



Partition Functions Of Twisted Supersymmetric Gauge Theories On Four-Manifolds Via u-Plane Integrals

Gregory Moore
Rutgers University

Mostly review. Includes new work
with Iurii Nidaiev & Jan Manschot

StringMath2018, Tohoku Univ.
June 20, 2018

1 Introduction

2 Cambrian: Witten Reads Donaldson

3 Carboniferous: SW Theory & u-plane Derivation
Of Witten's Conjecture.

4 Holocene: Three New Apps

5 Holomorphic Anomaly & Continuous Metric Dep.

6 Back To The Future



Donaldson Invariants Of 4-folds

X : Smooth, compact, oriented, $\partial X = \emptyset$, $(\pi_1(X) = 0)$

$P \rightarrow X$: Principal $SO(3)$ bundle.

X has metric $g_{\mu\nu}$. Consider moduli space of instantons:

$$\mathcal{M} := \{ A: F + * F = 0 \} \text{ mod } \mathcal{G}$$

Donaldson defines cohomology classes in \mathcal{M} associated to points and surfaces in X : $\mu(p)$ & $\mu(S)$

$$\wp_D(p^\ell S^r) := \int_{\mathcal{M}} \mu(p)^\ell \mu(S)^r$$

Independent of metric! \Rightarrow smooth invariants of X .

Combined with Freedman theorem: Spectacular!

Witten's Interpretation: Topologically Twisted SYM On X

Consider $\mathcal{N} = 2$ SYM theory on X for gauge group G

Witten's "topological twisting": Couple to special external gauge fields for certain global symmetries.

Result: Fermion fields and susy operators are differential forms; The twisted theory is defined on non-spin manifolds.

And there is a scalar susy operator with $Q^2 = 0$

Formally: Correlation functions of operators in Q -coho. localize to integrals over the moduli spaces of G -ASD connections (or generalizations thereof).

Witten's proposal: For $G = SU(2)$ correlation functions of the special operators are the Donaldson polynomials.

Local Observables

$$\wp \in \text{Inv}(\mathfrak{g}) \Rightarrow U = \wp(\phi) \quad \phi \in \Omega^0(\text{ad}P \otimes \mathbb{C})$$

$$H_*(X, \mathbb{C}) \xrightarrow{\text{Descent formalism}} H^*(\text{Fieldspace}; Q)$$



$$H^*(\mathcal{M})$$

$$\mathfrak{g} = \mathfrak{su}(2) \quad U(p) = \text{Tr}_2 \left(\frac{\phi^2}{8\pi^2} \right) \quad U(S) \sim \int_S \text{Tr}(\phi F + \psi^2)$$

Donaldson-Witten Partition Function

$$Z_{DW}(p, S) = \langle e^{U(p)+U(S)} \rangle_{\Lambda_0}$$

$$\int_X \text{Tr } F^2 = 8\pi^2 k$$

$$= \sum_{k \geq 0} \Lambda_0^{4k} \int_{\mathcal{M}_k} e^{\mu(p)+\mu(S)}$$

Strategy: Evaluate in LEET \Rightarrow Witten (1994) introduces the Seiberg-Witten invariants.

Major success in
Physical Mathematics.



What About Other N=2 Theories?

Natural Question: Given the successful application of $\mathcal{N} = 2$ SYM for $SU(2)$ to the theory of 4-manifold invariants, are there interesting applications of OTHER $\mathcal{N} = 2$ field theories?

Topological twisting just depends on $SU(2)_R$ symmetry and makes sense for any $\mathcal{N} = 2$ theory.

$$Z^{\mathcal{T}} := \langle e^{U(p)+U(s)} \rangle_{\mathcal{T}}$$

Also an interesting exercise in QFT to compute correlation functions of nontrivial theories in 4d.

SU(2) With Matter On X

$$\mathcal{R} = 2^{\oplus N_{fl}} \quad \text{Mass parameters } m_f \in \mathbb{C}, f = 1, \dots, N_{fl}$$

$$M \in \Gamma(S^+ \otimes E^{\oplus N_{fl}}) \quad \longrightarrow \quad w_2(P) = w_2(X)$$

$$\mathcal{N} = 2^*: \mathcal{R} = (\text{adj}_{\mathbb{C}} \oplus \text{adj}_{\mathbb{C}}^*) \quad m \in \mathbb{C}$$

$$M \in \Gamma(S^+ \otimes \mathcal{L}^{\frac{1}{2}} \otimes \text{ad}P) \quad \longrightarrow \quad c_1(\mathcal{L}) \equiv w_2(X)$$

UV Interpretation

$$\begin{aligned} Z(p, S) &= \langle e^{U(p)+U(S)} \rangle_{\mathcal{T}} \\ &= \sum_k \Lambda^k \int_{\mathcal{M}_k} e^{\mu(p)+\mu(S)} \mathcal{E} \end{aligned}$$

But now \mathcal{M}_k : is the moduli space of:

$$F^+ = \mathcal{D}(M, \bar{M}) \quad \gamma \cdot D M = 0$$

“Generalized monopole equations”

[Labastida-Marino; Losev-Shatashvili-Nekrasov]

U(1) case: Seiberg-Witten equations.



1 Introduction

2 Cambrian: Witten Reads Donaldson

3 Carboniferous: SW Theory & u-plane Derivation
Of Witten's Conjecture.

4 Holocene: Three New Apps

5 Holomorphic Anomaly & Continuous Metric Dep.

6 Back To The Future

Coulomb Branch Vacua On \mathbb{R}^4

$SU(2) \rightarrow U(1)$ by vev of
adjoint Higgs field ϕ :

Order parameter:
 $u = \langle U(p) \rangle \in \mathbb{C}$

Coulomb branch: $\mathcal{B} = \mathfrak{t} \otimes \mathbb{C}/W \cong \mathbb{C}$ $adP \rightarrow L^2 \oplus \mathcal{O} \oplus L^{-2}$

Photon: Connection A on L $U(1)$ VM: (a, A, χ, ψ, η)

a : complex scalar field on \mathbb{R}^4 :

Compute couplings in $U(1)$ LEET.

