Terminal singularities and F-theory

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2018, String Math

University of Pennsylvania

String compactifications

String compactifications

Mathematics

String compactifications

Mathematics (manifolds, complex, algebraic)

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Mathematics (manifolds, complex, algebraic)

F-theory,

String compactifications

Mathematics (manifolds, complex, algebraic)

F-theory,

Calabi-Yau

String compactifications | Mathematics (manifolds, complex, algebraic)

F-theory, | Calabi-Yau (elliptically fibered)

String compactifications — Mathematics (manifolds, complex, algebraic)

F-theory, — Calabi-Yau (elliptically fibered)

Local,

String compactifications — Mathematics (manifolds, complex, algebraic)

F-theory, — Calabi-Yau (elliptically fibered)

Local, Global

String compactifications — Mathematics (manifolds, complex, algebraic)

F-theory, — Calabi-Yau (elliptically fibered)

Local, Global and — Local TO Global principles

From F-theory: We learned: a correspondence

- dim Lie algebras and certain representations
- $\downarrow\uparrow$
- smooth elliptically fibered Calabi-Yau varieties

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⇝:

Rich geometry, physics, with singularities.

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Evidence:

- Prove $\mathcal{R}$, a formula relating [1] and [2] (*local to global principles*) when:
$\dim(X) = 3$, X, elliptic, Calabi-Yau with singularities ($\mathbb{Q}$-factorial terminal)

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Application

Birational extension of Kodaira's classification of singular fibers of relatively minimal elliptic surfaces to higher dimensions.

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Applications

(Strings $\leftsquigarrow$) Outlook in F-theory, SCFT, . . .

Based on work with collaborators:

L. Anderson, P. Arras, M. Cvetič, J. Gray, J. Halverson, C. Long, D. Morrison, P. Oehlmann, F. Ruehle, J. Shaneson, J. Tian, T. Weigand.

References

For Strings, (324) references in
"TASI Lectures on F-theory" Timo Weigand, Jun 5, 2018.

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Caveat.

(Some) Ingredients:

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- invariants of singularities:
  discrepancies (terminal $\cdots$), Minor-Tyurina's numbers, $\cdots$
- Div / Pic, AND $\mathbb{Q}$-factoriality
- Homology , cohomology, topological Euler characteristic, computed via Mayer-Vietoris
- Poincaré duality
- deformations

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Lesson learned from Strings and Math.

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$$\begin{array}{ccc} Y & \hookleftarrow & E_p \simeq T^2 \\ \downarrow \pi & & \downarrow \pi \\ B & \hookleftarrow & \mathcal{U} \ni p \end{array}$$

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$\mathcal{U}$ open dense,

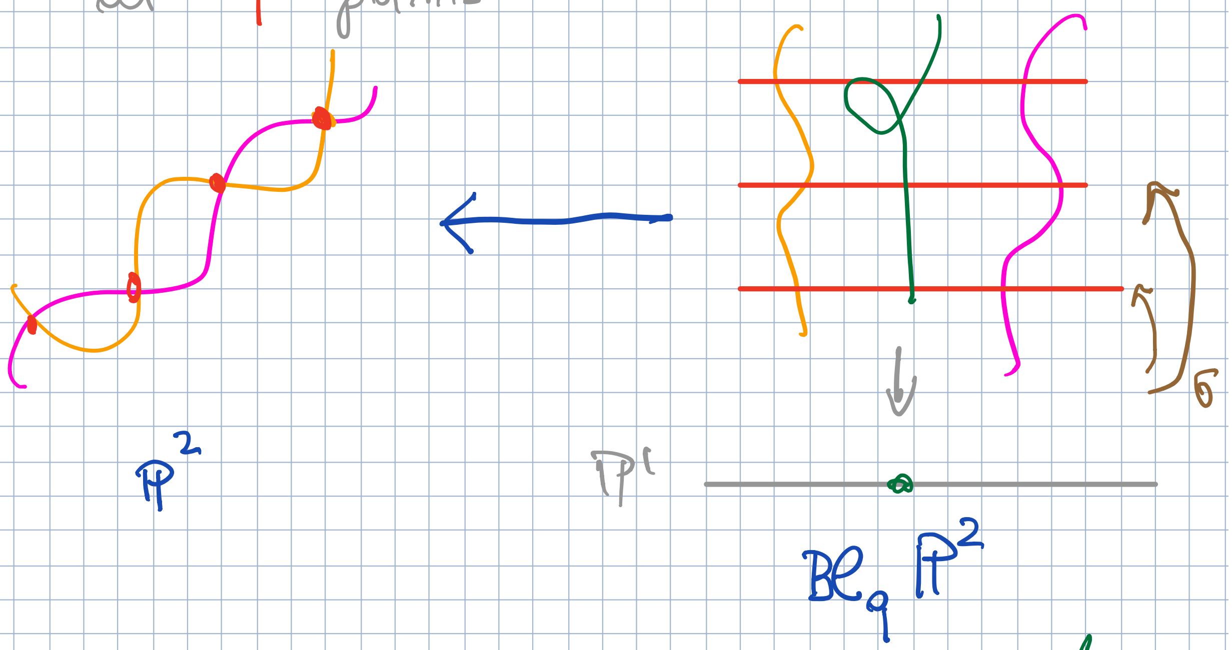
$\Sigma = B \setminus \mathcal{U}$: ramification locus. (Assume: $\Sigma \neq \emptyset$.)

Examples:

- $F = 0 \Leftrightarrow$ plane cubic, ✓ general
  $G = 0 \Leftrightarrow$ plane cubic

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$$
$$z \longmapsto [F(z), G(z)]$$

Not defined when $F = G = 0$, at 9 points



Claim: $\exists$ 12 nodal curves:

(one way) compute: $\chi_{top}(Bl_9 \mathbb{P}^2)$

Σ: 12 points.

