Holographic Orbifold CFTs

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Goal

We want to find new examples of conformal field theories in two dimensions that have holographic properties.

We are particularly interested in theories which have few light states.

I will describe such theories coming from orbifolds.

In a second part, I will talk about partition functions of these theories, and their phase transitions.

Part I

Spectrum

Partition functions and modular invariance

Define the partition function of a 2d CFT on S^1:

$$Z(\tau) := q^{-c/24}\bar{q}^{-c/24}\sum_{h,\bar{h}} N_{h,\bar{h}} q^h \bar{q}^{\bar{h}} \qquad q = e^{2\pi i\tau}$$

Here the complex parameter τ encodes the inverse temperature β and the spin potential μ,

$$\tau = i\frac{\beta}{2\pi} + \mu$$

The partition function is modular invariant:

$$Z\left(\frac{a\tau+b}{c\tau+d}\right) = Z(\tau) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

Cardy growth

Consider the *S*-transformation, that is

$$Z(\tau) = Z(-1/\tau)$$

This implies for the spectrum that we have *Cardy growth*

$$N_{h,\bar{h}} \sim e^{2\pi\sqrt{ch/6}} e^{2\pi\sqrt{c\bar{h}/6}}$$

This holds in the Cardy regime, that is $h, \bar{h} \gg c$.

This is exactly the growth that is needed to explain black hole entropy [Strominger,Vafa;...]

For holography, we are however more interested in a different regime: $c \to \infty$, with h not necessarily much bigger than c.

Light states

Modular invariance fixes the growth of very heavy states,

$$h \gg c \ .$$

We would like to understand lighter black holes, say $h \sim c$, or perturbative states, say $h \ll c$.

Let us call 'light' all states with

$$h \leq c/24$$

Modular invariance allows a lot of freedom for the choice of such light states.

The slogan is: 'The light spectrum fixes the heavy spectrum.'

Holomorphic CFTs

Let us specialize to *chiral* or *holomorphic* CFTs, that is theories where all correlation functions and partition functions are meromorphic. The partition function is then a holomorphic function with a pole only at $q = 0$,

$$Z(\tau) = q^{-c/24} \sum_{h=0}^{\infty} a_h q^h$$

It is completely fixed by its polar part at $q = 0$, that is by all $a_h, h = 0, \dots c/24$. In this sense the light spectrum completely fixes the heavy spectrum.

Explicitly we can write it as a polynomial in $j(\tau)$, the Hauptmodul of the modular group,

$$j(\tau) = q^{-1} + 744 + 196884q + \dots$$

Non-holomorphic theories

The reason this worked is that the space of weakly holomorphic modular function is finite dimensional for fixed central charge. This is no longer true if the partition function is not meromorphic.

The situation is a thus more complicated for non-holomorphic CFTs.

It is still true that the light spectrum constrains the heavy spectrum. This is *e.g.* at the core of the modular bootstrap [Hellerman; Hellerman,Schmidt-Colinet; Friedan,CAK; Collier,Lin,Yin;. . .].

For existence, see also [Maloney,Witten]

Growth of light states?

This means that from the point of view of just the partition function, we have (almost) complete freedom to choose the light spectrum.

We can therefore try to model different bulk behaviors for light states:

$$\log a_h \sim \begin{cases} \gtrsim h & : \text{Super-Hagedorn growth} \\ h & : \text{Hagedorn growth} \\ h^{(d-1)/d} & : QFT_d \ \text{growth} \\ 2\pi\sqrt{h/6} & : \text{gravitons only: extremal CFT} \end{cases}$$

The question is however: Can we actually construct CFTs that have such partition functions?

Symmetric Orbifolds

Let us start with the best know example of a holographic 2d CFT: the symmetric orbifold. Start with the N-fold tensor product of some CFT

$$V^{(N)} = \bigotimes_{i=1}^{N} V \qquad c_{V^{(N)}} = Nc$$

CFT is symmetric under permutations of factors $\Rightarrow$ We can orbifold by the symmetric group

$$V^{orb(S_N)} = V^{(N)}//S_N$$

The spectrum is given by

$$Z_{V^{orb(S_N)}}(\tau) = q^{-Nc/24} \sum_{h \geq 0} a_h^{(N)} q^h$$

[Dijkgraaf,Moore,Verlinde,Verlinde; Bantay]

Symmetric Orbifolds

What does the spectrum look like?