Then compute path integral with this action.

Then integrate over vacua.

Seiberg-Witten Theory: 1/2

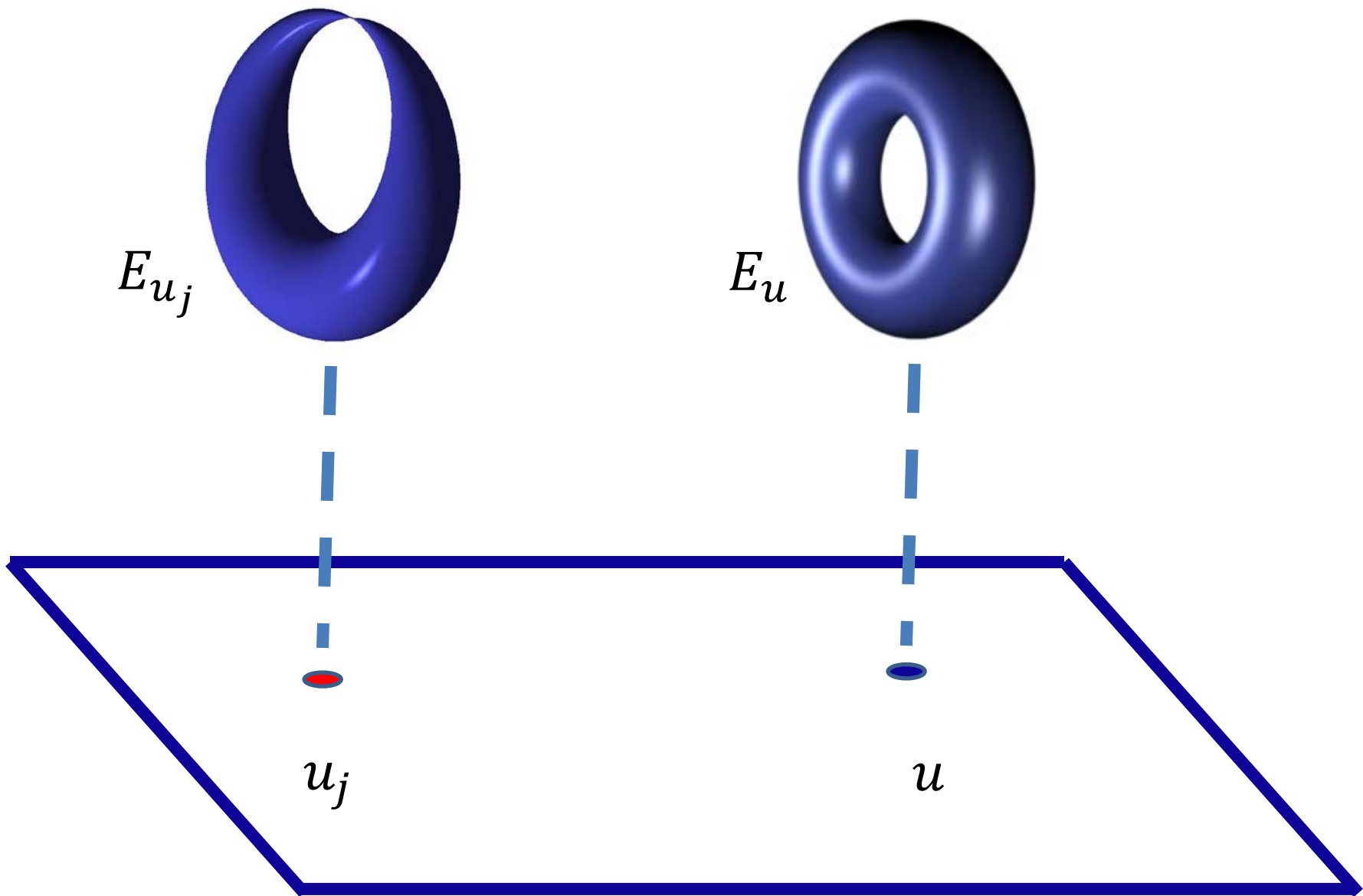
For $G=\text{SU}(2)$ SYM coupled to matter the LEET can be deduced from a holomorphic family of elliptic curves with differential:

$$E_u: \quad y^2 = 4x^3 - g_2x - g_3 \quad \frac{d\lambda}{du} = \frac{dx}{y} \quad u \in \mathbb{C}$$

g_2, g_3 are polynomials in u , masses, Λ , modular functions of τ_0

$$\Delta := 4(g_2^3 - 27g_3^2) : \text{polynomial in } u$$

$$\Delta(u_j) = 0 : \text{Discriminant locus}$$



Examples From SW '94

$$N_{fl} = 0: \quad y^2 = (x - u)(x - \Lambda^2)(x + \Lambda^2)$$

$$\mathcal{N} = 2^*: \quad y^2 = \prod_{i=1}^3 (x - \alpha_i) \quad \alpha_i = u e_i(\tau_0) + \frac{m^2}{4} e_i(\tau_0)^2$$

$$N_{fl} = 4: \quad y^2 = W_1 W_2 W_3 + \eta^{12} \sum_{i=1}^3 R_4^i W_i + \eta^{24} R_6$$

$$W_i = x - u e_i(\tau_0) - R_2 e_i(\tau_0)^2$$

Δ : 6th order polynomial in u with $\sim 3 \times 10^3$ terms

$$\mathfrak{m} = \begin{pmatrix} 0 & m_1 & & & & & & \\ -m_1 & 0 & & & & & & \\ & & 0 & m_2 & & & & \\ & & -m_2 & 0 & & & & \\ & & & & 0 & m_3 & & \\ & & & & -m_3 & 0 & & \\ & & & & & & 0 & m_4 \\ & & & & & & -m_4 & \end{pmatrix} \in \mathfrak{spin}(8)$$