- resolution X :

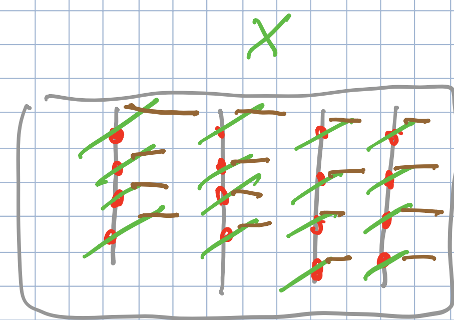
- $(T^2 \times T^2)/\mathbb{Z}/2\mathbb{Z} = \bar{X}$

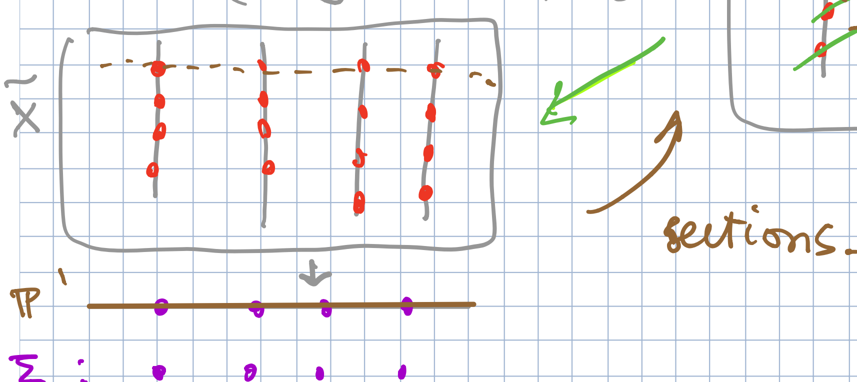
$$\downarrow$$

$$\mathbb{P}^1 = T^2/\mathbb{Z}/2\mathbb{Z}$$

Action:

$$(x, y) \mapsto (-x, -y)$$





- Many examples as hypersurfaces in toric varieties.

- Many examples as CICY

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- Birational geometry: X, such that $\dim(X) = n$ and $\kappa(X) = n - 1$ is birationally equivalent to a genus one fibration (Iitaka's fibration)

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- Study of Calabi-Yau X: $\exists$... with genus one fibrations
- F-theory "compactifications" ($\Sigma \leftrightarrow$ 7-branes of II-B on B.)

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Theorem (Nakayama)

Elliptic fibrations $X \overset{bir.}{\leftarrow}$ *Weierstrass models* W.

Example: $\dim(W) = 2$: $W : y^2 = x^3 + f(u)x + g(u), \quad u \in \mathbb{C}$;

$\Sigma : 4f^3 - 27g^2 = 0$.

$f(u)$ and $g(u)$ determine:
the type of singularity of W **AND** the fiber over Σ in the minimal resolution $X \to W$.

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Kodaira's classification of singular fibers of minimal elliptic surfaces :

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Kodaira's classification of singular fibers of minimal elliptic surfaces :

extended Dynkin diagrams when there is a section:

R. Miranda - elliptic surfaces

I.4: The List of possible singular fibers.

Kodaira

Before beginning to address the issues mentioned above, I would like to give a series of examples to illustrate some of the features of the theory to the reader. It will be useful for the purposes of illustration and communication for the reader to know the possible singular fibers which can occur, and Kodaira's names for them. This I present below, without proof, simply so that I can speak of them intelligently in the examples to follow.

(I.4.1) Table of possible singular fibers of a smooth minimal elliptic surface. The names are those used by Kodaira.

Name	Fiber
I_0	smooth elliptic curve
I_1	nodal rational curve
I_2	two smooth rational curves meeting transversally at two points
I_3	three smooth rational curves meeting in a cycle; a triangle
I_N, $N\geq 3$	N smooth rational curves meeting in a cycle, i.e., meeting with dual graph $\tilde{A}_N$
I_N^*, $N\geq 0$	N+5 smooth rational curves meeting with dual graph $\tilde{D}_{N+4}$
II	a cuspidal rational curve
III	two smooth rational curves meeting at one point to order 2
IV	three smooth rational curves all meeting at one point
IV^*	7 smooth rational curves meeting with dual graph $\tilde{E}_6$
III^*	8 smooth rational curves meeting with dual graph $\tilde{E}_7$
II^*	9 smooth rational curves meeting with dual graph $\tilde{E}_8$
${}_MI_N$, $N\geq 0$	topologically an I_N, but each curve has multiplicity N

All components of reducible fibers have self-intersection -2; the irreducible fibers have self-intersection 0, of course.

The dual graphs referred to above are those of the extended Dynkin diagrams. For ease of reference I'll give below tables of the Dynkin diagrams and the extended Dynkin diagrams.

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Let $\tilde{\pi} : \tilde{X} \to \tilde{B}$ be any genus one fibration. There is a (bir.) commutative diagram:

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$K_X = \pi^*(K_B + \Delta)$,

$X(\stackrel{bir}{\sim} \tilde{X})$ *with at most: $\mathbb{Q}$-factorial, terminal singularities*

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C. *$dim(\tilde{X}) \leqslant 3$.*

Applications

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2. If X is Calabi-Yau: $X \rightleftarrows (B, \Sigma)$.

Definition

X is $\mathbb{Q}$-factorial if any Weil divisor is $\mathbb{Q}$-Cartier.

Example

1. $\mathbb{Q}$-factorial: X, toric: every cone in the fan is simplicial.
2. $X \subset \mathbb{P}^4$ of equation $x_0 g_0 + x_1 g_1 = 0$ is NOT $\mathbb{Q}$-factorial

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$K_Y = f^*(K_X) + \sum_k b_k E_k$ with $b_k > 0$ ($b_k \geqslant 0$, $b_k > -1$) and E_k exceptional divisors

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$\rightsquigarrow$ "non-Calabi-Yau resolvable singularities"

Singularities play a central role in string theory

$\rightsquigarrow$ Gauge symmetries

$\rightsquigarrow$ Wrapping branes or strings on shrinking cycles

$\rightsquigarrow$ Transitions through singularities

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It is important, possible and interesting **to work directly with singularities**

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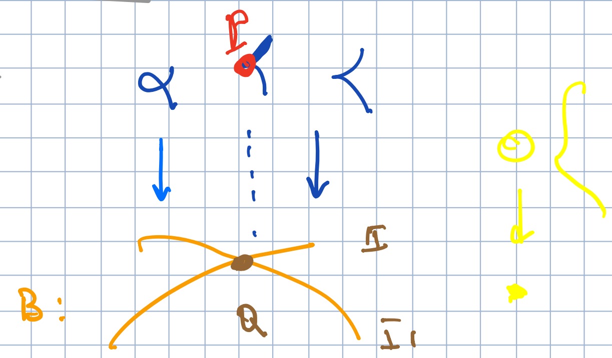
Transverse intersections of nodal divisors on 3-folds ($\mathbb{Q}$-factorial conifold).