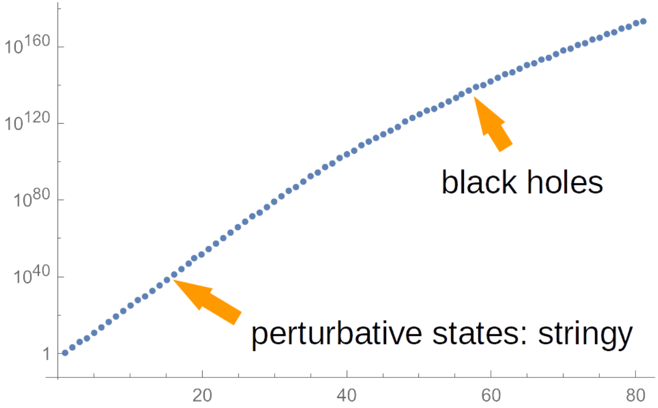
To leading order in large N, the spectrum of a symmetric orbifold CFT is given

$$\log a_h^N \sim \begin{cases} 2\pi h & : 0 < h < cN/12 \\ 2\pi\sqrt{cN(h - cN/24)/6} & : cN/12 < h \end{cases}$$

[CAK;Hartman,CAK,Stoica]

To leading order, it is independent of the seed theory V.

Symmetric Orbifolds



Elliptic Genus

Note that this is a statement on the partition function.

The behavior of the elliptic genus can be very different, due to cancellations. The symmetric orbifold of the elliptic genus of K3 for instance grows like

$$\log a_h \sim h^{1/2}$$

This allows to match the spectrum to the supergravity spectrum [de Boer]

For higher dimensional Calabi-Yaus, generically one finds Hagedorn growth [Benjamin,Kachru,CAK,Paquette]. There are however special cases where one finds supergravity growth [Belin,Castro,Gomes,CAK].

Permutation Orbifolds

For the symmetric orbifold we found Hagedorn growth. Can we find different behaviors?

One idea is to consider permutation orbifolds. Rather than orbifolding by the full symmetric group S_N, we can orbifold by some smaller permutation group G_N

$$V^{(N)}//G_N$$

Note that the G_N have to be large enough so that there is a large N limit [Hähl,Rangamani;Belin,CAK,Maloney]: they have to be *oligomorphic*.

S_N works, but $\mathbb{Z}_N$ for instance does not work.

Examples with different growth

What oligomorphic groups can we find? Arrange the N elements in a $\sqrt{N} \times \sqrt{N}$ matrix, and permute the rows and columns separately: This gives the direct product action $S_{\sqrt{N}} \times S_{\sqrt{N}}$, which is oligomorphic.

Direct product action: $S_{N^{1/d}} \times \cdots \times S_{N^{1/d}}$

$$\log a_h \gtrsim (d-1) h \log h$$

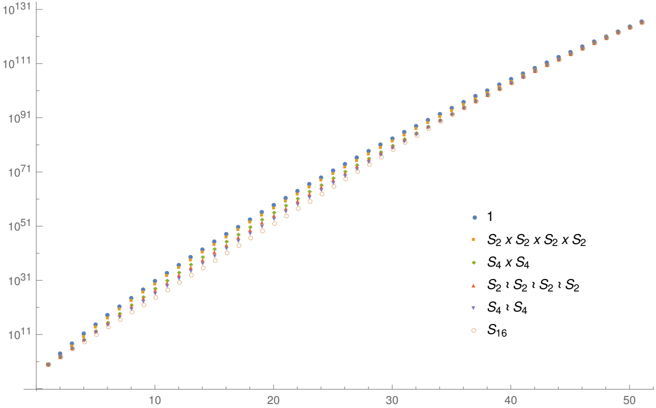
This is an example with super-Hagedorn growth.

