Hecke Relations in Rational Conformal Field Theory

(with Y. Wu, arXiv:1804.06860)

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OUTLINE

1. Introduction
2. Data
3. RCFT
4. Scalar Hecke operators
5. Hecke operators for RCFT Characters
6. Applications and Examples
7. Summary, Questions, Speculations

INTRODUCTION

This talk concerns a new relation between characters of two-dimensional rational conformal field theory (RCFT) given by Hecke operators.

The relation generalizes previously known Galois symmetry relations between the representations of the modular group provided by RCFT characters.

I will discuss these new relations, explain how they explain a number of scattered results in the literature and present some possible applications.

Example:

The Yang-Lee model is a non-unitary minimal model $\mathrm{M}(5,2)$ with two independent characters

$$\chi_0^{YL} = q^{-1/60} \sum_{n=0} c_0^{YL}(n) q^n$$

$$\chi_{1/5}^{YL} = q^{11/60} \sum_{n=0} c_{1/5}^{YL}(n) q^n$$

Affine G_2 is a unitary rational CFT with two independent characters (aka Fibonacci anyon in CMT)

$$\chi_0^{G_2} = q^{-7/60} \sum_{n=0} c_0^{G_2}(n) q^n$$

$$\chi_{17/60}^{G_2} = q^{17/60} \sum_{n=0} c_{17/60}^{G_2}(n) q^n$$

Although apparently unrelated, there is a subtle relation between the character coefficients:

DATA

coefficients of q expansion of h=1/5 character of Yang-Lee model c=-22/5

$n\backslash k$	1	2	3	4	5	6	7
0	0	1	1	1	1	2	2
1	3	3	4	4	6	6	8
2	9	11	12	15	16	20	22
3	26	29	35	38	45	50	58
4	64	75	82	95	105	120	133
5	152	167	190	210	237	261	295
6	324	364	401	448	493	551	604
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26	9054025	9574106	10121903	10700327	11309427	11952388	12629349
27	13343681	14095665	14888948	15723845	16604348	17530906	18507742

Table 4: Coefficients $c^{YL}_{1/5}(7n+k)$

coefficients of q expansion of vacuum character of affine G2 at level one (c=14/5) divided by 7

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260072/7	1/7
62202	
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166450	
266850	
422966	
662780	
7198178/7	1/7
1579853	
2406046	
3633082	
5443362	
8094202	
11952388	
122716344/7	2/7

RCFT

Hilbert space: $$\mathcal{H} = \bigoplus_{i,\bar{i}} \mathcal{N}_{i,\bar{i}} V_i \otimes \overline{V_{\bar{i}}}$$

Representation of chiral algebra, e.g. Virasoro for minimal models

Characters: $$\chi_i(\tau) = \mathrm{Tr}_{V_i} q^{L_0 - c/24}, \qquad q = e^{2\pi i \tau}$$

Partition function: $$Z(\tau) = \sum_{i \in \mathcal{I}, \bar{i} \in \bar{\mathcal{I}}} \mathcal{N}_{i,\bar{i}} \chi_i(\tau) \overline{\chi_{\bar{i}}(\tau)}$$

Examples: Ising model, Yang-Lee model, affine Lie algebras, Monster VOA, BM VOA (details later)

Modular Properties

Characters are weakly holomorphic weight zero vector-valued modular functions transforming according to

$$\rho : SL(2,\mathbb{Z}) \to GL(n,\mathbb{C})$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; (\tau \to -1/\tau) \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \; (\tau \to \tau + 1)$$

$$(\rho(S))^2 = (\rho(S)\rho(T))^3 = C \qquad C^2 = 1$$

$$N = \mathrm{order}(\rho(T)) \qquad \Gamma(N) \subset \ker(\rho)$$ (Bantay)

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | \; a = d = 1 \bmod N, \; b = c = 0 \bmod N \right\}$$

Example

Ising Model

$$\chi_0^I = \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) = q^{-1/48}(1 + q^2 + q^3 + 2q^4 + 2q^5 + \cdots),$$

$$\chi_{1/2}^I = \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) = q^{23/48}(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + \cdots),$$

$$\chi_{1/16}^I = \frac{1}{\sqrt{2}}\sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}} = q^{1/24}(1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + \cdots).$$

$$\rho(S) = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \qquad \chi_i(-1/\tau) = \sum_j \rho(S)_{ij}\chi_j(\tau)$$

$$\rho(T) = e^{-2\pi i/48}\mathrm{diag}(1, e^{2\pi i/2}, e^{2\pi i/16}) \qquad \chi_i(\tau+1) = \sum_j \rho(T)_{ij}\chi_j(\tau)$$

$$N = 48$$

Fusion Algebra and Verlinde Formula

$$\phi_i \times \phi_j = N_{ij}{}^k \phi_k$$

$$N_{ij}{}^k = \sum_m \frac{\rho(S)_{im}\rho(S)_{jm}\bar{\rho}(S)_{km}}{\rho(S)_{0m}} \in \mathbb{Z}_{\geq 0}$$

vacuum representation

For some RCFT the representations ρ and fusion matrices $N_{ij}{}^k$ are related by Galois symmetry.