It governs general behavior of the Jacobian of genus-one fibrations.

Example: dim $X = 3$.

1.



. X is smooth except at $P \in \pi^{-1}(Q)$.

• local equation around P:

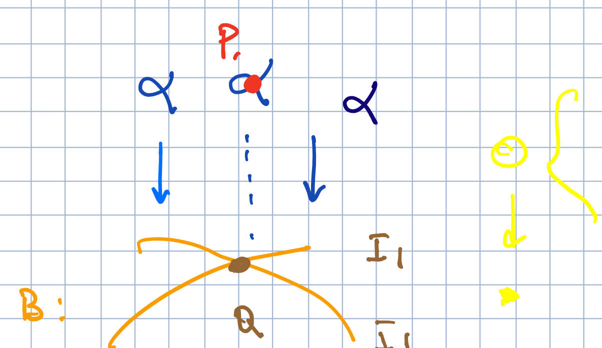
$$w_1^3 + w_2^2 + w_3^2 + w_4^2 = 0$$

• If $f : Y \to X$ is a resolution:

$$K_Y = f^*(K_X) + \Sigma a_i E_i \qquad a_i \geq 0$$

(terminal)

2.



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3. Algebras, representations, geometry

For W:

Kodaira fiber	I_n	II	III	IV	I_n^*	IV^*	III^*	II^*
algebra: A-D-E	$\mathrm{su}(n)$	$\{e\}$	$\mathrm{su}(2)$	$\mathrm{su}(3)$	$\mathrm{so}(2n+8)$	e_6	e_7	e_8

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Conjectures-Theorem (slice): *Brieskorn-Grothendieck*, 1970

$$X \xrightarrow{\pi} B$$

$$B \hookleftarrow \Sigma \hookleftarrow \Sigma^1$$

smooth, codim1

Stratified discriminant locus Σ

$$\begin{array}{ccccc} X & & & & \\ \downarrow{\scriptstyle \pi} & & & & \\ B & \hookleftarrow & \Sigma & \hookleftarrow & \Sigma^1 \\ & & & & \uparrow \\ & & & & \text{smooth, codim1} \end{array} \qquad \begin{array}{ccccc} X & & & & \\ \downarrow{\scriptstyle \pi} & & & & \\ B & \hookleftarrow & \Sigma \backslash \Sigma^1 & \hookleftarrow & \Sigma^2 \\ & & & & \uparrow \\ & & & & \text{smooth, codim2} \end{array}$$

etc.

Math: "A Brieskorn-Grothendieck program", local, global, local to global

$X \to B$, X $\mathbb{Q}$-factorial, klt, B smooth in codimension 2.

- Semi-simple Lie algebras $\mathfrak{g}$ and some of their representations
  $\leftrightsquigarrow$ geometry of genus one fibrations and degenerations of fibers,
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Implement it:

1. Verify consistency with physics expectations
2. Prove results in mathematics.

Analysis in the String Literature, $\dim(X) = 3$ (from $\sim$ mid 90s)

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Physics constructions use X smooth;

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verify consistencies for X Calabi-Yau (anomalies cancellations formulae)

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   Deform to nodal fibers.

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· Example (I₂) $\gamma_1 = \gamma_3$

$J = (1, 0, -1, 0, \dots 0) \in \mathbb{Z}^N$

· $\sum J_j [\Gamma_j] = [\Gamma_1] - [\Gamma_3] \leftrightarrow S^2 \in H_2(X)$

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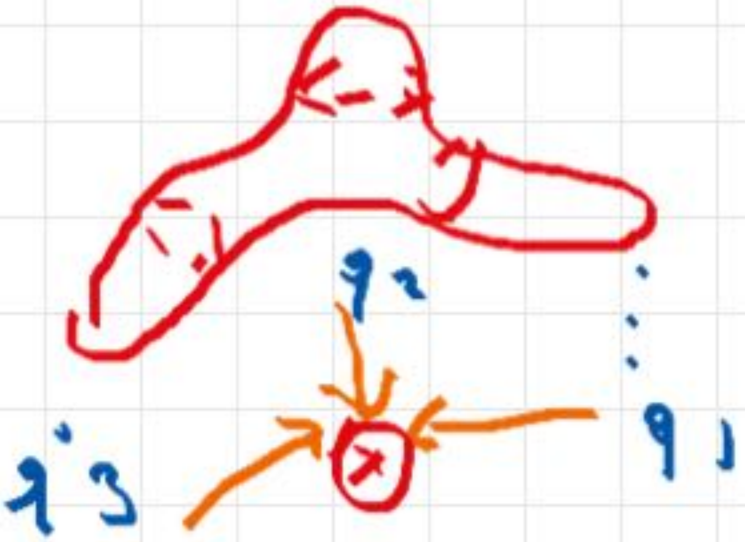

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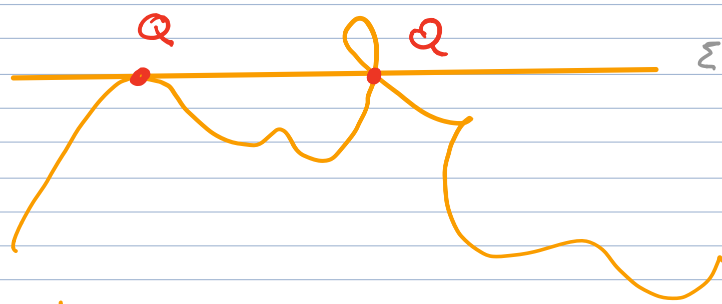
X X, CY $\mathbb{Q}$-Factorial, terminal.

$$\begin{array}{c} X \\ \downarrow \\ B \end{array}$$

General conditions :

B :



Σ_i : genus g

$$\left[\begin{array}{l} \Sigma \supset \Sigma^1 \\ \Sigma \supset \Sigma \setminus \Sigma_1 \supset \Sigma^2 \end{array}\right.$$

stratification.