$$R_2 \sim \text{Tr}_{8_i} \mathfrak{m}^2 \quad R_4^i \sim \text{Tr}_{8_i} \mathfrak{m}^4 \quad R_6 \sim \text{Tr}_{8_i} \mathfrak{m}^6$$

$$SL(2, \mathbb{Z}) \rightarrow \text{Aut}(\mathfrak{spin}(8)) \cong S_3$$

Local System Of Charges

Electro-mag. charge lattice: $\Gamma_u = H_1(E_u; \mathbb{Z})$

has nontrivial monodromy around
discriminant locus: $\Delta(u_j) = 0$

LEET: Requires choosing a duality frame:

$$\Gamma_u \cong \mathbb{Z}\gamma_e \oplus \mathbb{Z}\gamma_m \Rightarrow \tau(u)$$

$$S \sim \int_X \bar{\tau} F_+^2 + \tau F_-^2 + \dots$$



$$N_{fl} = 0$$



$$N_{fl} = 1$$

$$\mathcal{N} = 2^*$$

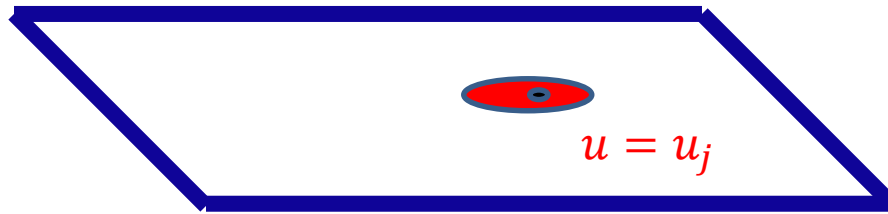


$$N_{fl} = 4$$

LEET breaks down at $u = u_j$ where $Im(\tau) \rightarrow 0$

Seiberg-Witten Theory: 2/2

LEET breaks down because there are new massless fields associated to BPS states



Charge 1 HM: $(M = q \oplus \tilde{q}^*, \dots)$

LEET in \mathcal{U}_j : $+$

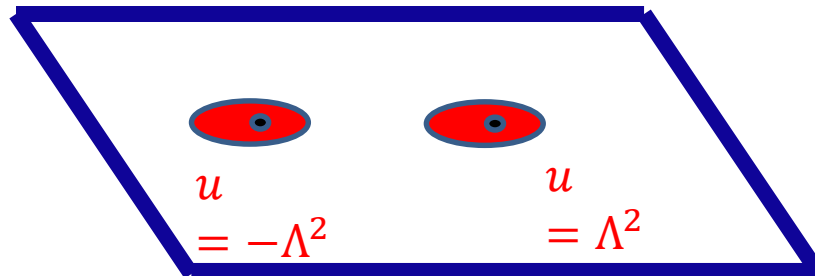
$U(1)_j$ VM: $(a, A, \chi, \psi, \eta)_j$

Evaluate Z_{DW} Using LEET

$$Z_{DW}(p, S) = \langle e^{U(p)+U(S)} \rangle_{\Lambda}$$

$$= \sum_k \Lambda^k \int_{\mathcal{M}_k} e^{\mu(p)+\mu(S)} \varepsilon$$

$$Z_{DW} = Z_u + \sum_{u_j} Z_j^{SW}$$



u -Plane Integral Z_u

Can be computed explicitly from QFT of LEET

Vanishes if $b_2^+ > 1$. When $b_2^+ = 1$:

$$Z_u = \int du d\bar{u} \mathcal{H} \Psi$$

\mathcal{H} is holomorphic and metric-independent

Ψ : Sum over line bundles for the U(1) photon.

(Remnant of sum over SU(2) gauge bundles.)

$$\Psi \sim \sum_{\lambda=c_1(L)} e^{-i \pi \bar{\tau}(\bar{u}) \lambda_+^2 - i \pi \tau(u) \lambda_-^2}$$

NOT holomorphic and metric-DEPENDENT

Comments On Z_u

Z_u is a very subtle integral.

It requires careful regularization
and definition.

It is also related to integrals from number theory
such as “ Θ -lifts” and mock modular forms.
It is very nearly an integral of a total derivative...

But first let's finish writing down the
full answer for the partition function.

Contributions From \mathcal{U}_j

Path integral for $U(1)_j$ VM + HM: LEET: Need unknown couplings: $\mathbf{C}(\mathbf{u})^{\lambda^2} P(u)^\sigma E(u)^\chi$

$$\sum_{\lambda \in \frac{1}{2}w_2(X) + H^2(X, \mathbb{Z})} SW(\lambda) e^{2\pi i \lambda \cdot \lambda_0} R_\lambda(p, S)$$

$$R_\lambda(p, S) = \text{Res} \left[\left(\frac{da_j}{a_j^{1 + \frac{d(\lambda)}{2}}} \right) e^{2pu + S^2 T(u) + i \left(\frac{du}{da_j} \right) S \cdot \lambda} \mathbf{C}(\mathbf{u})^{\lambda^2} P(u)^\sigma E(u)^\chi \right]$$

