Dynamics of Adjoint QCD in 2+1 Dimensions

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June 18, 2018

There are many new non-perturbative results about gauge theories in 3 (2+1) dimensions. My goal here is to review some very particular model which exhibits many general ideas. We will then see that there are also interesting implications for gauge theories in 4 dimensions.

The presentation is mostly based on collaborations with Davide Gaiotto, Jaume Gomis, Anton Kapustin, Nathan Seiberg, as well as some work in progress.

What are the different phases of Quantum Field Theory that we will encounter today ?

- **Trivial Gapped**: No massless excitations, no topological theory, trivial (product) wave function
- **Topological Field Theory (TFT)**: no massless excitations, but some long range entanglement and topological order (such as anyons). Nontrivial ground state wave function.
- **Massless phases**: This could be due to a Conformal Field Theory, or due to Nambu-Goldstone particles.

A particularly interesting case study is the theory of an $SU(N)$ adjoint *Majorana* fermion λ_α coupled to $SU(N)$ gauge fields with a Chern-Simons term at level k.

The Lagrangian is

$$\mathcal{L} = \frac{-1}{4g^2}\mathrm{Tr}F^2 + \frac{k}{4\pi}\mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right) + i\bar{\lambda}\not{D}\lambda + \frac{m}{4\pi}\bar{\lambda}\lambda$$

Note that m is a real parameter. The model has no ordinary global symmetries. We take $k \geq 0$ without loss of generality. Consistency requires

$$\frac{N}{2} + k \in \mathbb{Z} \; .$$

An interesting special case is $N \in 2\mathbb{N}$, which allows to set $k = m = 0$. In this case we have time reversal symmetry and $T^2 = (-1)^F$.

This time reversal symmetry cannot be consistently gauged. In other words, we cannot really put the theory on a non-orientable (pin) manifold.

This is a standard discrete 't Hooft anomaly and the bulk (4 dimensional) anomaly inflow term is the η invariant in four dimensions. This anomaly is valued in $\mathbb{Z}_{16}$ [see Witten's 1508.04715 for an explanation of the origin of $\mathbb{Z}_{16}$].
An important point is that the Majorana fermion contributes $\pm 1 \mod 16$ depending on an overall choice of orientation. We will take it to be $1 \mod 16$.

The time reversal symmetry anomaly of adjoint QCD is given by

$$\nu = N^2 - 1 \bmod 16$$

which for even N takes the values

$$\nu = \begin{cases} 3 \bmod 16 & \text{if} \quad N = 2 \bmod 4 \\ -1 \bmod 16 & \text{if} \quad N = 0 \bmod 4 \end{cases}$$

This time reversal anomaly is already sufficient to rule out a trivial gapped vacuum at $m = k = 0$.

Such a time reversal anomaly can be accounted for by having in the deep IR, for instance, some time-reversal invariant TQFT, and/or a sigma model with some theta term, and/or by having massless fermions etc.

A crucial new tool in our analysis would be to exploit the one-form symmetry of this model. Indeed, the center of $SU(N)$ does not act on the matter field λ_α which means that the model has a $\mathbb{Z}_N$ one-form symmetry. (We use the terminology of [Kapustin-Seiberg], [Gaiotto-Kapustin-Seiberg-Willett].)

The background field for this $\mathbb{Z}_N$ one-form symmetry is a two-form $\mathbb{Z}_N$ gauge field B,

$$[B] \in H^2(M_3, \mathbb{Z}_N) \ .$$

The coupling to B may be inconsistent in the sense that the partition function will depend not just on $[B]$ but also on the gauge choice for B.

To classify the possible anomalies of this type we write local functionals made out of B in four dimensions. In this sense we can view the adjoint QCD model as the boundary of some SPT phase for a two-form gauge field in 3+1 dimensions.

The allowed (gauge invariant on closed manifolds) local terms in four dimensions are given by (actually we need to use the Pontryagin square)

$$\frac{2\pi i P}{2N}\int_{\mathcal{M}_4} B \cup B \ ,$$

where the distinct choices (for even N) are labeled by $P = 0, 1, ..., 2N-1$. However, on spin manifolds, only $P \bmod N$ matters. This is true for both odd and even N.

