Perverse schobers and applications

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Perverse schobers and applications

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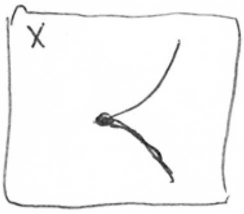
"Perverse $\underbrace{\text{sheaves of}}$ triangulated $\underbrace{\text{categories}}$"

stacks : over-used term

Schober = stack in German.

① Perverse sheaves (X, S) Stratified $\mathbb{C}$-manifold

$S = \{X_\alpha\}$ smooth, $\overline{X}_\alpha$ may be singular



$Sh(X,S)$ S-constructible sheaves $\mathcal{F}$

$\mathcal{F}|_{X_\alpha}$ loc. const

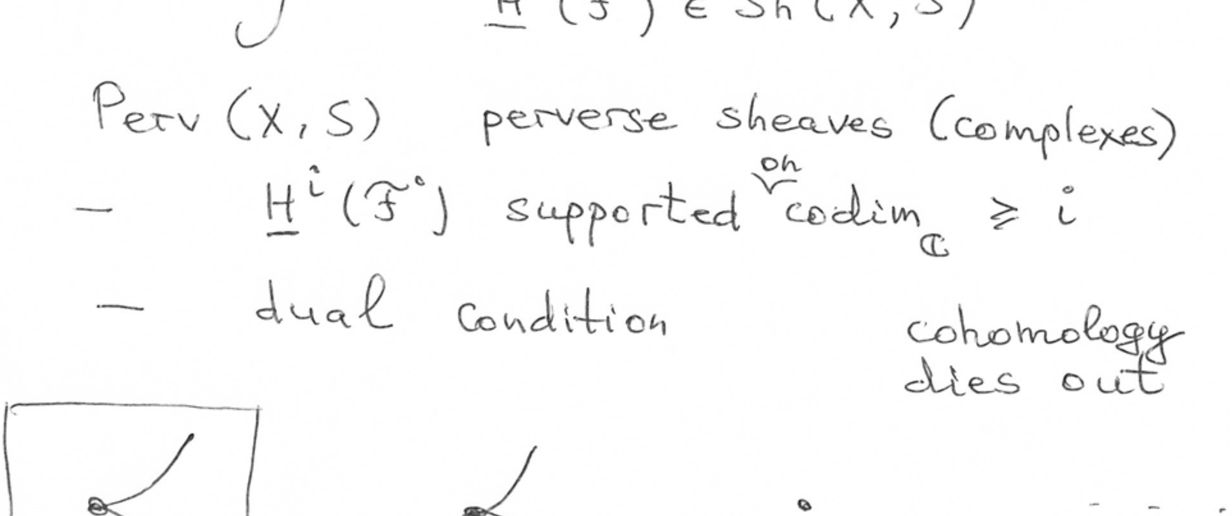
$D^b(X,S)$: constructible complexes $\mathcal{F}^\bullet$

$\underline{H}^i(\mathcal{F}^\bullet) \in Sh(X,S)$

abelian $\quad \mathrm{Perv}(X,S)$ perverse sheaves (complexes)

- $\underline{H}^i(\mathcal{F}^\bullet)$ supported on $\mathrm{codim}_{\mathbb{C}} \geq i$
- dual condition

cohomology dies out



$\underline{H}^0$ everywhere $\qquad \underline{H}^1$ here $\qquad \underline{H}^2$ here $\qquad H^3 = 0$

② "Categorification"

Grothendieck group K

Numbers

?

Vector spaces V (systems of) - - - ? - - -> Triang. Categories $\mathcal{V}$ (systems of)

$(a,b) \in \mathbb{C}$

$\mathrm{Hom}(A,B) \in \mathrm{Vect}$

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[dl] \\ C \arrow[u, "+1"] & \end{tikzcd}$$

C "=" Cone(f) : categorified difference

$V = K(\mathcal{V}) \otimes \mathbb{C}$

$[A]-[B]+[C]=0$

additive relations - - - ? - - -> systems of triangles

<u>Want</u>: Categorify perverse sheaves

Sheaf of (triangul.) categories OK
dg, stable..

Complexes of categories ??