$\Sigma \setminus \Sigma_1$: general.

Correspondence, Calabi-Yau with $\mathbb{Q}$-factorial terminal singularities

Σ, stratified $\longleftrightarrow$ Algebras, representations and multiplicity

$\Sigma^1 \rightarrow$ Algebras, representations: adj$(g-1)$

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$\Sigma^2 \rightarrow$ Tyurina's number of P.

Number	Type	$\mathfrak{g}$	ρ_0	$\rho_{Q_1^\ell}$	$\rho_{Q_2^\ell}$	$(\dim \mathrm{adj})_{ch}$	$(\dim \rho_0)_{ch}$	$\dim(\rho_{Q_1^\ell})_{ch}$	$\dim(\rho_{Q_2^\ell})_{ch}$
1	I_1	$\{e\}$		–	–	0	0	0	0
2	I_2	su(2)		–	fund	2	0	0	2
3	I_3	su(3)		–	fund	6	0	0	3
4	$I_{2k},\ k \geqslant 2$	sp(k)	Λ_0^2	–	fund	$2k^2$	$2k^2-2k$	0	$2k$
5	$I_{2k+1},\ k \geqslant 1$	sp(k)	$\Lambda^2 + 2 \times$ fund	$\frac{1}{2}$ fund	fund	$2k^2$	$2k^2+2k$	k	$2k$
6	$I_n,\ n \geqslant 4$	su(n)		Λ^2	fund	n^2-n	0	$\frac{1}{2}(n^2-n)$	n
7	II	$\{e\}$		–		0	0	0	
8	III	su(2)		$2 \times$ fund		2	0	4	
9	IV	sp(1)	$\Lambda^2 + 2 \times$ fund	$\frac{1}{2}$ fund		2	4	1	
10	IV	su(3)		$3 \times$ fund		6	0	9	
11	I_0^*	$\mathfrak{g}_2$	**7**	–		12	6	0	
12	I_0^*	spin(7)	vect	–	spin	18	6	0	8
13	I_0^*	spin(8)		vect	spin$_\pm$	24	0	8	8
14	I_1^*	spin(9)	vect	–	spin	32	8	0	16
15	I_1^*	spin(10)		vect	spin$_\pm$	40	0	10	16
16	I_2^*	spin(11)	vect	–	$\frac{1}{2}$ spin	50	10	0	16
17	I_2^*	spin(12)		vect	$\frac{1}{2}$ spin$_\pm$	60	0	12	16
18	$I_n^*,\ n \geqslant 3$	so($2n+7$)	vect	–	NM	$2(n+3)^2$	$2n+6$	0	NM
19	$I_n^*,\ n \geqslant 3$	so($2n+8$)		vect	NM	$2(n+3)(n+4)$	0	$2n+8$	NM
20	IV^*	$\mathfrak{f}_4$	**26**	–		48	24	0	
21	IV^*	$\mathfrak{e}_6$		**27**		72	0	27	
22	III^*	$\mathfrak{e}_7$		$\frac{1}{2}$**56**		126	0	28	
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In 1, 5, 7 : $\mathbb{Q}$-factorial terminal singularities at P;
topologically the fiber through P is the same as the general fiber over Σ_1.
In 7 the singularity induces a non-trivial representation associated to Q_2^ℓ.

4. Local, global invariants, local to global principles, $\mathcal{R}$ and applications

Definition (The charged dimension of ρ)

$\mathfrak{h}$ Cartan of $\mathfrak{g}$. $(\dim\rho)_{ch} = \dim(\rho) - \dim(ker\,\rho|_{\mathfrak{h}})$.

Example: $(\dim \mathrm{adj})_{ch} = \dim\mathfrak{g} - \dim\mathfrak{h} = \dim\mathfrak{g} - rk\mathfrak{h} = \ .$

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$(\mathcal{U}, 0) \subset \mathbb{C}^{n+1}$, isolated hypersurface singularity $P = 0$, defined by $f = 0$.

Definition (The Milnor number of P)

$$m(P) = \dim_{\mathbb{C}}(\mathbb{C}\{x_1, \ldots, x_{n+1}\}/< \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n+1}} >) \,.$$

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$$\tau(P) = \dim_{\mathbb{C}}(\mathbb{C}\{x_1, \ldots, x_{n+1}\}/ < f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n+1}} >).$$

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Saito: $\tau(P) = m(P) \leftrightarrow P$ is a weighted hypersurface singularity.

The formula $\mathcal{R}$: local to global.
Application: Global to local, *Birational* Kodaira classification of "singular fibers".

Theorem

$30K_B^2 + \frac{1}{2}\left(\chi_{top}(X) + \sum_P m(P)\right)$, *$P$ singular of X with Milnor number $m(P)$, is independent of the choice of the particular minimal model X.*

Theorem (Local to Global (simplified version))

$30K_B^2 + \frac{1}{2}\left(\chi_{top}(X) + \sum_P m(P)\right) =$
$= (g-1)(\dim \mathrm{adj})_{ch} + (g'-g)(\dim \rho_0)_{ch} + \sum_Q (\dim \rho_Q)_{ch} + \sum_P \tau(P)$

Birational extension of Kodaira's classification of singular fibers of relatively minimal elliptic surfaces to higher dimensions (codimension one and two strata):

Theorem

$30K_B^2 + \frac{1}{2}\left(\chi_{top}(X) + \sum_P m(P)\right)$, *P singular of X with Milnor number* $m(P)$, *is independent of the choice of the particular minimal model X.*

Theorem (Local to Global (simplified version))

$$30K_B^2 + \frac{1}{2}\left(\chi_{top}(X) + \sum_P m(P)\right) =$$
$$= (g-1)(\dim \mathrm{adj})_{ch} + (g'-g)(\dim \rho_0)_{ch} + \sum_Q (\dim \rho_Q)_{ch} + \sum_P \tau(P)$$

Birational extension of Kodaira's classification of singular fibers of relatively minimal elliptic surfaces to higher dimensions (codimension one and two strata):

Theorem

The dimensions of the representations *are birational invariants of the minimal model of the elliptic fibrations.*

In fact,

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$\mathrm{KaDef}(X)$, $\mathrm{CxDef}(X)$ are birational invariants of the minimal model.