Therefore the anomaly in the one-form symmetry is given by an integer $\mathrm{mod}\ N$. It cannot depend on the mass of λ and we find after a short computation that

$$P = k + \frac{N}{2} \mod N$$

for our adjoint QCD theory.

The infrared has to match this anomaly, but how can it? Certainly a trivial vacuum cannot do it. Also many massless theories would not be able to match this anomaly. One option is that the infrared theory has a TQFT.

It would be useful to remember that, for example, $SU(N)_m$ Chern-Simons theory has such a $\mathbb{Z}_N$ symmetry (generated by some of the Wilson lines) and the anomaly is

$$P = m \mod N \ .$$

Many other TQFTs also have such one-form symmetries with various anomalies.

So far we have seen that for $N \in 2\mathbb{N}$ and $m = k = 0$ there is a time reversal symmetry with an anomaly and for generic choices of N, k (and for all m) there is an anomaly in the one-form symmetry.

These two facts appear to be already very constraining. We will need two more observations before we suggest a “solution” to this theory.

Observation 1: Decoupling at $|m| \gg g^2$ or $k \gg N$.

In some "corners" of the parameter space the theory becomes weakly coupled. We can then use semi-classics to find the long-distance behaviour. Let us map out these corners.

If $|m| \gg g^2$ the quarks decouple even before the interactions set in. However, one has to be careful integrating them out as there is a famous non-decoupling effect [Redlich, Niemi-Semenoff] proportional to $m/|m| = sgn(m)$.
This shifts k according to

$$k \to k + sgn(m)\frac{N}{2} \ .$$

Therefore for large positive m and ANY k, N we flow to the TQFT

$$SU(N)_{k+\frac{N}{2}} \ ,$$

and for large negative m we flow to the TQFT

$$SU(N)_{k-\frac{N}{2}} \ .$$

These two TQFTs have the same one-form symmetry anomaly, since $k+N/2 = k-N/2 \mod N$ which is a nice consistency check.

These are different TQFTs and there must be therefore some sort of phase transition at $m \sim g^2$. Maybe even more than one transition. Clearly these transitions are non-Landau-Ginzburg. Essentially this is guaranteed by the one-form symmetry anomaly.

Our next weak coupling limit is

$$k \gg N$$

The gauge field A now has a mass kg^2 and therefore it decouples before the interactions set in:

$$kg^2 \gg g^2 N \ .$$

The model therefore dramatically simplifies since we can remove the kinetic term of the gauge field, which is the same as integrating out the heavy gauge field.

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right) + i\bar{\lambda}\not{D}\lambda + \frac{m}{4\pi}\bar{\lambda}\lambda \ .$$

Now normalizing the fields canonically we see that all the interactions scale like $1/\sqrt{k}$. Therefore for small $1/k$ the remaining light fields are weakly interacting and there is a weakly coupled *Conformal Field Theory* if we tune m. (Qualitatively, we tune m to zero.)

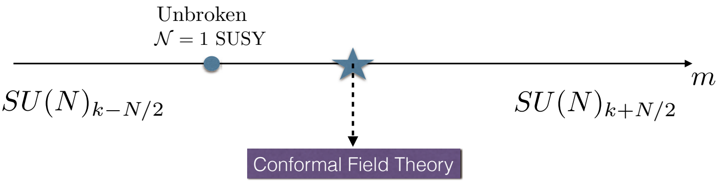
Observation 2: $\mathcal{N} = 1$ Supersymmetry for $m = -kg^2$.

This theory then has $\mathcal{N} = 1$ supersymmetry in 3 dimensions (i.e. two supercharges). [Witten] has shown that the index does not vanish for $k \geq N/2$ and vanishes otherwise. The index for $k \geq N/2$ is the same as the number of lines in $SU(N)_{k-N/2}$ TQFT.

Therefore, for $k \geq N/2$, when $m = -kg^2$ we are still in the asymptotic phase $SU(N)_{k-N/2}$. For $k < N$ there must be at least a massless Majorana fermion at $m = -kg^2$. The one-form symmetry anomaly implies that this cannot be the whole story, though.