Special coordinate: $a_j \quad u = u_j + \kappa_j a_j + \mathcal{O}(a_j^2)$

$$d(\lambda) = \frac{(2\lambda)^2 - c^2}{4} \quad c^2 = 2\chi + 3\sigma$$

Deriving C,P,E From Wall-Crossing

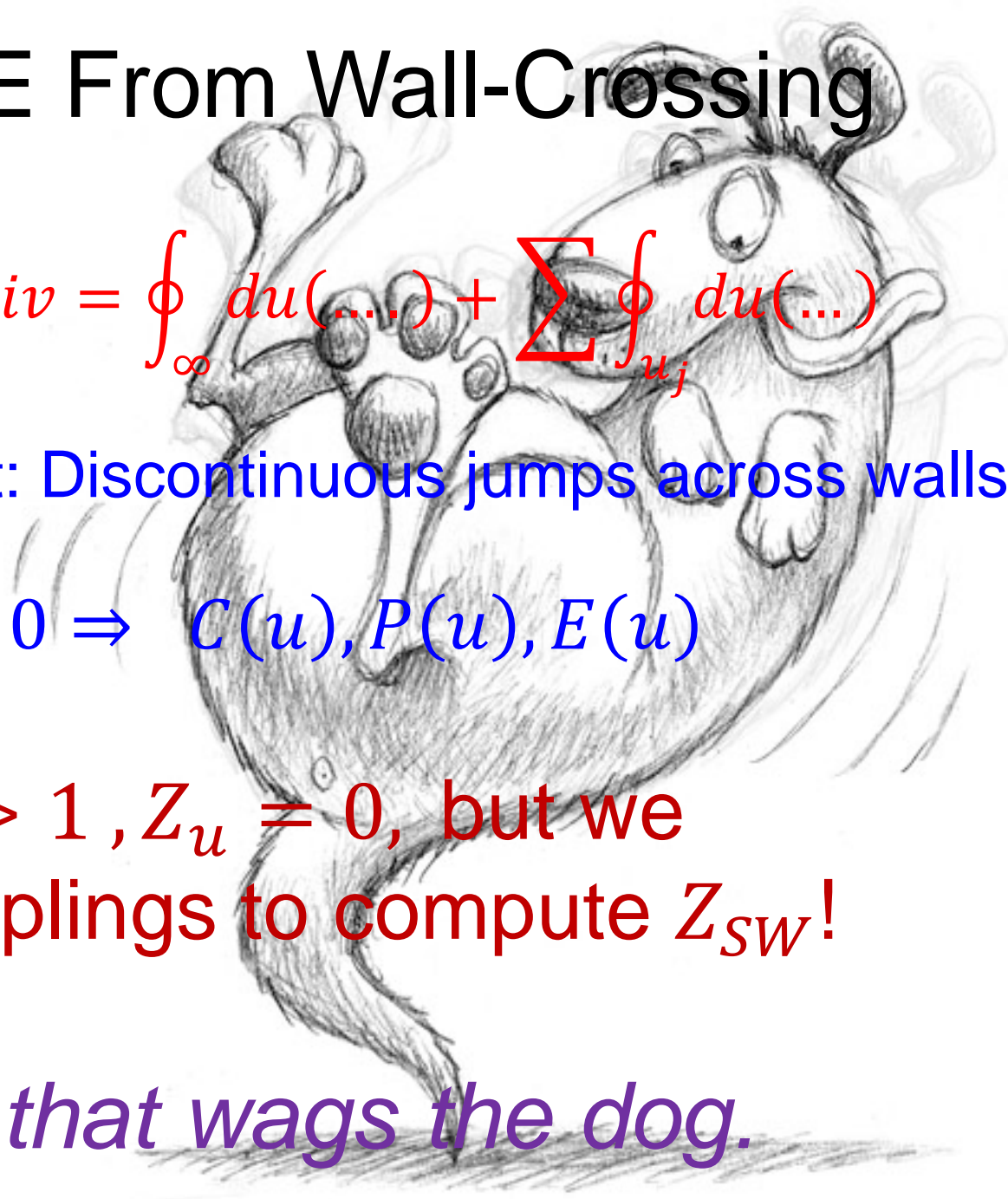
$$\frac{d}{dg_{\mu\nu}} Z_u = \int \text{Tot deriv} = \oint_{\infty} du(\dots) + \sum \oint_{u_j} du(\dots)$$

Z_u piecewise constant: Discontinuous jumps across walls.

$$\Delta_j Z_u + \Delta Z_j^{SW} = 0 \Rightarrow C(u), P(u), E(u)$$

Then for $b_2^+ > 1$, $Z_u = 0$, but we know the couplings to compute Z_{SW} !

Z_u is the tail that wags the dog.



Witten Conjecture

\Rightarrow a formula for all X with $b_2^+ > 0$. For $b_2^+(X) > 1$ we derive "Witten's conjecture":

$$Z_{DW}(p, S) = 2^{c^2 - \chi_h} \left(e^{\frac{1}{2}S^2 + 2p} \sum_{\lambda} SW(\lambda) e^{2\pi i \lambda \cdot \lambda_0} e^{S \cdot \lambda} + \right. \\ \left. + i\chi_h e^{-\frac{1}{2}S^2 - 2p} \sum_{\lambda} SW(\lambda) e^{2\pi i \lambda \cdot \lambda_0} e^{-iS \cdot \lambda} \right)$$

$$\chi_h = \frac{\chi + \sigma}{4}$$

Example: $X = K3$:

$$c^2 = 2\chi + 3\sigma$$

$$Z = \Lambda_0^6 \sinh\left(\frac{(\Lambda_0 S)^2}{2} + 2\Lambda_0^2 p\right)$$

SWST

SWST = Seiberg-Witten Simple Type

$SW(\lambda) \neq 0$ ONLY when moduli space of solutions to Seiberg-Witten is zero-dimensional.

All known X with $b_2^+ > 1$ are of SWST

Generalization To $N_{fl} > 0$

$$b_2^+ > 1 \quad Z(p; S; m_f) = \sum_{j=1}^{2+N_{fl}} Z(p, S; m_f; u_j)$$

$$Z(p, S; m_f; u_j) = \sum_{\lambda} SW(\lambda) e^{2\pi i \lambda \cdot \lambda_0} R_j(p, S)$$

X is SWST \Rightarrow

$R_j(p, S)$ is computable explicitly as a function of $p, S, m_f, \Lambda(\tau_0)$ from first order degeneration of the SW curve.