(3) <u>Quiver descriptions</u>: results of the form

$$\mathrm{Perv}(X, S) \sim \mathrm{Rep}(\text{some Quiver w. relations})$$

<u>Example</u> $\mathrm{Perv}(D, 0) \sim$ either

(a) $$\Phi \underset{v}{\overset{u}{\rightleftarrows}} \Psi \qquad \mathrm{Id}_\Phi - vu,\ \mathrm{Id}_\Psi - uv \text{ invertible}$$

(b) $$E_- \underset{\delta_-}{\overset{\gamma_-}{\leftrightarrows}} E_0 \underset{\delta_+}{\overset{\gamma_+}{\rightleftarrows}} E_+ \qquad \begin{array}{l} \gamma_-\delta_- = \mathrm{Id} \\ \gamma_+\delta_- \text{ invert.} \end{array} \quad + \text{ symm.}$$

(a) categorifies to:

<u>Spherical functors</u>

$$\mathcal{C}_0 \underset{v = u^* \text{ adjoint}}{\overset{u}{\rightleftarrows}} \mathcal{C}_1 \qquad \begin{array}{l} \mathrm{Cone}\{uv \to \mathrm{Id}_{\mathcal{C}_1}\} \\ \mathrm{Cone}\{\mathrm{Id}_{\mathcal{C}_0} \to uv\} \end{array} \text{ equivalence}$$

$\underline{\underline{\text{def}}}$: perverse schobers on $(D, 0)$.

(4) Why do we want perverse schobers?

They appear (or can be used) as:

1. Coefficients for Fukaya categories
2. Data encoding derived categories of flopped varieties
3. Structure describing:
   categorified Picard-Lefschetz theory
   "algebra of the infrared" of Gaiotto-Moore-Witten

⑤ Fukaya categories and perversity

$H_n(M, \mathbb{C})$ or H^n

with

(ℓ, ℓ') intersect. form

$\underline{H}^n_K(\underline{\mathbb{C}}_M) = R_K$

Purity: $\underline{H}^{\neq n}_K(\mathbb{C}_M) = 0$

in many cases (graphs, Nadler's arboreals,)

$H^0(K, R_K) =$

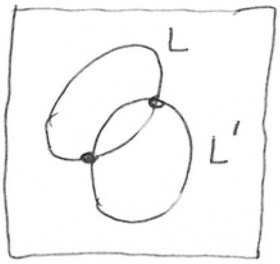
$= H_K(M, \mathbb{C})$

local cohom.

$H^n(M, \mathcal{F})$

sheaf? complex?...

For purity on K's $\mathcal{F}$ must be perverse



(M^{2n}, ω) sympl, compact

$\mathbb{F}(M)$

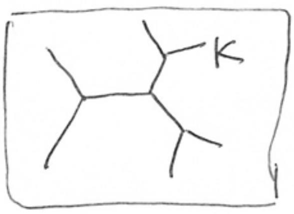
Objects $L \subset M$ Lagr.

$\mathrm{Hom}(L, L') = (\mathbb{C}^{L \cap L'}, d)$

when transverse

Non-local!

Kontsevich: localization on (singular) Lagr. skeleton $K \subset M$



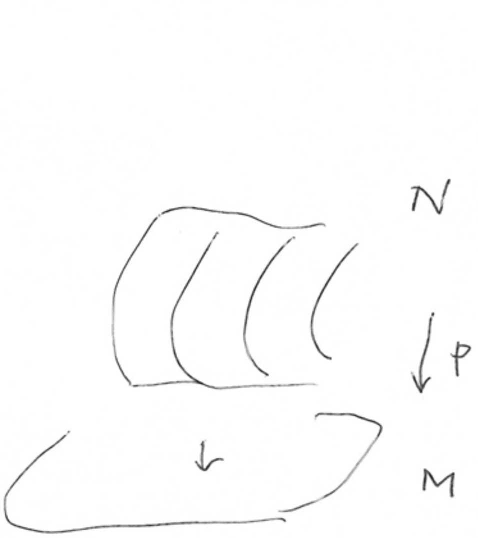
$\mathcal{R}_K$ sheaf of dg-cat on K

$\Gamma(K, \mathcal{R}_K) = \mathbb{F}$(Liouville neighb.)

Local Fukaya

?