We therefore suggest the following phase diagram for $k \geq N/2$ ("large k"):

$$SU(N) + \lambda_\alpha, \text{ level } k \qquad k \geq N/2$$



Some consistency checks:

- $SU(2)$ with $k = 3$. The $\mathbb{Z}_2$ one-form symmetry is anomaly free. We gauge it and get $SO(3)$ gauge theory at level $k = 3/2$ with a single real fermion in the three-dimensional representation. This theory has a bosonic dual. Our Conformal Field Theory thus has a bosonic description

$$O(2) + \phi \ ,$$

The $O(2)$ has Chern-Simons terms at levels $-3, -3$. This is just a gauged version of the XY transition.

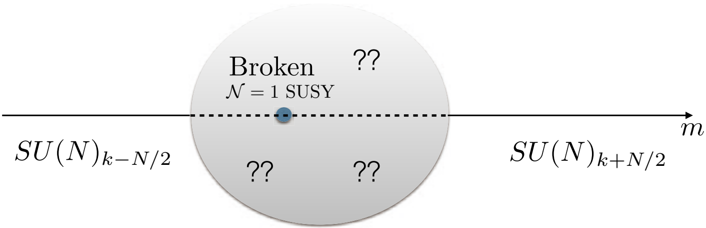
- Similarly, we may consdier $SU(2)$ with $k = 1$. In this case the Conformal Field Theory has a bosonic description

$$O(1) + \phi \ .$$

The $O(1)$ has Chern-Simons terms at level -3. This is just a gauged version of the Ising transition.

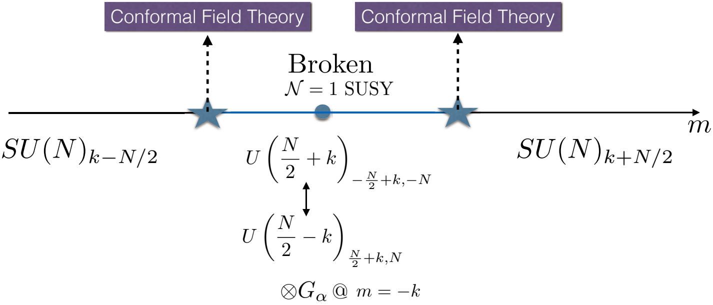
What if $0 \leq k < N/2$? For $m = -kg^2$ there is at least one massless Majorana fermion (the Goldstino). In addition, we need to saturate the one-form symmetry anomaly and the time reversal anomaly for $k = 0$.

$$SU(N) + \lambda_\alpha, \text{ level } k \qquad 0 \leq k < N/2$$



Here one needs to make a leap. We propose the following phase diagram:

$$SU(N) + \lambda_\alpha, \ \text{level } k \qquad\qquad 0 \leq k < N/2$$



This proposal miraculously passes many consistency checks:

- One-Form Symmetry: $U\left(\frac{N}{2}-k\right)_{\frac{N}{2}+k,N}$ has $\mathbb{Z}_N$ one-form symmetry and it can be checked that it has exactly the right one-form symmetry anomaly. (I will give a derivation, soon.)
- Time Reversal Symmetry: From level-rank duality,

$$U\left(\frac{N}{2}-k\right)_{\frac{N}{2}+k,N} \simeq U\left(\frac{N}{2}+k\right)_{-\frac{N}{2}+k,-N} ,$$

we see that this TQFT is time-reversal invariant if $k=0$, as required!!

Let us therefore consider the time reversal invariant (spin) TQFT

$$U\left(\frac{N}{2}\right)_{\frac{N}{2},N}$$

(even N). The time-reversal anomaly of this theory [Tachikawa-Yonekura,Cheng...] [see Barkeshli's talk for related matters]

$$\nu = 2(-1)^{N/2+1} ,$$

and from the Goldstino G_α we get at $\nu(Goldstino) = 1$. So the total infrared time reversal anomaly at $k = m = 0$

$$\nu_{IR} = 1 + 2(-1)^{N/2+1} \bmod 16 = \begin{cases} 3 \bmod 16 & \text{if} \quad N = 2 \bmod 4 \\ -1 \bmod 16 & \text{if} \quad N = 0 \bmod 4 \end{cases}$$

This is in agreement with the ultraviolet anomaly.