Galois Symmetry

K=Field obtained by adjoining matrix elements of $\rho(\gamma)$ to $\mathbb{Q}$

De Boer & Goeree: K is a finite Abelian extension of $\mathbb{Q}$

Kronecker-Weber: $K \subseteq \mathbb{Q}[\zeta_m]$ minimal m=conductor

N=conductor (also of RCFT). $\mathrm{Gal}(\mathbb{Q}[\zeta_N]) \cong (\mathbb{Z}/N\mathbb{Z})^\times$

Primitive Nth root of unity

Group of units in $\mathbb{Z}/N\mathbb{Z}$

Coste-Gannon: For each $\ell \in (\mathbb{Z}/N\mathbb{Z})^\times$

$$f_{N,\ell} : \rho(T) \to \rho(T)^\ell$$

$$f_{N,\ell} : \rho(S)_{i,j} \to \varepsilon_\ell(i)\rho(S)_{\pi_\ell(i),j} = \varepsilon_\ell(j)\rho(S)_{i,\pi_\ell(j)} \qquad \varepsilon = \pm 1$$

$$\rho(S) \to G_\ell \rho(S)$$

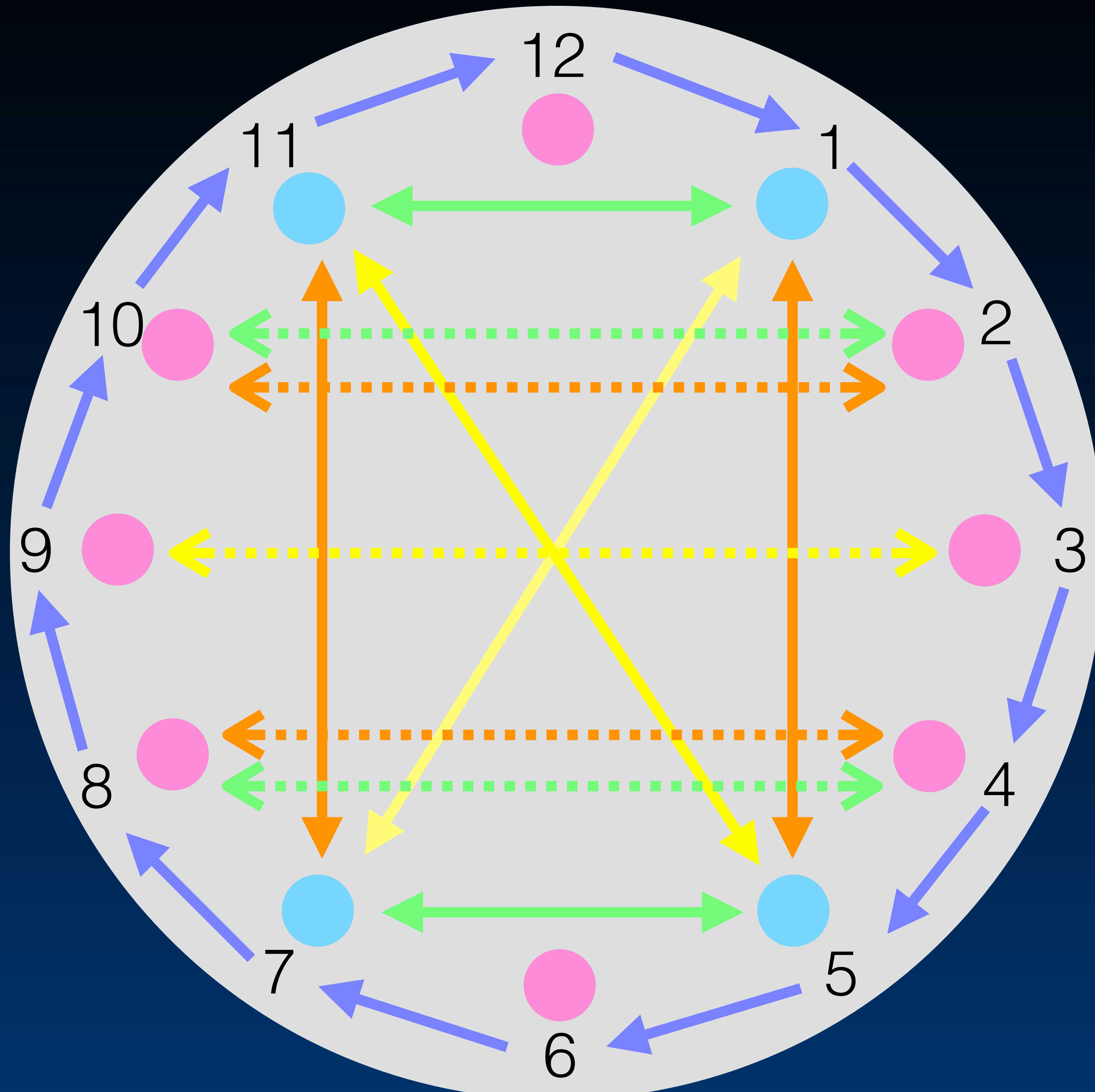
permutation of indices

$\mathbb{Z}/12\mathbb{Z}$

$(\mathbb{Z}/12\mathbb{Z})^{\times}$

$5 \quad 7 \quad 11$

$f_5 \quad f_7 \quad f_{11}$



Examples of Galois RCFT Relations

$$\rho(S)^{YL} = \begin{pmatrix} -\frac{1}{2\sin(\pi/5)} & \frac{1}{2\sin(2\pi/5)} \\ \frac{1}{2\sin(2\pi/5)} & \frac{1}{2\sin(\pi/5)} \end{pmatrix} \qquad \rho(S)^{G_2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rho(S)^{YL}$$

$$\xrightarrow{f_{60,7}}$$