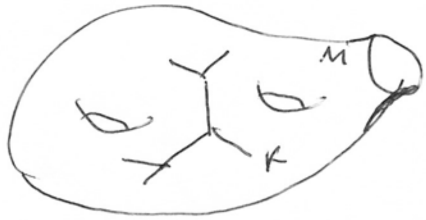
To study F(N) via fibrations



Perverse sheaves satisfy purity better than sheaves

Ex. (a) M top. surface

$\mathcal{F}$ constr. complex. TFAE



K = graph

(i) $\forall$ graph $K \subset S$

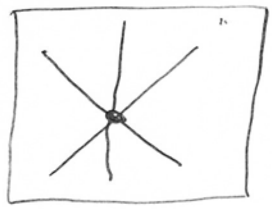
$$\underline{H}_K^{\neq 1}(\mathcal{F}) = 0$$

(ii) $\mathcal{F}$ is perverse

(b) $M = \mathbb{C}^n$ $K = \mathbb{R}^n$ $\quad S$ stratif. by arrangement of hyperplanes $/\mathbb{R}$

$\mathcal{F} \in \mathcal{D}^b(M, S)$

same equivalence.



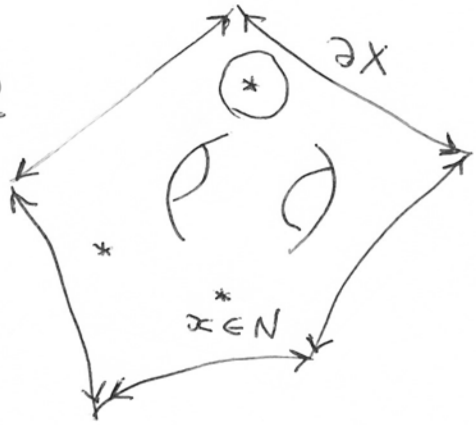
So need to categorify

perverse sheaves

⑥ Schobers and topol. Fukaya cat. with coefficients on surfaces

X surface

$S \sim N = \{\text{allowed sing. pts}\}$



$\mathrm{Sch}(X,N) \xrightarrow{K} \mathrm{Perv}(X,N)$

(∞) cat of schobers on X with singularities $\subset N$

Naively
* A local system of cat. on $X-N$
* + a spherical functor near $\forall x \in N$

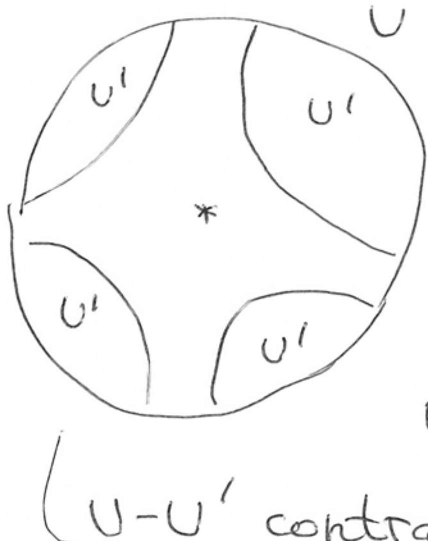
(glued together)

More sheaf-like: data of $\mathfrak{S}$ cat. $\mathfrak{S}(U,U')$

$H^1(U,U';\mathcal{F})$ perv

$H^{\neq 1}=0$

exact seq. of cohom. for a triple



$U \supset U'$ Milnor pair

U = disk

$U' = \coprod$ disks

$U-U'$ contractible
$\cup\!\!\!\!\cdot$ at most one $x \in N$

+ recollement (SOD) for some triples

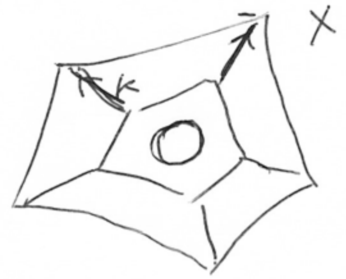
Th. (w. Dyckerhoff, Schechtman, Soibelman)