The transitions from the "quantum" phase to the semi-classical phases have a new dual description, for example, the left transition could be described with

$$U\left(\frac{N}{2}-k\right)+\hat{\lambda}$$

with levels $\frac{3}{4}N+\frac{k}{2}, N$. $\hat{\lambda}$ is a dual fermion in the adjoint representation. This is a new adjoint-adjoint Fermion-Fermion duality. This dual description shows that the one-form symmetry anomaly must match in the quantum phase.

[This duality has nontrivial content in the deep infrared if the transitions are second order, which we do not know for sure is true.]

Let us consider the simplest example: $SU(2)$ gauge theory with a massless adjoint fermion and no Chern-Simons term. According to our discussion above, the theory flows in the deep infrared to

$$(k = m = 0) \;\; SU(2) + \lambda \implies U(1)_2 + G_\alpha \ ,$$

where $U(1)_2$ is an Abelian pure Chern-Simons theory with one nontrivial anyon of spin $1/4$ and G_α is a decoupled free Majorana fermion.

This adjoint theory, which is time reversal invariant and massless, does NOT confine. Indeed, the fundamental Wilson line in the original theory maps to the spin $1/4$ anyon and it is deconfined.

(In this special case no phase transitions occur as we change m, except that the Majorana fermion is lifted at nonzero m.)

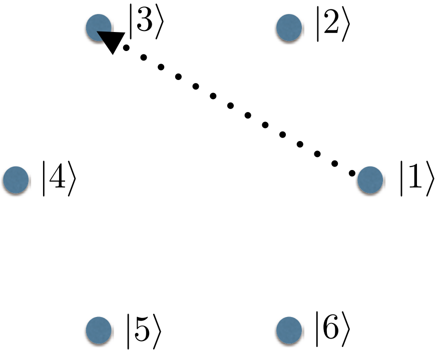
We will now draw a parallel between the problem of $3d$ dynamics that we have been analyzing so far and the problem of $4d$ dynamics and domain walls of Super Yang-Mills theory (as well as a few other theories with two-index matter fields). This will turn out to be a fruitful analogy as we will be led to some new results about $4d$ gauge theories.

Four-dimensional $\mathcal{N} = 1$ Super Yang-Mills theory with gauge group $SU(N)$ has N trivial gapped ground states. They are related by a (partially) spontaneously broken $\mathbb{Z}_{2N}$ symmetry. There is also a $\mathbb{Z}_N$ one-form symmetry which is unbroken.

We label the vacua by $|i\rangle$, $i = 1, ..., N$. We can therefore arrange for a domain wall connecting $|i\rangle$ and $|j\rangle$. The infrared theory on the wall only depends on

$$i - j$$

and time reversal symmetry relates this wall with the wall between $|N - i + 2\rangle$ and $|N - j + 2\rangle$, which is the same as the wall between $|j\rangle$ and $|i\rangle$.



$|3\rangle$
$|2\rangle$
$|4\rangle$
$|1\rangle$
$|5\rangle$
$|6\rangle$

In particular, for even N, the wall between $|1\rangle$ and $|N/2+1\rangle$ is related to itself by time reversal and hence it is time reversal invariant.

Of course, any wall that jumps over $N/2-1$ points is time reversal invariant and we choose the anchor point $|1\rangle$ without loss of generality.

The Acharya-Vafa conjecture for the theory on the wall between $|1\rangle$ and $|i+1\rangle$ is is

$$U(i)_{N-i,N} \ .$$

This conjecture is nicely consistent with the wall at $i = N/2$ being described by a time reversal invariant TQFT and by the wall from $|1\rangle$ to $|i+1\rangle$ being related by time reversal to the wall from $|1\rangle$ to $|N-i+1\rangle$.

It is interesting to observe that the quantum phase of our 3d model exactly coincides with the Acharya-Vafa theory. In fact, the consistency requirements from the quantum phase are analogous to those from the Acharya-Vafa theory.

This agreement between the domain wall theory and the phases of the pure 3d model is not accidental. These two systems have the same symmetries and the same anomalies, so it is not surprising that they develop the same phases. (See also [Dierigl-Pritzel])

Using these ideas, we will now make a proposal for the domain walls theories of four-dimensional gauge theories with other *simply connected* gauge groups. We simply identify the Acharya-Vafa theories with the quantum phases in the corresponding 3d adjoint QCD models. This problem has been resistant to other approaches for quite some time.