$$\rho(T)^{YL} = \mathrm{diag}(e^{2\pi 11/60}, e^{-2\pi i/60}) \qquad \rho(T)^{G_2} = (\rho(T)^{YL})^7$$

Similarly Yang-Lee $\xrightarrow{f_{60,13}}$ F_4

Three character RCFT Ising $\xrightarrow{f_{48,47}}$ Baby Monster

These are relations between modular representations.
We will extend them to relations between **characters**.

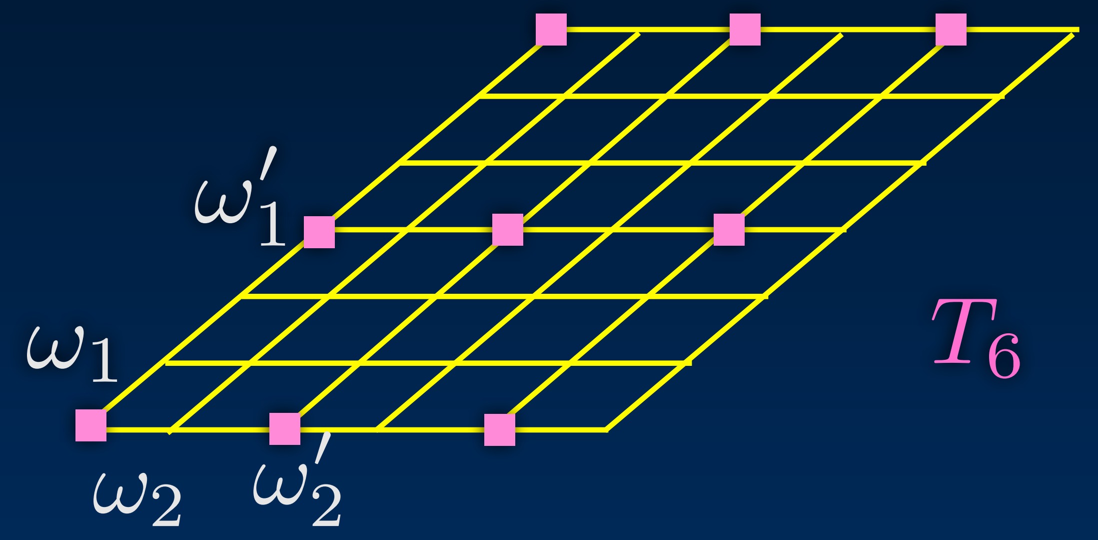
Scalar Hecke

Modular forms can be thought of in two ways:

Functions of $\tau \in \mathbb{H}$ $\quad f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$

Functions of rank 2 lattices $\quad F(\lambda L) = \lambda^{-k} F(L)$

$$F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \omega_2^{-k} f(\omega_1/\omega_2)$$



Hecke operator: $\quad (T_n F)(L) = \sum_{\substack{L' \subset L \\ |L/L'|=n}} F(L')$

$$\omega_1' = a\omega_1 + b\omega_2$$
$$\omega_2' = c\omega_1 + d\omega_2 \qquad ad - bc = n$$

Hecke Operators

$$\mu = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mu\tau = \frac{a\tau + b}{c\tau + d} \qquad (f|_k\mu)(\tau) = \frac{(\det(\mu))^{k/2}}{(c\tau + d)^k} f(\mu\tau)$$

$$(T_p f)(\tau) = p^{k/2-1} \sum_{\mu \in SL(2,\mathbb{Z})\backslash\mathcal{M}_p} (f|_k\mu)(\tau)$$ (p prime for simplicity)