Application: *Birational* Kodaira classification of "singular fibers"

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[↑ Hodge duality, and Poincaré duality.]

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Need: Poincaré dualtiy (with singularities).

Also $m(P)$ is a birational invariant of the minimal model

Conjecture

(1) It always holds.

(2) It holds for Multiple fibers as well.

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(2) For Calabi-Yau $\rightleftarrows$ discrete gauged symmetry (Anderson-Grassi-Gray-Ohelmann)

Applications

Cancellations of anomalies unveiled and extended to the singular cases.

The anomaly cancellation requirements lead to:

1. (Global:) $0 = (n_H - n_V + 273 - 29 n_T)$ and $0 = (\frac{9-n_T}{8})$
   n_T: tensor multiplets, n_V vector multiplets, n_H hypermultiplets.
2. (Local:) Conditions on Tr_{adj} in adjoint representation, Tr_ρ in *suitable* representations ρs, with multiplicities, and local geometries around Σ^1 and Σ^2.

Dictionary:

$$\begin{aligned} n_V &= \dim(G), \\ n_T &= h^{1,1}(B) - 1 \rightsquigarrow \mathrm{KaDef}(X) \\ n_H &= H_{ch} + \mathrm{CxDef}(X) + 1, \end{aligned}$$

$X \to B$, X, Calabi-Yau, smooth, B smooth.

The anomaly cancellation requirements lead to:

$$h^{2,1}(X) + 1 + H_{ch} - \dim(G) = 273 - 29n_T.$$

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AND

Analysis in the String Literature, $\dim(X) = 3$ (from $\sim$ mid 90s)

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$9 - n_T = K_B^2$ (always verified, Noether's formula)

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Applications: Strings

$\leftrightarrows$ propose and extend cancellations of anomalies in the presence of singularities.

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We also verify it in other, different, examples.

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Opportunities with Singularities.

Towards: higher dimensions.

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Consistent quantum theory:

$\rightsquigarrow$

the "anomalies" of this theory MUST VANISH.

Schwarz: (N=1 theories in six dimensions with a semisimple group G)

The anomaly polynomial:

$$\kappa \cdot A \cdot trR^4 + B \cdot (\mathrm{tr}\, R^2)^2 + \frac{1}{6}\,\mathrm{tr}\, R^2 \sum X_i^{(2)} - \frac{2}{3} \sum X_i^{(4)} + 4 \sum_{i<j} Y_{ij}$$

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$$X_i^{(n)} = \mathrm{Tr}_{\mathrm{adj}}\, F_i^n - \sum_\rho n_\rho\, \mathrm{Tr}_\rho\, F_i^n \qquad\qquad Y_{ij} = \sum_{\rho,\sigma} n_{\rho\sigma}\;\; \mathrm{Tr}_\rho\, F_i^2\;\; \mathrm{Tr}_\sigma\, F_j^2,$$

n_ρ is a *suitable* multiplicity in the matter representation, and $n_{i,j}$ multiplicity of

representation (ρ, σ) of $G_i \times G_j$.

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2. Dictionary:

$$n_V = \dim(G),$$
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$$n_H = H_{ch} + h^{2,1}(X) + 1,$$

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The anomaly vanishes if:

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AND

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Green-Schwarz says (**??**) vanishes if it can be written as:

$$\left(\mathbf{D}\operatorname{tr} R^2 + \sum \mathbf{D_i}\operatorname{tr} F_i^2\right)\cdot\left(\mathbf{E}\operatorname{tr} R^2 + \sum \mathbf{E_i}\operatorname{tr} F_i^2\right). \tag{3}$$

The anomaly vanishes if:

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Sadov say (**??**) can be taken as:

$$\frac{1}{2}\left(\frac{1}{2} K_B \operatorname{tr} R^2 + 2 \sum \boldsymbol{\Sigma}_i \operatorname{tr} F_i^2\right) \cdot \left(\frac{1}{2} K_B \operatorname{tr} R^2 + 2 \sum \boldsymbol{\Sigma}_i \operatorname{tr} F_i^2\right).$$

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$$\begin{aligned}
9 - n_T &= K_B^2 \quad (\text{always verified, Noether's formula}) \\
-6 K_B \cdot \boldsymbol{\Sigma}_i (\mathrm{tr}\, F_i^2) &= -\,\mathrm{Tr}_{\mathrm{adj}}\, F_i^2 + \sum_\rho n_\rho\, \mathrm{Tr}_\rho\, F_i^2 \\
3 \boldsymbol{\Sigma}_i{}^2 (\mathrm{tr}\, F_i^2)^2 &= -\,\mathrm{Tr}_{\mathrm{adj}}\, F_i^4 + \sum_\rho n_\rho\, \mathrm{Tr}_\rho\, F_i^4 \\
\boldsymbol{\Sigma}_i \cdot \boldsymbol{\Sigma}_j (\mathrm{tr}\, F_i^2)(\mathrm{tr}\, F_j^2) &= \sum_{\rho,\sigma} n_{\rho\sigma}\ \ \mathrm{Tr}_\rho\, F_i^2\ \ \mathrm{Tr}_\sigma\, F_j^2
\end{aligned}$$

Grothendieck
Brieskorn, ICM, Nice, 1970

Theorem: $\bar{G}$ Lie group/$\mathbb{C}$, of type A-D-E.