(a) $\forall$ graph $K \subset X$ we have

$$\begin{array}{ccccc} \sigma \in & \mathrm{Sch}(X, N) & \xrightarrow{\text{Groth, gr}} & \mathrm{Perv}(X, N) & \mathcal{F} \\ & {\scriptstyle R_K}\downarrow & & \downarrow & \downarrow \\ & \mathrm{Sh.\,of\,Cat}_K & \xrightarrow{\text{Groth}} & \mathrm{Sh}_K & \underline{H}^1_K(\mathcal{F}) \end{array}$$

(b) If K is spanning for (X, N)

hits all corners
contains N
$K \sim X$



then $\Gamma(K, R_K(\sigma)) = F(X, \sigma)$

is canonically independ. on K

categorifies $H^1(X, \partial X, \mathcal{F})$

pure boundary

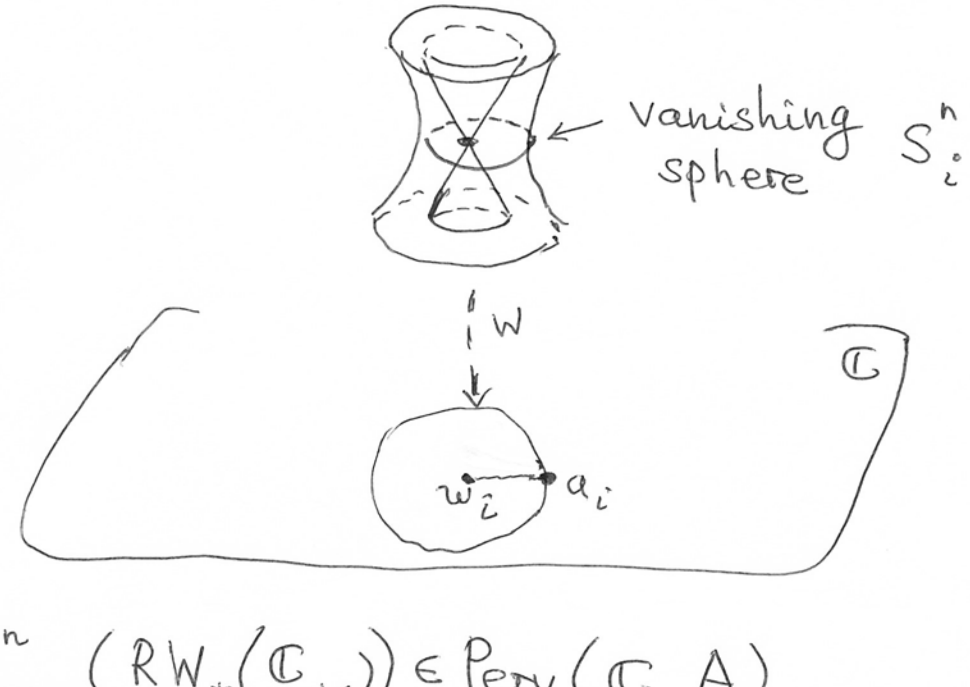
⑦ Lefschetz schober and Fukaya-Seidel category.

$Y^{n+1} \xrightarrow{W} \mathbb{C}$ proper holomorphic Morse function (Lefschetz fibr.)

Kahler CY

$\cup$

$A = \{w_1, \ldots, w_N\}$ crit. values



$\underline{H}^n_{perv}(RW_*(\underline{\mathbb{C}}_Y)) \in \mathrm{Perv}(\mathbb{C}, A)$

On $\mathbb{C} - A$: local system of H^n(fibers)

$\Phi_{w_i} \simeq \mathbb{C}$ spanned by $[S_i^n]$

! Categorifies to a perverse schober

$\mathcal{L}_W \in \mathrm{Sch}(\mathbb{C}, A)$

$\mathcal{L}_W\big|_{\mathbb{C}-A}$ = loc. system of Fuk (fibers)

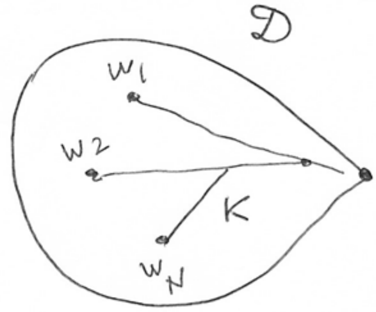
Near w_i: spherical functor

$$D^b\,\mathrm{Vect} \longrightarrow \mathrm{Fuk}(W^{-1}(a_i))$$

$$\mathbb{C} \longmapsto S_i^{n-1} \quad \text{spherical object}$$

$\mathcal{D} \subset \mathbb{C}$ disk $\supset A$

with one corner



Th $\mathbb{F}(\mathcal{D}, \mathcal{L}_W) =$

$= FS(W)$

Fukaya-Seidel cat.