Action on Fourier coefficients:

$$f = \sum_n a(n) q^n \qquad (T_p f)(\tau) = \sum_n a^{(p)}(n) q^n$$

$$a^{(p)}(n) = \begin{cases} p^k a(pn) & \text{if } p \nmid n, \\ p^{k-1}\big(p\, a(pn) + a(n/p)\big) & \text{if } p|n. \end{cases}$$

In math these are often applied to weight k>0 modular forms and used to study cusp forms which are eigenfunctions of the Hecke operators. E.g. the unique weight 12 cusp form (vanishing as $\tau \to i\infty$)

$$\Delta = \eta^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} c(n) q^n$$

$$T_n \Delta = c(n) \Delta$$

The modularity theorem used in the proof of Fermat's last theorem associates a weight 2 Hecke eigenform to each elliptic curve over the rationals.

In RCFT we are interested in the action on weight 0, weakly holomorphic modular functions which are never Hecke eigenfunctions.

To generalize Hecke operators to RCFT characters:

Use Hecke operators for $\Gamma(N)$ (Rankin, Chap. 9)

Use modular representation properties of RCFT characters.

This leads to the following formula for Hecke images of RCFT characters for (p,N)=1:

N.B.

On Fourier coefficients:
$$\chi_i(\tau) = \sum_n b_i(n) q^{n/N}$$
$$(\mathsf{T}_p\chi)_i(\tau) = \sum_n b_i^{(p)}(n) q^{n/N}$$

$$b_i^{(p)}(n) = \begin{cases} p b_i(np) & p \nmid n \\ p b_i(np) + \sum_b \rho_{ij}(\sigma_p) b_j(n/p) & p | n \end{cases}$$

The $(\mathsf{T}_p\chi)_i$ are again modular forms for $\Gamma(N)$ but transform under a different representation of $SL(2,\mathbb{Z})$

Representation of Hecke image $\quad \rho^{(p)}(S) = \rho(\sigma_p S), \qquad \rho^{(p)}(T) = \rho(T^{\bar{p}})$

where σ_p is the pre-image of $\begin{pmatrix} \bar{p} & 0 \\ 0 & p \end{pmatrix}$ under the mod N map $SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}/N\mathbb{Z})$ and $\bar{p}p = 1 \bmod N$

Example: $N = 60, p = 7, \bar{p} = 43 \qquad \sigma_7 = \begin{pmatrix} 27343 & -33780 \\ 480 & -593 \end{pmatrix}$

The change of representation under Hecke is the same as that under Galois for $\ell = p, \ (p, N) = 1$.

The equivalence relies on the identities

$$f_{N,p}(\rho(S)) = \rho(\sigma_{\bar{p}} S)$$
$$f_{N,p}(\rho(T)) = \rho(T^p)$$

Applications and Examples

The example from the first DATA slide:

$$\chi_0^{YL} = q^{-1/60} G(q) = q^{-1/60} \sum_{n=0}^{\infty} c_0^{YL}(n) q^n$$

$$\chi_{1/5}^{YL} = q^{11/60} H(q) = q^{11/60} \sum_{n=0}^{\infty} c_{1/5}^{YL}(n) q^n$$

Then we have the Hecke relation $\chi^{G_2} = \mathsf{T}_7 \chi^{YL}$

$$c_0^{G_2}(n) = \begin{cases} 7c_{1/5}^{YL}(7n-1) & \text{if } 7 \nmid n, \\ 7c_{1/5}^{YL}(7n-1) + c_0^{YL}(\frac{n}{7}) & \text{if } 7 | n; \end{cases} \qquad \rho^{YL}(\sigma_7) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$c_{2/5}^{G_2}(n) = \begin{cases} 7c_0^{YL}(7n+2) & \text{if } 7 \nmid (n-1), \\ 7c_0^{YL}(7n+2) - c_{1/5}^{YL}(\frac{n-1}{7}) & \text{if } 7 | (n-1). \end{cases}$$

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662780	
7198178/7	1/7
1579853	
2406046	
3633082	
5443362	
8094202	
11952388	
122716344/7	2/7

The Hecke action preserves the dimension of the representation of the modular group. We thus look for relations between RCFT characters for models with the same number of independent characters.

The characters of a RCFT with n independent characters satisfy an n*th* order Modular Linear Differential Equation (MLDE). Alternatively, we can use MLDE to search for possible characters of new RCFTs.