Another example: Wreath product action: $S_{N^{1/d}} \wr \cdots \wr S_{N^{1/d}}$

$$\log a_h \gtrsim h / \log(\log(\cdots \log(h) \cdots))$$

[CAK,Mühlmann]

Examples ($N = 16$)



a_h for some oligomorphic groups

A lower bound for permutation orbifolds

It turns out that one can not do much better than Hagedorn growth:

For oligomorphic permutation orbifolds we have

$$\log a_h \gtrsim h/\log h$$

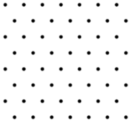
[Belin,CAK,Maloney]

We therefore want to investigate more general orbifolds.

Let us turn to lattice CFTs and their orbifolds.

Lattice CFTs

Consider a lattice Λ of rank d which is *even* and *self-dual*, that is $\Lambda^* = \Lambda$.



From this we can construct a chiral (holomorphic) CFT with central charge d.

To have few light states, we want a lattice with few short vectors.

An *extremal lattice* is an even self-dual lattice with the minimal number of short vectors:

$$\Theta(\tau) = 1 + 0 \cdot q + \ldots + \# q^{\lfloor d/24 \rfloor + 1} + \ldots$$

where $\Theta(\tau) = \sum_{v \in \Lambda} q^{\langle v, v \rangle / 2}$ is the lattice theta function.

What extremal lattices are known?

What extremal lattices have been constructed?

- $d = 24$: Leech lattice is the unique extremal lattice
- $d = 48, 72$: 5 examples are known [Nebe,Sloane].
- d large enough: cannot exist [Mallows,Odlyzko,Sloane]

Extremal lattices are not good enough yet for our purposes: They still have a large number of free boson descendants at light weight.

$$Z_\Lambda(\tau) = \frac{\Theta_\Lambda(\tau)}{\eta(\tau)^d}$$

To eliminate those, we want to orbifold by symmetries of the lattice.

Mathematical description of Orbifolds

As we are considering chiral theories, we are doing chiral (asymmetric) orbifolds. We need to be careful and do this mathematically rigorous.

Mathematically, a 2d CFT is described by

- a vertex operator algebra V: physically, the symmetry algebra (or W-algebra) of the CFT
- its modules M: physically, the primary fields

We want to restrict to rational V that also satisfy some additional conditions: we want V to be *tame*.

If the CFT is chiral, then V itself describes the entire CFT: V is the only irreducible module. For simplicity, we restrict to this case.

Lattice Symmetries

Let us lift a lattice automorphism $\sigma \in Aut(\Lambda)$ to a VOA automorphism $\hat{\sigma}$:

$$\hat{\sigma}(e_v) = u(v) e_{\sigma v} \qquad v \in \Lambda$$

We introduced a phase $u(v)$ which must be compatible with the cocycle of the lattice vertex operator.

Sometimes $u(v)$ can be chosen trivially, sometimes not. For example, if $\langle v, \sigma^{ord(\sigma)/2} v \rangle \notin 2\mathbb{Z}$, we need to pick a $u(v)$ such that $ord(\hat{\sigma}) = 2 ord(\sigma)$.

The order of the VOA automorphism group is then bigger than the geometric group. See *e.g.* [Narain,Sarmadi,Vafa; Harvey,Moore] More generally, we can also try to pick more complicated lifts $u(v)$, which lead to extensions of the symmetry group.

Holomorphic Orbifolds

To orbifold a VOA V by a group G, first take the invariant subspace V^G (‘untwisted sector’). If V is tame, then V^G is again a tame vertex operator algebra.

However, it will have many more irreducible modules: If *e.g.* V is holomorphic, then $V^{\mathbb{Z}_n}$ has n^2 irreducible modules.

If we want to recover a holomorphic CFT, we can try to extend V^G by adjoining modules (‘twisted sectors’)

$$V^{orb(G)} = V^G \oplus M$$

such that $V^{orb(G)}$ is again holomorphic. Under what conditions is that possible?

Cyclic orbifolds

If $G = \mathbb{Z}_n$, say generated by $\hat{\sigma}$, [van Ekeren,Möller,Scheithauer] give a complete mathematical description. Effectively the only condition is level-matching [Vafa].

The 'vacuum anomaly', that is the ground state in the twisted sector, needs to be compatible with the graded modes.