Can consider $Y \xrightarrow{W} X$

Riem. surface

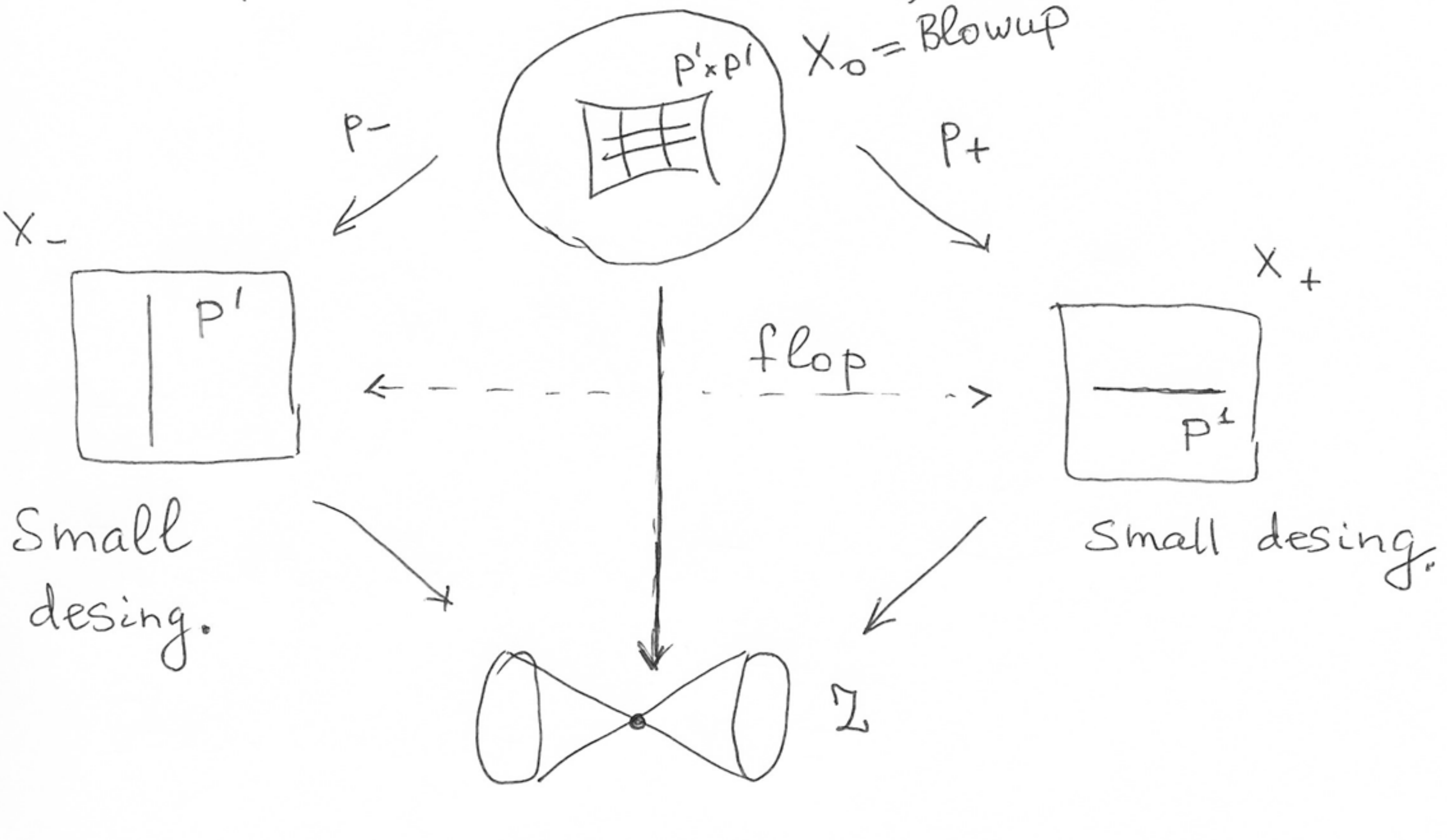
more corners etc.