Developed in physics literature by Anderson&Moore, Eguchi&Ooguri, Mathur, Mukhi & Sen, Naculich, Bantay, …and in math by Kaneko & Zagier, Franc, Mason, Gannon, Kaneko, Arike, Nagatomo, Sakai, …

Modular Linear Differential Equations

quasi modular weight 2
Eisenstein series

Ramanujan-Serre: $\mathcal{D}_k = d/d\tau - \frac{1}{6} i\pi k E_2$

$$\mathcal{D}_k : M_k(\Gamma) \to M_{k+2}(\Gamma) \qquad \mathcal{D}^n = \mathcal{D}_{2n-2}\mathcal{D}_{2n-4}\cdots\mathcal{D}_2\mathcal{D}_0$$

nth order MLDE: $$\mathcal{D}^n f + \sum_{k=0}^{n-1} \phi_k(\tau)\mathcal{D}^k f = 0$$

weight 2(n-k)

What properties should the coefficient functions have? RCFT characters have poles only at q=0 but this need not be true for the ϕ_k

From standard theory of differential equations we have

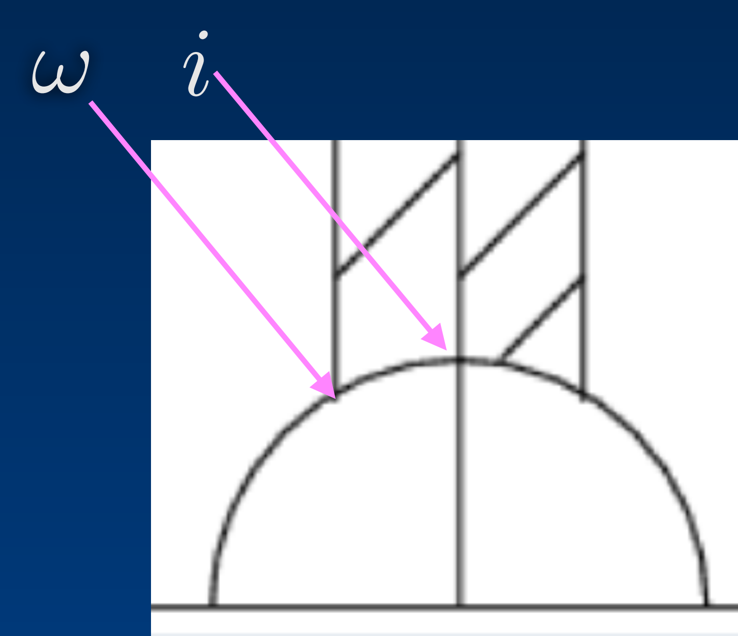
$$\phi_k = (-1)^{n-k} W_k / W \qquad W = W_n$$

no poles — may have zeroes

$$W_k = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ \mathcal{D} f_1 & \mathcal{D} f_2 & \cdots & \mathcal{D} f_n \\ \vdots & \vdots & & \vdots \\ \mathcal{D}^{k-1} f_1 & \mathcal{D}^{k-1} f_2 & \cdots & \mathcal{D}^{k-1} f_n \\ \mathcal{D}^{k+1} f_1 & \mathcal{D}^{k+1} f_2 & \cdots & \mathcal{D}^{k+1} f_n \\ \vdots & \vdots & & \vdots \\ \mathcal{D}^{n} f_1 & \mathcal{D}^{n} f_2 & \cdots & \mathcal{D}^{n} f_n \end{vmatrix},$$

$$\ell(W) = 6\left(\frac{1}{2}\mathrm{ord}_i(W) + \frac{1}{3}\mathrm{ord}_\omega(W) + \sum_{p\in\mathcal{F}}{}' \mathrm{ord}_p(W)\right)$$ =number of zeros of W

$$\mathrm{ord}_\infty(W) + \frac{\ell(W)}{6} = \frac{n(n-1)}{12}$$



Mathur-Mukhi-Sen classified n=2, $\ell(W) = 0$ solutions

with $\mathrm{D} = \frac{1}{2\pi i}\frac{d}{d\tau}$ we have

Second order: $\mathrm{D}^2 f - \frac{E_2}{6}\mathrm{D} f - \frac{\mu E_4}{4} f = 0$

MMS: $X = \{YL, A_1, A_2, G_2, D_4, F_4, E_6, E_7, E_{7\frac{1}{2}}\}$

Yang-Lee model

Affine level 1 characters Deligne exceptional series

Characters of Intermediate Vertex Subalgebra (Kawasetsu)

Hecke relations in addition to $\chi^{G_2} = \mathsf{T}_7 \chi^{YL}$

$$\chi^{F_4} = \mathsf{T}_{13}\chi^{YL} \qquad \chi^{E_7} = \mathsf{T}_7 \chi^{A_1} \qquad \chi^{E_{7\frac{1}{2}}} = \mathsf{T}_{19}\chi^{YL}$$

(Proof uses Sturm bound)

Two Character Models

Character with leading singularity as $q \to 0$

$$\chi_0 \sim q^{-c_{eff}/24}$$

$$\chi_{h_{eff}} \sim q^{-c_{eff}/24+h_{eff}}$$

$$h_{eff}(X) = \{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\}$$

Farey series: $F_5 = \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}$

We don't have an explanation for this curious fact.