$$Z(p, S; m_f; \Lambda) =$$

$$\alpha^\chi \beta^\sigma \sum_{j=1}^{2+N_{fl}} \mu_j^{\chi h} \nu_j^{\chi h + \sigma} e^{2p u_j + S^2 T_j} F_j(S)$$

$$\mu_j = g_2(u_j)^3 / \prod_{t \neq j} (u_j - u_t) \quad \nu_j \sim (g_2(u_j))^{1/4}$$

$$F_j(S) = \sum_{\lambda} SW(\lambda) e^{i \pi \lambda \cdot c_1(s)} \cos(\nu_j \lambda \cdot S)$$

$$y^2 = 4x^3 - g_2(u, m_j)x - g_3(u, m_j)$$

1 Introduction

2 Cambrian: Witten Reads Donaldson

3 Carboniferous: SW Theory & u-plane Derivation
Of Witten's Conjecture.

4 Holocene: Three New Apps

5 Holomorphic Anomaly & Continuous Metric Dep.

6 Back To The Future



Application 1: S-Duality Of $N_{fl} = 4$

$Z(p, S; \tau_0)$ is expected to have modular properties:

$$\alpha^\chi \beta^\sigma \sum_{j=1}^6 \mu_j^{\chi h} \nu_j^{\chi h + \sigma} e^{2p u_j + S^2 T_j} F_j(S)$$

$$\mu_j = g_2(u_j)^3 / \prod_{t \neq j} (u_j - u_t) \quad \nu_j \sim \left(g_2(u_j) \right)^{\frac{1}{4}}$$

Sum over j gives **symmetric rational**
function of $u_j \Rightarrow Z$ will be modular:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

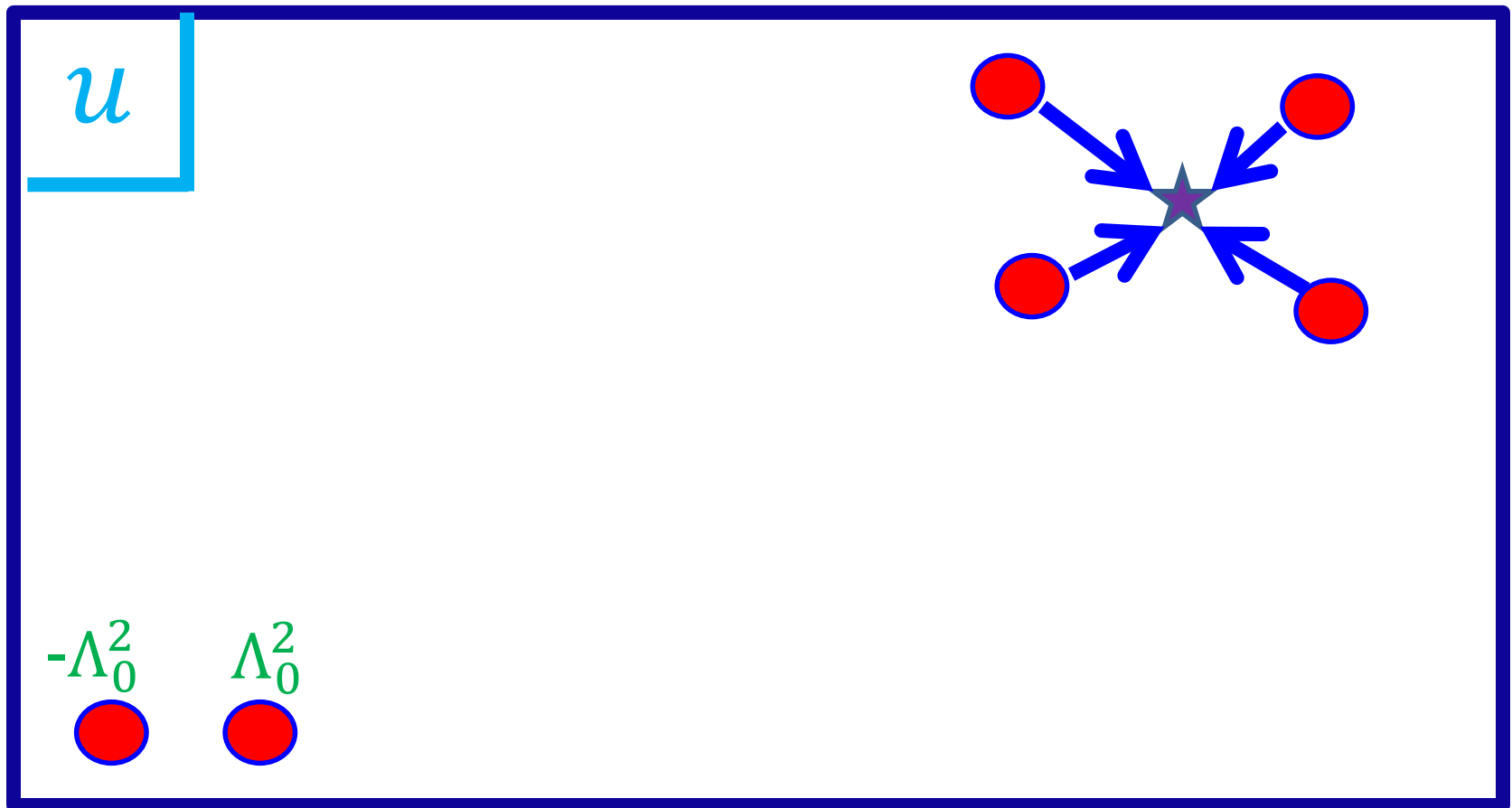
$$Z \left(\frac{p + c(c\tau_0 + d)S^2}{(c\tau_0 + d)^2}, \frac{S}{(c\tau_0 + d)^2}; \frac{a\tau_0 + b}{c\tau_0 + d}; \gamma \cdot \mathfrak{m} \right)$$