(8) Perverse schobers and flops

$\mathrm{Perv}(\mathbb{D}, 0)$ also described by

$$\left\{ E_- \underset{\delta_-}{\overset{\gamma_-}{\rightleftarrows}} E_0 \underset{\delta_+}{\overset{\gamma_+}{\rightleftarrows}} E_+ \right\} \quad \begin{array}{l} \gamma_-\delta_- = \mathrm{Id} \\ \gamma_+\delta_- \text{ invert, etc.} \end{array}$$

Flop (conifold transition)



3d quadratic sing = Cone $(P^1 \times P^1)$

$$D^b(X_-) \underset{p_-^*}{\overset{Rp_{-*}}{\leftrightarrows}} D^b(X_0) \underset{p_+^*}{\overset{Rp_{+*}}{\rightleftarrows}} D^b(X_+)$$

Satisfy same relations (Bondal-Orlov 2002)

$$D^b(X_-) \xrightarrow{R(p_+)_* \circ p_-^*} D^b(X_+) \quad \text{equivalence}$$

Similarly for a more general class of flops (rel. 1d flops of van den Bergh)

Web of several 3d flops

$\downarrow$

Local systems of $\mathcal{D}^b$ on $\mathbb{C}^n$ - Hyperplane arrang.

X_1 X_2 X_3

[Donovan - Wemyss]

Hom. Min. Model Progr.

? Extend to schobers on full $\mathbb{C}^n$ (Conj. with A. Bondal, V. Schechtman)

In 1d ("schobers" on $(D, 0)$)

slightly different concept from before

$$\mathfrak{S} = \{ \mathcal{E}_- \underset{\delta_-}{\overset{\gamma_-}{\rightleftarrows}} \mathcal{E}_0 \underset{\delta_+}{\overset{\gamma_+}{\rightleftarrows}} \mathcal{E}_+ \}$$ w. relations

categories

$\exists$ cat. analogs of $\mathbb{H}^i(\mathcal{D},\mathcal{F})$, $\mathbb{H}^i_c(\mathcal{D},\mathcal{F})$

perv. sheaf

$$\Big\{ \underset{0}{E_0} \xrightarrow{\gamma_+,\gamma_-} \underset{1}{E_+ \oplus E_-} \Big\}$$

$$\Big\{ \underset{1}{E_+ \oplus E_-} \xrightarrow{\delta_+,\delta_-} \underset{2}{E_0} \Big\}$$

<u>Th.</u> (w. Bondal, Schechtman). For a flop as above

$\mathbb{H}^0(\mathcal{D},\sigma) \cong \mathrm{Perf}_Z$ perfect complexes

$\mathbb{H}^2_c(\mathcal{D},\sigma) \cong \mathcal{D}^b_{coh}(Z)$ full coherent $\mathcal{D}^b$

$\qquad\qquad \cong \mathrm{dgFun}(\mathrm{Perf}_Z, \mathrm{dg\,Vect})$

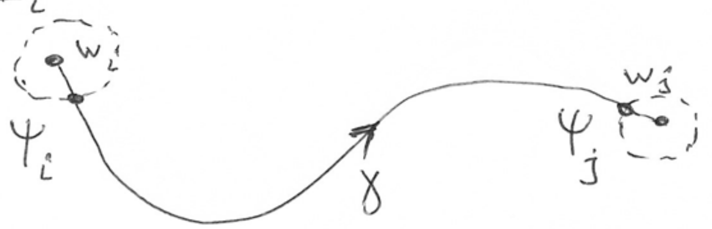
"categorical Poincare duality"

⑨ Categorified Picard–Lefschetz theory

$\sigma \in Sch(\mathbb{C}, A)$, where $A = \{w_1, \dots, w_N\}$

$\Phi \underset{v}{\overset{u}{\rightleftarrows}} \Psi$ on a disk

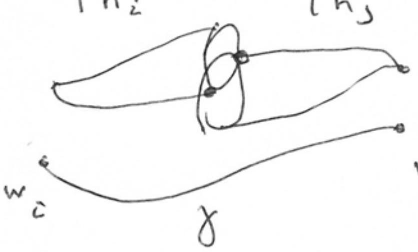
Category Φ_i



$$
\begin{tikzcd}
M_{ij}(\gamma) : \Phi_i \arrow[r] \arrow[d, "u_i"'] & \Phi_j \quad \text{functor} \\
\Psi_i \arrow[r, "\text{transl.}"] & \Psi_j \arrow[u, "v_i"']
\end{tikzcd}
$$