We only discuss super Yang-Mills theory with simply connected gauge groups because otherwise some of the vacua may not be trivial gapped and one needs to work out the answers case by case.

$\mathcal{N}=1$ Super Yang-Mills with gauge group $Sp(N)$. It has $N+1$ trivial gapped vacua. The domain wall theory connecting $|1\rangle$ and $|i+1\rangle$ is proposed to be

$$Sp(i)_{N-i+1} \ .$$

This obeys

$$Sp(i)_{N-i+1} \simeq Sp(N-i+1)_{-i} \ .$$

This exactly reflects the duality between the wall connecting $|1\rangle$ and $|i+1\rangle$ and the wall connecting $|1\rangle$ and $|N-i+2\rangle$. It passes various additional nontrivial consistency checks.

$\mathcal{N}=1$ Super Yang-Mills with gauge group $Spin(N)$. It has $N-2$ trivial gapped vacua. Here one encounters a slightly more exotic theory on the wall.
The analysis of the dynamics of the 3d theory with gauge group $Spin(N)$ proceeds by first solving the 3d dynamics with gauge group $SO(N)$ and then gauging the magnetic symmetry. The first and second step are done, respectively in [Gomis-Seiberg-ZK, Cordova-Hsin-Seiberg].

The ultraviolet magnetic symmetry becomes a charge conjugation symmetry in the quantum phase, which is why one finds a Chern-Simons theory with gauge group $O(m)$ with some m.

Our proposal for the theory connecting $|1\rangle$ and $|i+1\rangle$ is

$$O(i)^{1}_{N-i-2,N-i+1}\ .$$

The two levels denote the SO level and $\mathbb{Z}_2$ level, respectively. The superscript 1 signifies that we have to add a term proportional to the SW class $w_3(O(m))$, which intuitively can be thought of as an interaction term between the $\mathbb{Z}_2$ and $SO(m)$ gauge fields.

The level rank duality

$$O\left(i\right)^1_{N-i-2,N-i+1} \simeq O\left(N-i-2\right)^1_{-i,-i-3}$$

in particular implies that the wall connecting $|1\rangle$ and $|N/2\rangle$ (for even N) is time reversal invariant, as it should be. Also, more generally, the wall connecting $|1\rangle$ and $|i\rangle$ is equivalent up to time reversal to the wall connecting $|1\rangle$ and $|N-i\rangle$, as the symmetries of super Yang-Mills theory require.

Similar ideas can be applied to 4d $Sp(N)$ gauge theory coupled to a massless Weyl fermion in the anti-symmetric representation and $Spin(N)$ gauge theory coupled to a massless Weyl fermion in the symmetric representation. We just state the results here, in that order:

- There are $N-1$ trivial gapped vacua due to chiral symmetry breaking. The domain wall theory interpolating between $|1\rangle$ and $|i+1\rangle$ is

$$Sp\left(i\right)_{N-1-i} \leftrightarrow Sp\left(N-1-i\right)_{-i} \ .$$

- There are $N+2$ trivial gapped vacua due to chiral symmetry breaking. The domain wall theory interpolating between $|1\rangle$ and $|i+1\rangle$ is

$$O\left(i\right)^{1}_{N+2-i,N+1-i} \leftrightarrow O\left(N+2-i\right)^{1}_{-i,-i+1} \ .$$

Conclusions

- Discrete anomalies associated to interesting discrete topological classes severely constrain the dynamics of gauge theories in 2+1 dimensions. The same is also true for gauge theories in 1+1 and 3+1 dimensions. There are many recent interesting papers and progress on such ideas.
- A conjecture for the infrared dynamics of 2+1 dimensional adjoint QCD. Some of the ideas here should be certainly testable on the lattice.

- We used these ideas to infer the domain wall theories in Yang-Mills theory with gauge groups $Sp(N)$ and $Spin(N)$ and a massless Weyl fermion in a two-index representation.
- QCD with $N_f > 1$ Majorana adjoint fermions? Non-simply connected groups? Other simply connected groups? Domain walls in $\mathcal{N} = 1^*$, QCD etc...
- Can we shed light on these non-SUSY quantum phases with brane constructions? See for instance recent works or [Armoni et al. , Argurio-Bertolini...]

Thank you for the attention !!