Note that the grading of the modes is given by the order of $\hat{\sigma}$. As mentioned above, if the simplest lift of σ is anomalous, we can try to find a more complicated lift with a higher order, which is potentially non-anomalous.

For non-cyclic (e.g. non-Abelian) orbifolds, the story is more complicated
[Dijkgraaf,Witten;Dijkgraaf,Pasquier,Roche;. . . Evans,Gannon]

$\mathbb{Z}_2$ orbifolds

The most famous orbifold is the Monster CFT [Frenkel,Lepowsky,Meurman]:

Start with the CFT of the Leech lattice, and then orbifold by the $\mathbb{Z}_2$ symmetry $x \mapsto -x$. This eliminates the 24 states at weight 1.

The resulting CFT is in fact an extremal CFT [Höhn;Witten] at $c = 24$:

$$Z(\tau) = j(\tau) - 744 = q^{-1} + 0 + 196884q + \dots$$

We want to study cyclic orbifolds of extremal lattices in $d = 48$ and $d = 72$. Note that now the $\mathbb{Z}_2$ orbifold no longer produces an extremal CFT, since more than 1 state at weight 2 survives.

What $\mathbb{Z}_n$ orbifolds of $d = 48, 72$ lattice CFTs can we find?

Cyclic orbifolds for $d = 48$

For $d = 48$, four extremal lattices are known [Nebe]:

$$\Gamma_{48p}, \qquad \Gamma_{48q}, \qquad \Gamma_{48n}, \qquad \Gamma_{48m}$$

Scanning through all cyclic subgroups, we find around 100 new holomorphic CFTs [Gemünden,CAK]

a_h	0	1	2
extremal spectrum :	1	0	1
lattice CFT $V_{\Gamma_{48x}}$:	1	48	1224
best orbifold CFT:	1	0	48

Cyclic orbifolds for $d = 72$

For $d = 72$, one extremal lattice is known [Nebe]:

$$\Gamma_{72}$$

We find around 50 new holomorphic CFTs.

a_h	0	1	2	3
extremal spectrum :	1	0	1	1
lattice CFT $V_{\Gamma_{72}}$:	1	72	2700	70080
best orbifold CFT $V_{\Gamma_{72}}/\mathbb{Z}_{182}$:	1	0	36	408

Holomorphic VOAs

[Schellekens] conjectured that there are exactly 71 holomorphic VOAs with $c = 24$. This is very close to being proven through the work of [Dong, van Ekeren, Frenkel,Kawasetsu, Lam, Lepowsky,Lin,Mason,Meurman, Möller,Sagaki, Scheithauer, Shimakura...]

For $c = 48, 72, \dots$ much less is known. The Siegel Maass formula implies there is an exponentially large number of lattice VOAs. Our constructions give us at least some idea of what VOAs are out there.

Moreover, many of our constructions have no spin 1 fields, and therefore likely have finite automorphism groups. It might be interesting to look for moonshine.

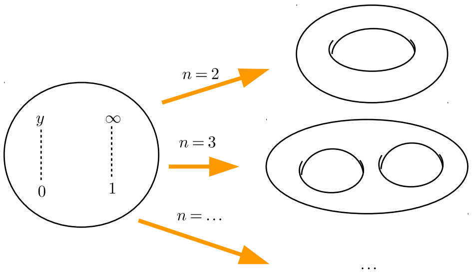
Part II

An Application: Partition functions

Higher genus partition functions

What can we do with our orbifold theories? Let us try to compute partition functions, and check for phase transitions.

One motivation: Compute Rényi entropies and entanglement entropies. Consider two intervals on the sphere with cross ratio *y*, and go to the *n*-fold cover.



Genus 1

The genus 1 partition function is

$$\lim_{N\to\infty} q^N Z(\tau) = 1 + \sum_h a_h q^h$$

In the large N limit, depending on how fast a_h grows, we will have a finite radius of convergence.

One example: Symmetric orbifold

- $a_h \sim e^{2\pi h}$
- The radius of convergence is $q_0 = e^{-2\pi}$.
- Phase transition at $\beta = 2\pi$ (or $y = 1/2$)

This is in fact the Hawking-Page transition.