$$= (c\tau_0 + d)^w Z(p, S; \tau_0; \mathfrak{m})$$

$$w = \frac{2\chi + 3\sigma}{2}$$

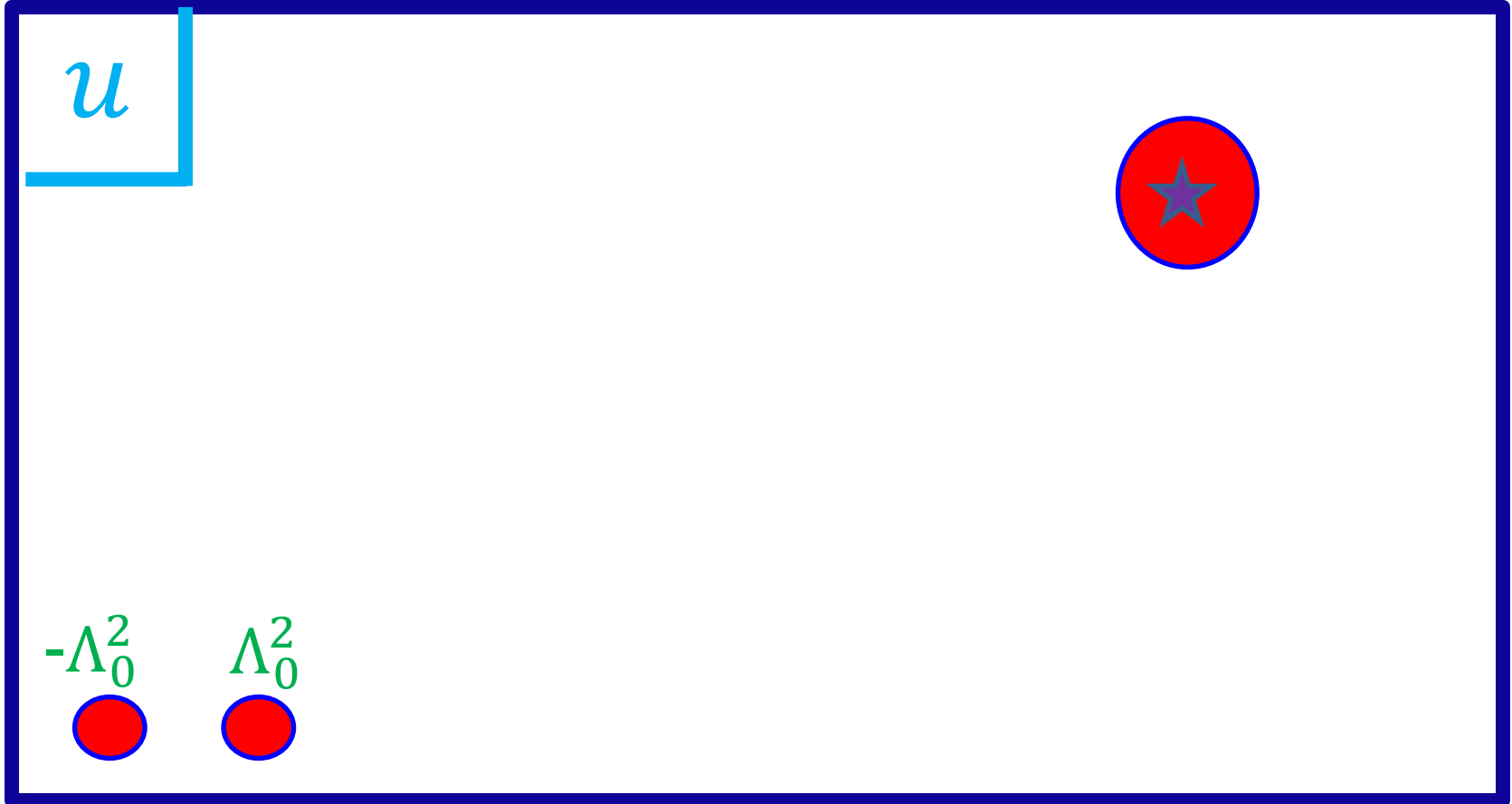
Application 2: Collision Of Mutually Local Singularities

Interesting things when mass parameters m_j
take values so that u_j collide.



Application 2: Collision Of Mutually Local Singularities

Interesting things when mass parameters m_j
take values so that u_j collide.



$$e^{i\pi\tau_0} \sim \frac{\Lambda_0^4}{m_1 m_2 m_3 m_4} \quad m_j = M + \delta m_j$$

$$W^{eff} \sim \sum_f \delta m_f \tilde{Q}^f Q_f$$

$\Rightarrow Z$ should be $SU(N_{fl})$ –equivariant integral over moduli space of $U(1)$ –multimonopole equations:

$$F^+ = \sum_f \bar{M}^f M_f \quad \gamma \cdot DM_f = 0$$

In the limit $\delta m_j \rightarrow 0$ the dominant term in Z is:

$$J(\tilde{S}) \sum_f \prod_{t \neq f} (\delta m_f - \delta m_t)^{-\chi_h} e^{\delta m_f \tilde{p}}$$

$$J(\tilde{S}) = e^{\tilde{S}^2/2} \sum_{\lambda} e^{i \pi \lambda \cdot c_1(\mathfrak{g})} SW(\lambda) \cos(\lambda \cdot \tilde{S})$$

This agrees with computations of Dedushenko, Gukov and Putrov of the equivariant integrals on $U(1)$ -multimonopole space.