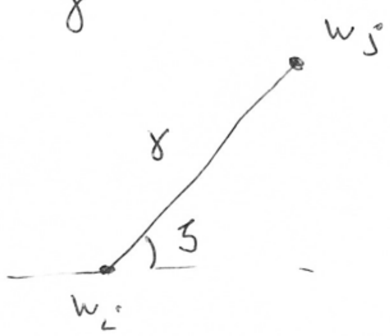
<u>Ex.</u> For $\sigma = \mathcal{L}_W$ $\quad Y \xrightarrow{W} \mathbb{C}$ holom. Morse

$\Phi_i = D(\mathrm{Vect})$ "1-dim. category"



$$M_{ij}(\gamma) \cong - \otimes \underbrace{\mathbb{C}^{Th_i \cap Th_j}}_{\text{Floer cx}}$$

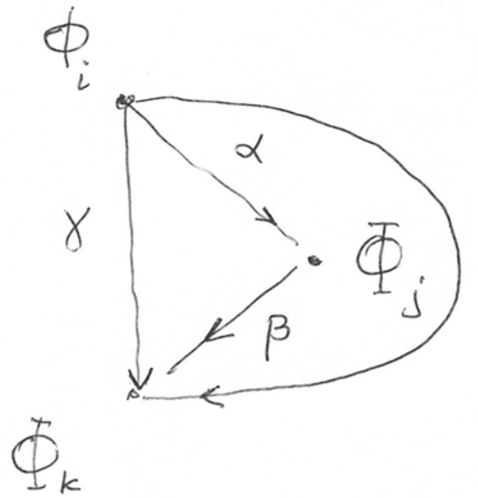
γ straight



$$M_{ij} = - \otimes \mathbb{C}^{\#\, \zeta\text{-grad. traj}}$$

$\mathrm{Re}(\zeta^{-1} W)$

Moving γ past a singular pt.:



gives an exact triangle of functors

$$M_{jk}(\beta)\, M_{ij}(\alpha) \xrightarrow{?} M_{ij}(\gamma) \longrightarrow M_{ij}(\gamma')$$

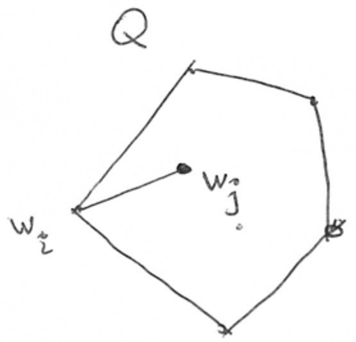
"Picard-Lefschetz" triangle

(10) Relation to "algebra of infrared".

$W : Y \longrightarrow \mathbb{C}$ holom. Morse

$\cup$

$A = \{w_1, \dots, w_N\}$ $\quad Q = \mathrm{Conv}(A)$ polygon



straight lines, convexity

GMW: – a Lie_∞-algebra $\mathfrak{g}_A$

(category)

– an A_∞-algebra R_A

– a morphism $\varphi : \mathfrak{g}_A \longrightarrow \mathrm{Hoch}^\bullet(R_A)$

+ a MC element

$\eta \in \mathfrak{g}_A^1$ s.t.

deform. of R_A along

$\varphi(\eta) = FS(W)$

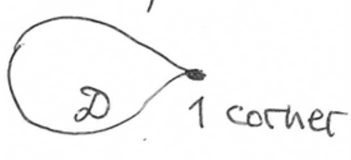
K., Kontsevich, Soibelman: interpret via secondary polytopes

+ φ a qis

Observation and work in progress w. Y. Soibelman:

This can be done for any schober

$\mathfrak{S} \in Sch(\mathbb{C}, A)$ to describe $\mathbb{F}(\mathcal{D}, \mathfrak{S})$



via some

$\eta \leftarrow\!\!-\!\!\rightarrow$ system of PL triangles