Hawking-Page transition

In the bulk, this is the transition that occurs when the vacuum solution no longer dominates over the BTZ black hole solution.

Schematically we have:

$$q^N Z(\beta) = \underset{\substack{\Uparrow \\ \text{AdS vacuum}}}{1} + \underset{\substack{\Uparrow \\ \text{Black Hole}}}{e^{N(\beta - \frac{4\pi^2}{\beta})}}$$

This is a simplified version of the Farey tail [Dijkgraaf,Maldacena,Moore,Verlinde].

For $N \to \infty$, there is a phase transition at $\beta = 2\pi$: Expression diverges!

Similarly, there is a phase transition at $y = 1/2$ for the entanglement entropy [Ryu,Takayanagi]

How universal is this phase transition?

What about other theories? Do they still have Hawking-Page transitions?

The answer is again related to the light spectrum:

If the light spectrum is *sparse*:

$$\log a_h \lesssim 2\pi h \qquad h \leq c/24$$

then the genus 1 partition function has the same Hawking-Page transition. [Hartman,CAK,Stoica]

$\Rightarrow$ The Hawking-Page transition is quite universal!

What about genus 2?

What happens for higher genus? The work of [Faulkner;Hartman] suggests that there should still be a phase transition at $y = 1/2$ if we are close enough to classical gravity. Let us see what happens for genus 2.

Somewhat schematically, the genus 2 partition function is given by:

$$Z_2(y) = \sum_{h_1,h_2,h_3} C^2_{h_1,h_2,h_3} y^{h_1+h_2+h_3}$$

The radius of convergence now depends both on the spectrum, and on the growth of the average three point function.

Correlation functions

We need to compute correlation functions. For permutation orbifolds, in principle one knows how to do this, but in practice it is quite tedious [Lunin,Mathur; Pakman,Rastelli,Razamat]

However, in the large N limit things simplify considerably: As expected from holography, symmetric orbifolds become generalized free theories [Lunin,Mathur]. The same is true for suitable permutation orbifolds [Belin,CAK,Maloney].

To leading order in N, correlation functions are Wick contractions of factors. They are thus given by combinatorial expressions.

Correlation functions

A general state in the large N limit of a symmetric orbifold theory is a symmetrized combination of tensor factors, almost all of which are in the vacuum.

Three normalized states with K_i non-vacuum factors each. This is the analogue of $\langle : \phi^{K_1} :: \phi^{K_2} :: \phi^{K_3} : \rangle$:

$$C_{K_1 K_2 K_3} = \frac{\sqrt{K_1! K_2! K_3!}}{\left(\frac{K_1+K_2-K_3}{2}\right)! \left(\frac{K_1-K_2+K_3}{2}\right)! \left(\frac{-K_1+K_2+K_3}{2}\right)!} + O(N^{-1/2}) \, .$$

Crucially, this grows exponentially:

$$C_{K,K,K} \sim 2^{\frac{3K}{2}}$$

A new phase transition

If the state has weight $h = Kh_1$, we have $C_{hhh} \sim 2^{3h/2h_1}$

The radius of convergence is smaller than $y = \frac{1}{2}$ if there is an operator with

$$h_1 < 0.17\ldots$$

Interpretation: There is a new phase. For genus 2, phase transitions are much less universal! [Belin, CAK, Zadeh]

Note that even though we did this for the symmetric orbifold, we get the same result if we consider *e.g.* a free theory with just one scalar with weight h_1.

Bulk side: Eigenvalues of Laplacian

Can we check this from the bulk side?

This makes a prediction for the eigenvalues of the Laplace operator on AdS_3. Consider a scalar of mass squared $\mu(h)$ on AdS_3 with conformal boundary specified by y:

$$(-\Delta + \mu(h))\phi = 0$$

Eigenvalue problem for Laplacian Δ on AdS_3.

Recent work: [Dong,Maguire,Maloney,Maxfield] found that indeed $-\Delta + \mu(h)$ has negative eigenvalues for $h_1 < 0.17\ldots$, which implies that the vacuum solution becomes unstable for $y < 1/2$.

$\Rightarrow$ This precisely confirms the prediction.

Thank you!