Application 3: AD3 Partition Function

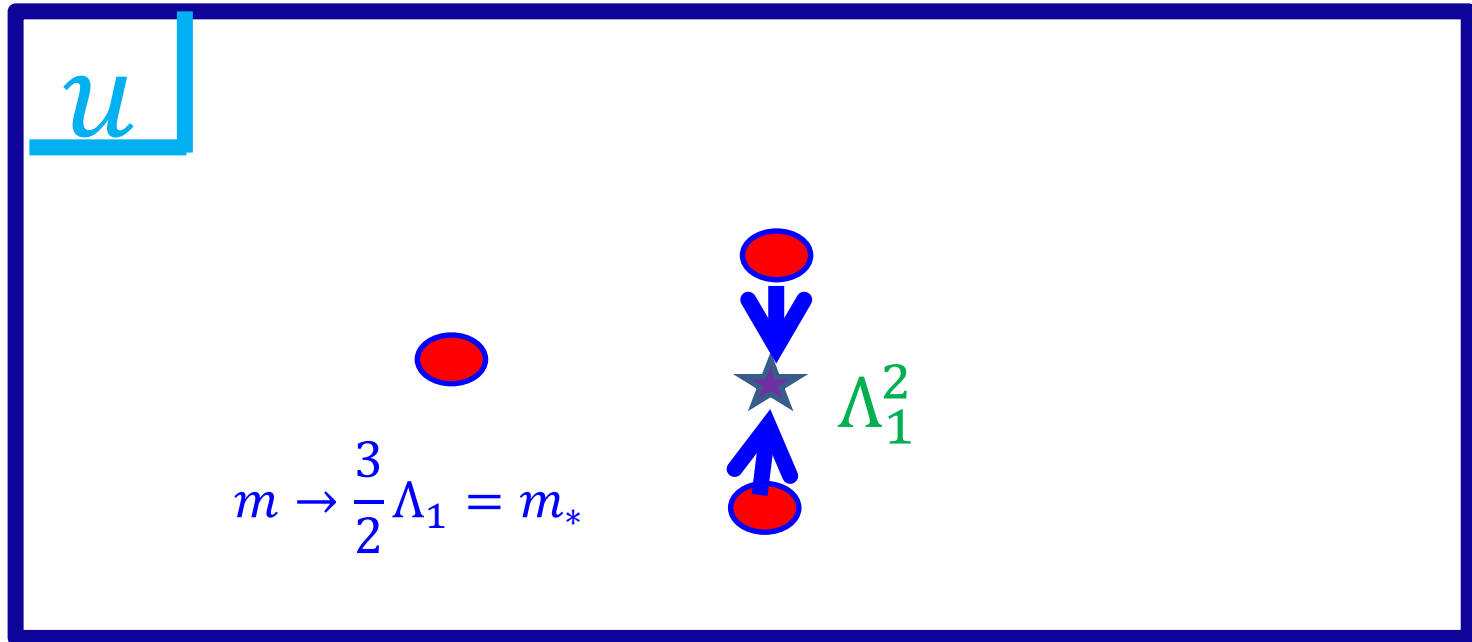
Consider $N_{fl} = 1$. At a critical point $m = m_*$ two singularities u_{\pm} collide at $u = u_*$ and the SW curve becomes a cusp: $y^2 = x^3$ [Argyres, Plesser, Seiberg, Witten]

Two mutually nonlocal BPS states have vanishing mass:

$$\oint_{\gamma_1} \lambda \rightarrow 0 \quad \oint_{\gamma_2} \lambda \rightarrow 0 \quad \gamma_1 \cdot \gamma_2 \neq 0$$

Physically: No local Lagrangian for the LEET :
Signals a nontrivial superconformal field theory
appears in the IR in the limit $m \rightarrow m_*$

AD3 From $SU(2)$ $N_{fl} = 1$



SW curve in the scaling region:

$$y^2 = x^3 - 3 \Lambda_{AD}^2 x + u_{AD}$$

$$\Lambda_{AD}^2 \sim (m - m_*)$$

Call it the “AD3-Family over the u_{AD} -plane”

AD3 Partition Function - 1

No obvious UV definition of the invariant.

The $SU(2) N_{fl} = 1$ u-plane integral has a nontrivial contribution from the scaling region $u_{\pm} \rightarrow u_*$

$$\lim_{m \rightarrow m_*} Z_u - \int du d\bar{u} \lim_{m \rightarrow m_*} \text{Measure}(u, \bar{u}; m) \neq 0$$

Limit and integration commute except in an infinitesimal region around u_*

Attribute the discrepancy to the contribution of the AD3 theory

AD3 Partition Function - 2

1. Limit $m \rightarrow m_*$ exists. (No noncompact Higgs branch.)
2. The partition function is a sum over all Q-invariant field configurations.
3. Scaling region near u_* governed by AD3 theory.

$\lim_{m \rightarrow m_*} Z^{SU(2), N_{fl}=1}$ ``contains'' the AD3 partition function

Extract it from the scaling region. Our result:

$$Z^{AD3} = \lim_{\Lambda_{AD} \rightarrow 0} \left(Z_u^{AD3-family} + Z_{SW}^{AD3-family} \right)$$

Claim: This is the AD3 TFT on X for $b_2^+ > 0$.

AD3 Partition Function: Evidence 1/2

Existence of limit is highly nontrivial. It follows from “superconformal simple type sum rules”:

Theorem [Marino, Moore, Peradze, 1998] If the superconformal simple type sum rules hold:

a.) $\chi_h - c^2 - 3 \leq 0$

b.) $\sum_{\lambda} SW(\lambda) e^{2\pi i \lambda \cdot \lambda_0} \lambda^k = 0 \quad 0 \leq k \leq \chi_h - c^2 - 4$

Then the limit $m \rightarrow m_*$ exists

It is now a rigorous theorem that SWST \Rightarrow SCST

AD3 Partition Function: Evidence 2/2

For $b_2^+ > 0$ recover the expected selection rule:

$$\langle U(p)^\ell U(S)^r \rangle \neq 0 \text{ only for } \frac{6\ell + r}{5} = \frac{\chi_h - c^2}{5}$$

Consistent with background charge for AD3 computed by Shapere-Tachikawa.

Explicitly, if X of $SWST$ and $b_2^+ > 1$:

Explicitly, if X of $SWST$ and $b_2^+ > 1$:

$$Z^{AD3} = \frac{1}{\mathfrak{B}!} K_1 K_2^\chi K_3^\sigma \sum_{\lambda} e^{2\pi i \lambda \cdot w_2} SW(\lambda) \cdot$$

$$\cdot \{(\lambda \cdot S)^{\mathfrak{B}-2} (24(\lambda \cdot S)^2 + \mathfrak{B}(\mathfrak{B}-1)S^2)\}$$

$$\mathfrak{B} = \chi_h - c^2 = -\frac{7\chi + 11\sigma}{4}$$

Surprise! p drops out: $U(p)$ is a "null vector"

Should We Be Surprised?