Mathur, Mukhi & Sen classified n=2, $\ell(W)=0$ solutions, but solutions exist for $\ell(W)>0$ and are Hecke images of $\ell(W)=0$ solutions since T_p changes $\mathrm{ord}_\infty(W)$

p	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71
$\ell^{YL}(p)$	0	2	0	2	0	2	2	0	0	2	0	2	2	2	6	6	8

Table 1: Number of zeros in the modular Wronskian for Hecke images under T_p of Yang-Lee characters for small values of p.

- These Hecke images have negative coefficients
- These Hecke images have positive coefficients and as RCFT characters appear in work of Naculich and Hampapurma & Mukhi

p	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47	49	53	57
$\ell^{I}(p)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	6	6

Table 1: Number of zeros in the modular Wronskian for Hecke images under T_p of Ising characters for small values of p.

Other Examples

Three character theories:

No classification I am aware of. Examples include

Minimal models: $\mathcal{M}_{4,3}\ (\text{Ising})\,,\ \mathcal{M}_{5,2}^{\otimes 2}\ (\text{YL}^{\otimes 2}),\ \mathcal{M}_{7,2}$

Hampapura-Mukhi explored three-character RCFT w/o Kac-Moody symmetry and found examples with c=47/2, 164/5, 236/7. These are all Hecke images:

$$\chi^{c=47/2} = \mathsf{T}_{47}\chi^{(4,3)} \qquad \text{BabyMonster}$$

$$\chi^{c=164/5} = \mathsf{T}_{41}\chi^{(5,2)^{\otimes 2}}$$

$$\chi^{c=236/7} = \mathsf{T}_{59}\chi^{(7,2)}$$

Duality of H-M implied by Hecke relations

One can use Hecke images to construct families of possible RCFT characters—non-holo analogs of extremal CFT of Höhn/Witten. If

$$p = 7, 13, 47, 53 \bmod 60$$

then $T_p \chi^{YL}$

1. Has non-negative integer coefficients in q expansion.

2. The vacuum appears with degeneracy one.

3. The fusion coefficients from Verlinde are non-negative.

Consistency with Virasoro requires

$$\chi_{N,p} = \mathsf{T}_{60*N+p}\chi^{YL} + \sum_{k=0}^{N-2} d(k)\mathsf{T}_{p+60k}\chi^{YL}$$

$$d(k) \geq c(k) \qquad \frac{1}{\prod_{n=2}^{\infty}(1-q^n)} = \sum_{n=0}^{\infty} c(n)q^n$$

Summary and Questions

There is a hidden symmetry relating characters of many different RCFTs based on the mathematical theory of Hecke operators.

The relation generalizes previously known Galois symmetry relations between the representations of the modular group provided by RCFT characters.

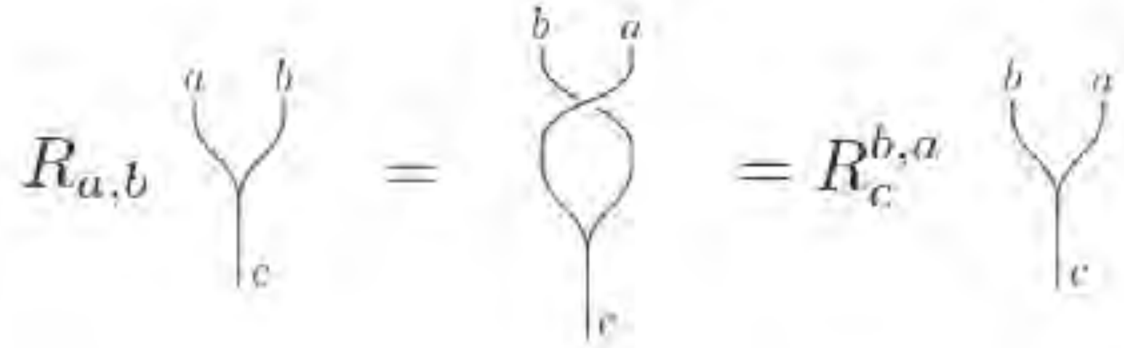
These Hecke relations appear to be very common. Other examples we are exploring where they appear include RCFT with $n>3$ characters, rational Gaussian models/lattice VOAs and Gepner models.

