \exp 2\pi i \tau$.

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- Hierarchical equations with respect to hyperbolic Laplacian.
There exists a natural family $\mathcal{MI}^E$ of non-holomorphic modular forms satisfying all the desired properties (+ more).
A new class of non-holomorphic modular forms

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Idea of the construction: single-valued machine. Take real and imaginary parts of iterated integrals of Eisenstein series

$$G_{2k} = -\frac{b_{2k}}{4k} + \sum_{n \geq 1} \sigma_{2k-1}(n)q^n$$

in such a way as to make them modular.
Modular analogue of Bloch-Wigner dilogarithm:

\[
\text{Im} \int_T^\infty \mathcal{G}_{2a} \mathcal{G}_{2b} - \text{Re} \left( \int_T^\infty \mathcal{G}_{2a} \right) \times \int_T^\infty \mathcal{G}_{2b} \\
+ \text{correction terms involving integrals of cusp forms}
\]
Example

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+ correction terms involving integrals of cusp forms

Related to

- Universal Mixed elliptic motives
- Mock modular forms
- Weak harmonic Maass forms
- Subtle questions in number theory
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- Does the two-tower principle (Grothendieck) play a role: periods in genus 0, 1 generate periods in all higher genera?