$\exists\, S \subset X \subset \bar{G}$, $\dim S = 2$,

$\chi_{|X} : X \longrightarrow \mathbb{C}^r$ such that

(i) S has an isolated Du Val singularity, of type A-D-E.

(ii) $\chi_{|X} : X \longrightarrow \mathbb{C}^r$ is a semiuniversal deformation of S

Example:

$G = SU(2)$ $\tilde{G} = SL(2, \mathbb{C})$

$X = \tilde{G}$

$S = (\chi^{-1} \cdot \chi(e))$

Text

$$= \left\{ \begin{bmatrix} 1+\alpha & \beta \\ \gamma & 1+\delta \end{bmatrix} \text{ s.t. } \begin{matrix} \alpha+\delta=0 \\ \alpha\delta-\gamma\beta=0 \end{matrix} \right\}$$

$$S \longleftrightarrow \{ z_1^2 + z_2 z_3 = 0 \} \subset \mathbb{C}^3$$

Theorem:

1. $\exists$: $W \rightsquigarrow W_0$ deformation s.t.

$$\begin{array}{ccc} W & \rightsquigarrow & W_0 \\ \pi \downarrow & & \downarrow \pi_0 \\ M & = & M \end{array}$$

$\pi_0^{-1}(q)$: model (I_1), $\forall\, q \in \Sigma$

(Recall : $\Sigma = \pi^{-1}(0)$)

2. $J^{(-2)} \overset{\text{def}}{=} \{ J \text{ s.t. } \langle J, J \rangle = -2 ,\ a(J) = 0 \}$

$\rightarrow \# J^{(-2)}$ = roots of Lie algebra

(of the Dynkin diagram classically associated to $\pi^{-1}(0)$.)

3. $\exists\ \{\alpha_1, \dots \alpha_r\} \subset J^{(-2)}$ s.t.

$\langle \alpha_j, \alpha_k \rangle$ is the Negative Cartan

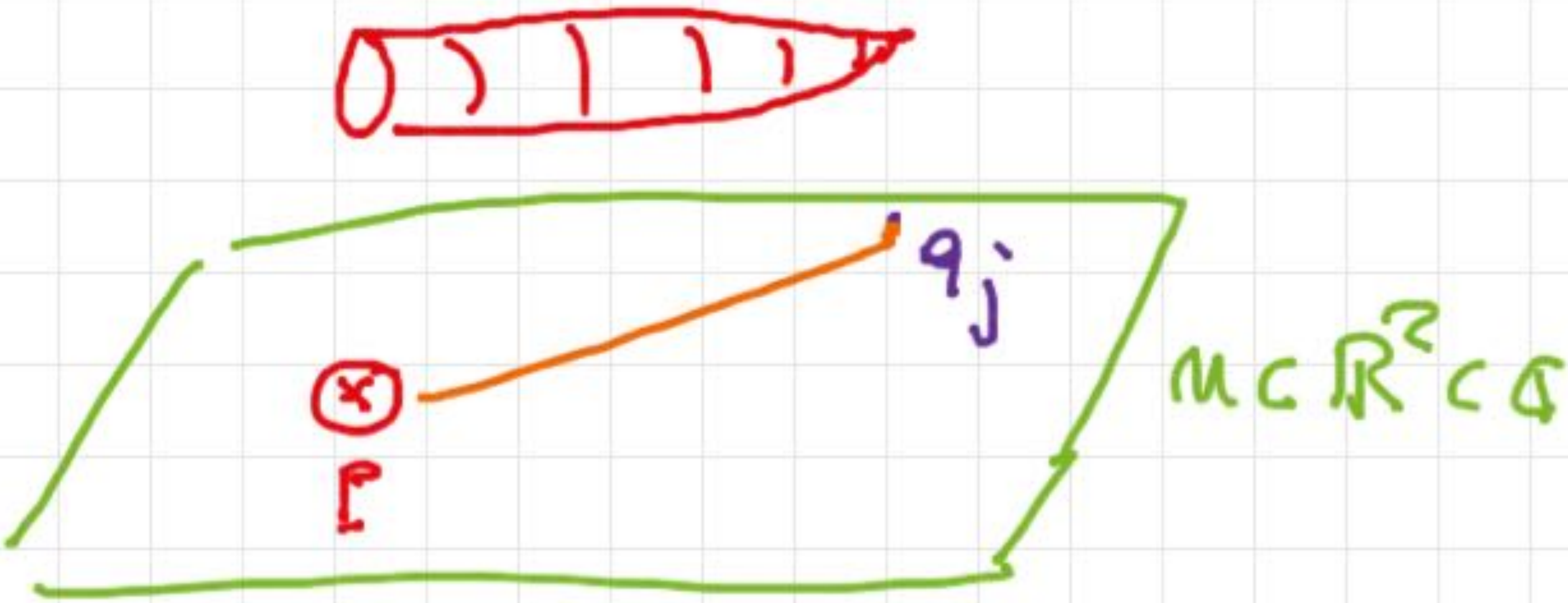
↳ We also derive (computer aided):
entire representation structure
of the Lie algebras.

- Application of 4:
to higher dimensional elliptic
fibrations

- What is new:
(De Wolfe - Zwiebach)
(Bonora - Savelli)