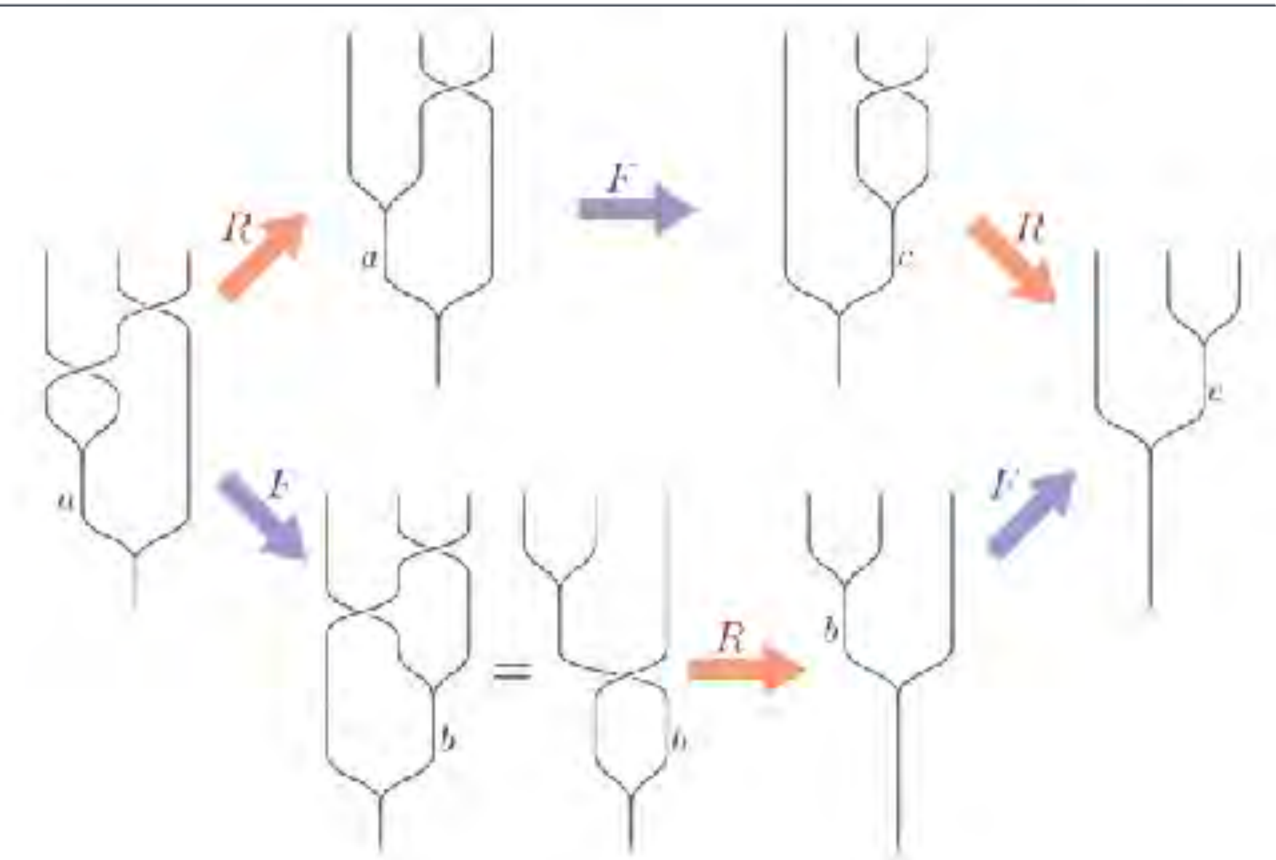
Do these Hecke operators have a natural physical origin?

Is there a nice theory of how Hecke operators relate the divisors of the Wronskians of MLDE?

Do these Hecke relations relate the full RCFT or just their characters? If the former, this indicates there are new symmetries acting on the space of RCFTs with a strong number theoretic flavor.

Note that for Yang-Lee and affine G2 theories with characters related by T_7 the braiding and fusion matrices are related by the associated Frobenius transformation $f_{60,7}$.





Hexagon relation for R,F

(See Barkeshli's talk)

Yang-Lee $\qquad g = (1+\sqrt{5})/2 \xrightarrow{f_{60,7}} -1/g \qquad$ Affine G2

$$F_{\phi}^{\phi\phi\phi} = \begin{pmatrix} -g & -ig^{1/2} \\ -ig^{1/2} & g \end{pmatrix} \xrightarrow[f_{60,7}]{} F_{\tau}^{\tau\tau\tau} = \begin{pmatrix} g^{-1} & g^{-1/2} \\ g^{-1/2} & -g^{-1} \end{pmatrix}$$

RCFTs and their fusion algebras, modular tensor categories, characters etc. appear in many places in condensed matter physics:

Boundary modes of QHE systems and topological insulators.

Tool for computing and studying Entanglement Entropy.

Quantum computation.

It will be interesting to see if these new Hecke relations have implications in the real world.

THANK YOU

Hecke Operators for $\Gamma(N)$

Double Coset: $\Gamma_1\alpha\Gamma_2 = \{\gamma_1\alpha\gamma_2 | \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}$

Γ_1, Γ_2 congruence subgroups

$$f[\Gamma_1\alpha\Gamma_2]_k = \sum_j f|_k\delta_j \qquad \Gamma_1\alpha\Gamma_2 = \bigcup_j \Gamma_1\delta_j$$

$$M_k(\Gamma_1) \to M_k(\Gamma_2)$$

Apply this with $\Gamma_1 = \Gamma_2 = \Gamma(N)$

$$\alpha = \alpha_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$$

Hecke Operators for $\Gamma(N)$

Define $\beta_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, $U_\nu = \begin{pmatrix} 1 & \nu N \\ 0 & p \end{pmatrix}$

Hecke Operators for $\Gamma(N)$, (p,N)=1

$$(T_p f)(\tau) \equiv f[\Gamma(N)\alpha_p\Gamma(N)]_k(\tau) = \sum_{\delta \in \Delta_N^{(p)}} (f|_k\delta)(\tau)$$

Rankin
Chap. 9

$$\Delta_N^{(p)} = \{\sigma_p\beta_p; U_b, 0 \leq b \leq p-1\}$$

where σ_p is the pre-image of $\begin{pmatrix} \bar{p} & 0 \\ 0 & p \end{pmatrix}$ under the mod

N map $SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}/N\mathbb{Z})$ and $\bar{p}p = 1 \bmod N$

Hecke Operators for $\Gamma(N)$

Example: $N = 60, p = 7, \bar{p} = 43$ $\sigma_7 = \begin{pmatrix} 27343 & -33780 \\ 480 & -593 \end{pmatrix}$

The term $(f|_k\sigma_p)(p\tau)$ does not allow for any simple action on Fourier coefficients.

Main problem with extending Hecke operators to vector-valued modular forms/functions:

Representation ρ is defined on elements of $SL(2,\mathbb{Z})$ but Hecke operators require action of elements of $GL(2,\mathbb{Z})$.

Brunier-Stein, Raum Weil representations, representation on $\mathbb{C}(\Delta_N^{(p)})$

Hecke Operators for RCFT Characters

Recall for RCFT characters $\chi_i, \quad i = 1, 2, \cdots n = \dim(V)$

$$\chi_i(\gamma\tau) = \sum_j \rho(\gamma)_{ij}\chi_i(\tau) \qquad \gamma \in SL(2,\mathbb{Z}) \qquad \Gamma(N) \subset \ker(\rho)$$

In particular $\chi_i(\sigma_p p\tau) = \sum_j \rho(\sigma_p)_{ij}\chi_j(p\tau)$

σ_p is only defined up to the action of $\Gamma(N)$ but since $\Gamma(N) \subset \ker(\rho)$, $\rho(\sigma_p)$ is well defined.

We can simply reinterpret the Hecke operators for $\Gamma(N)$ to get Hecke operators on RCFT characters with vector structure.

Hecke for RCFT

$$(\mathsf{T}_p f)_i(\tau) := \sum_{\delta\in\Delta_N^{(p)}} f_i(\delta\tau) = \sum_j \rho_{ij}(\sigma_p) f_j(p\tau) + \sum_{b=0}^{p-1} f_i\left(\frac{\tau+bN}{p}\right)$$

$$(p,N)=1$$

On Fourier coefficients: $\chi_i(\tau) = \sum_n b_i(n) q^{n/N}$

$$(\mathsf{T}_p\chi)_i(\tau) = \sum_n b_i^{(p)}(n) q^{n/N}$$

$$b_i^{(p)}(n) = \begin{cases} p b_i(np) & p\nmid n \\ p b_i(np) + \sum_b \rho_{ij}(\sigma_p) b_j(n/p) & p|n \end{cases}$$

N.B. unconventional normalization preserves integrality of coefficients at weight k=0.

Hecke for RCFT

The proof that the components are again modular forms for $\Gamma(N)$ is the standard one from e.g. Diamond-Shurman

Acting with $\gamma_2 \in \Gamma(N)$ permutes the orbit space

$$\Gamma(N)\backslash\Gamma(N)\alpha_p\Gamma(N)$$

by right multiplication and gives an equivalent set of orbit representatives δ .

While acting with the generators S,T of $SL(2,\mathbb{Z})$ gives

$$\Delta_N^{(p)} \circ T = T^{\bar{p}} \circ \Delta_N^{(p)},$$

$$\Delta_N^{(p)} \circ S = \sigma_p S \circ \Delta_N^{(p)}$$

Representation of Hecke image

$$\rho^{(p)}(S) = \rho(\sigma_p S), \qquad \rho^{(p)}(T) = \rho(T^{\bar{p}})$$