Vanishing cycles determine

thimbles (prongs) in $H_1(X; E_p)$



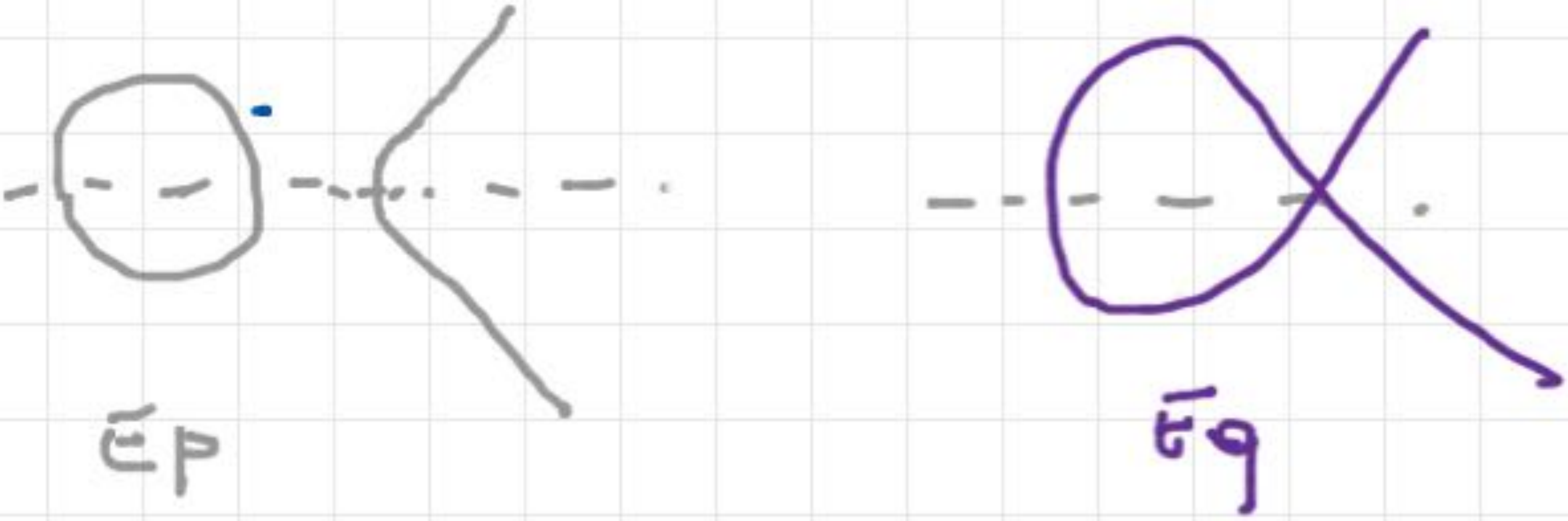
Example: $X = W$ weierstrass equation:

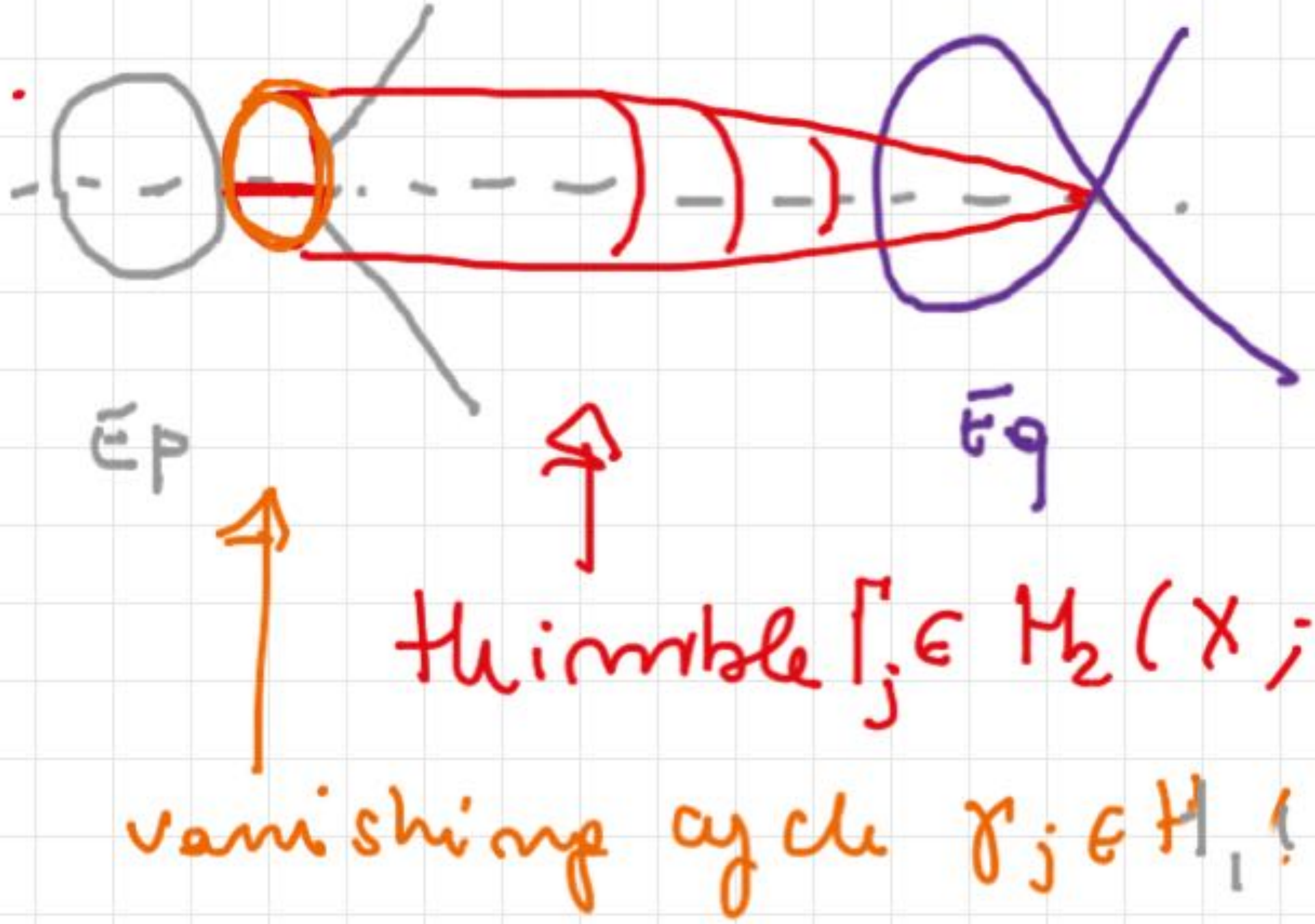
$\downarrow$ $\quad y^2 = x^3 + fx + g$

u

- E_p: double cover of $\mathbb{C}$ branched at 3 roots ⊕ ∞ of RHS

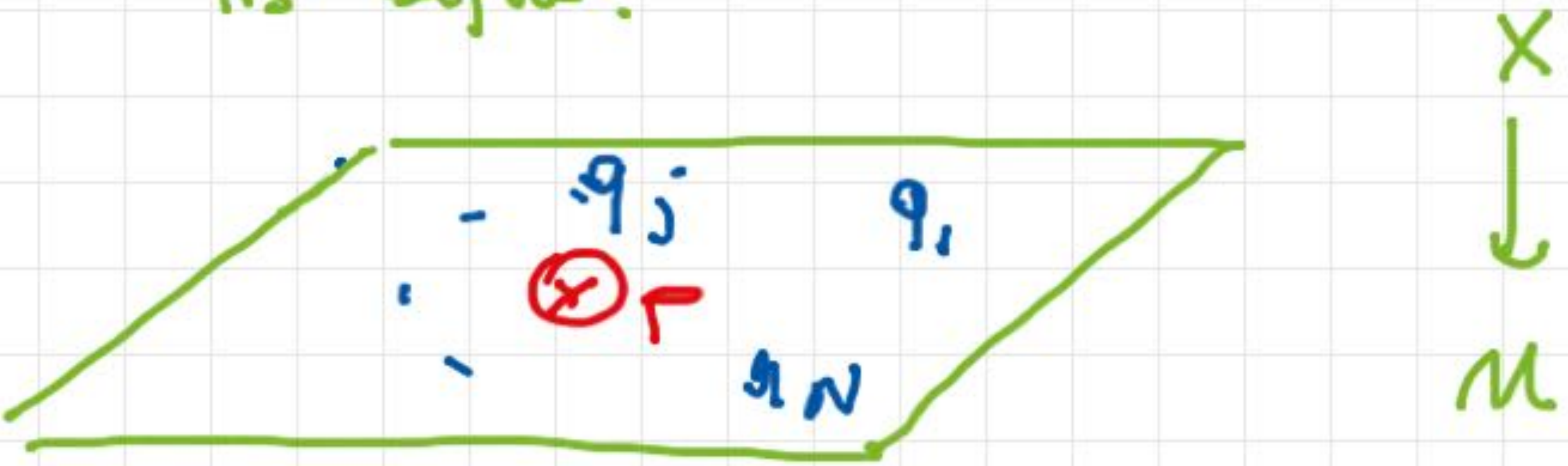
- E_{q_j}: two roots merge.





<u>Proposition</u>: We can explicitly determine γ_j, Γ_i and the above intuitive picture can be made accurate.

As before:



$\pi^{-1}(p) = E_p$ smooth

$\pi^{-1}(q_j) =$ nodal curve $j = 1, \ldots, N$

Def: $J = (J_1, \ldots, J_N) \in \mathbb{Z}^N$, junction.

$\rightarrow [J]_p = \sum_{j=1}^{N} J_j [\Gamma_j] \in H_2(X, E_p)$

$\rightarrow \partial [J_p] = \sum J_j [\gamma_j] \in H_1(E_p, \mathbb{Z})$

Note: $\partial [J_p] = a(J)$

asymptotic charge

· Example (I_2) $\gamma_1 = \gamma_3$

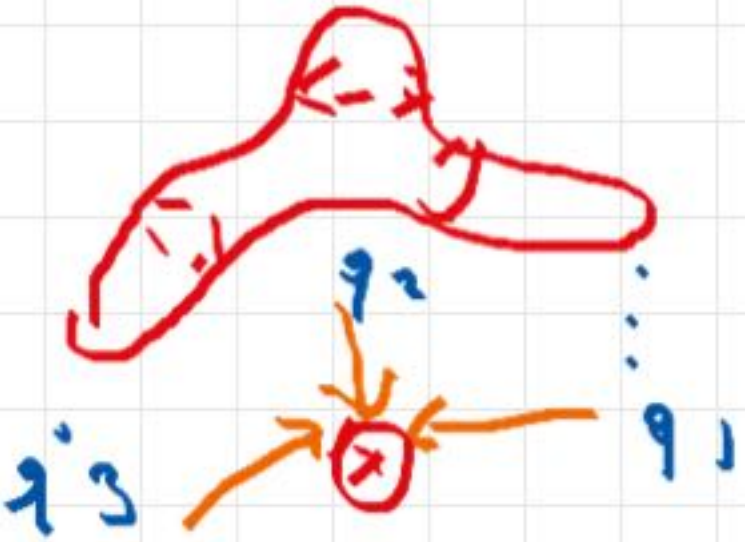
$J = (1, 0, -1, 0, \dots 0) \in \mathbb{Z}^N$

· $\sum J_j [\Gamma_j] = [\Gamma_1] - [\Gamma_3] \leftrightarrow S^2 \in H_2(X)$

· $\partial(J) = \gamma_1 - \gamma_3 = 0 \in H_1(X)$

· Example (III) : $\gamma_1 + \gamma_2 + \gamma_3 = 0$

$J = (1, 1, 1, 0 \text{ — } 0)$



· $\sum J_j [\Gamma_j] = [\Gamma_1 + \Gamma_2 + \Gamma_3] \leftrightarrow S^2 \in H_2(X)$

· $\partial(J) = \gamma_1 + \gamma_2 + \gamma_3 = 0$

Theorem: $a(\delta)=0$

$\to \sum \delta_j [\Gamma_j] \in H_2(X, \mathbb{Z})$.

Def: $\langle J, J\rangle = \sum_{j>k\geq 2} J_k J_j \, \gamma_k \cdot \gamma_j - \sum J_k^2$

intersection in $H_1(\Sigma, \mathbb{Z})$

intersection over Σ.

Theorem: If $a(J)=0$:

$\langle J, J\rangle = [J][J]$, topological intersection in $H_2(X, \mathbb{Z})$.

Note: there is an induced pairing:

$$\langle J, K\rangle = \frac{1}{2}\langle J+K, J+K\rangle - \langle J, J\rangle - \langle K, K\rangle.$$

$\rightarrow$ If $a(J) = a(K) = 0 \rightarrow \langle J, K\rangle = [J]\cdot[K]$