Already for $N_{fl} = 0$ and SWST it is also true that $U(p)^2 - \Lambda_0^4$ is a null vector.

Without a good physics reason why these should be null vectors one suspects that there are (standard) 4-manifolds not of SWST.

1 Introduction

2 Cambrian: Witten Reads Donaldson

3 Carboniferous: SW Theory & u-plane Derivation
Of Witten's Conjecture.

4 Holocene: Three New Apps

5 Holomorphic Anomaly & Continuous Metric Dep.

6 Back To The Future

Z_u Wants To Be A Total Derivative

$$Z_u = \int du d\bar{u} \mathcal{H}(u) \Psi$$

$$\Psi = \sum_{\lambda} \frac{d}{d\bar{u}} \left(\mathcal{E}(r_{\lambda}^{\omega}, b_{\lambda}) \right) e^{-i\pi\tau\lambda^2 - iS \cdot \frac{du}{da} + 2\pi i(\lambda - \lambda_0) \cdot w_2}$$

$$\mathcal{E}(r, b) = \int_b^r e^{-2\pi t^2} dt \quad r_{\lambda}^{\omega} = \sqrt{y}\lambda_+ - \frac{i}{4\pi\sqrt{y}} S_+ \frac{du}{da}$$

Lower limit b_{λ} must be independent of \bar{u}
but can depend on λ, ω, u .

[Korpas & Manschot; Moore & Nidaiev]

$$Z_u \stackrel{?}{=} \int du d\bar{u} \frac{d}{d\bar{u}} \left(\mathcal{H}(u) \Theta^{\omega} \right)$$

Not obvious: No obvious choice for b_λ with both convergence and modular invariance.

BUT: The difference for two metrics CAN be evaluated by residues!

$$Z_u^\omega - Z_u^{\omega_0} = \int du d\bar{u} \frac{d}{d\bar{u}} (\mathcal{H} \widehat{\Theta}^{\omega, \omega_0})$$

$$\widehat{\Theta}^{\omega, \omega_0} = \sum_\lambda \left(\varepsilon(r_\lambda^\omega) - \varepsilon(r_\lambda^{\omega_0}) \right) e^{-i \pi \tau(u) \lambda^2 + \dots}$$

$$r_\lambda^\omega = \sqrt{y} \lambda_+ - \frac{i}{4\pi \sqrt{y}} S_+ \frac{du}{da}$$

Modular completion of indefinite theta function of Vigneras, Zagier, Zagier

Holomorphic Anomaly & Metric Dependence

For theories with a manifold of superconformal couplings, $\tau(u) \rightarrow \tau_0$ when $u \rightarrow \infty$

$$\widehat{\Theta}^{\omega, \omega_0}(\tau(u), \dots) \rightarrow \widehat{\Theta}^{\omega, \omega_0}(\tau_0, \dots)$$

Contour integral at $\infty \Rightarrow$ For $b_2^+ = 1$, many “topological” correlators:

Nonholomorphic in τ_0

Depend **continuously** on metric

[Moore & Witten, 1997 -- albeit *sotto voce*]

Special Case Of $\mathbb{C}\mathbb{P}^2$

No walls: $\omega \in H^2(X, \mathbb{R}) \cong \mathbb{R}$, so we only see holomorphic anomaly.

\Rightarrow *Path integral* derivation of the holomorphic anomaly.

Vafa-Witten Partition Functions

VW twist of N=4 SYM formally computes the “Euler character” of instanton moduli space.

(Not really a topological invariant. True mathematical meaning unclear, but see recent work of Tanaka & Thomas; Gholampour, Sheshmani, & Yau.)

Physics suggests the partition function is **both** modular (S-duality) **and** holomorphic.

Surprise! Computations of Klyachko and Yoshioka for $X = \mathbb{C}P^2$ show that the holomorphic generating function is only mock modular.

But a nonholomorphic modular completion exists.

*This has never been properly derived
from a path integral argument.*

But we just derived completely analogous results for the $N_{fl} = 4$ theory from a path integral, suggesting VW will also have continuous metric dependence.

Indeed, continuous metric dependence in VW theory for $SU(r > 1)$ has been predicted in papers of Jan Manschot using rather different methods.

We hope that a similar path integral derivation can be found for the $\mathcal{N} = 4$ Vafa-Witten twisted theory.

1

Introduction

2

Cambrian: Witten Reads Donaldson

3

Carboniferous: SW Theory & u-plane Derivation
Of Witten's Conjecture.

4

Holocene: Three New Apps

5

Holomorphic Anomaly & Continuous Metric Dep.

6

Back To The Future

Future Directions

There is an interesting generalization to invariants for families of four-manifolds.

Couple to “topologically twisted supergravity”

$$Z \in H^*(BDiff(X))$$

Generalization to all theories in class S:
Many aspects are clear – this is under study.

u-plane for class S: General Remarks

UV interpretation is not clear in general.

These theories might give new 4-manifold invariants.

The u-plane is an integral over the base \mathcal{B} of a Hitchin fibration with a theta function associated to the Hitchin torus. It will have the form

$$Z_u = \int_{\mathcal{B}} du d\bar{u} \mathcal{H} \Psi$$

\mathcal{H} is holomorphic and metric-independent

Ψ : NOT holomorphic and metric-DEPENDENT
“theta function”

Class S: General Remarks

$$\mathcal{H} = \alpha^\chi \beta^\sigma \det \left(\frac{da^i}{du_j} \right)^{1 - \frac{\chi}{2}} \Delta_{phys}^{\frac{\sigma}{8}}$$

Δ_{phys} a holomorphic function on \mathcal{B} with first-order zeros at the loci of massless BPS hypers

α, β will be automorphic forms on Teichmuller space of the UV curve \mathcal{C}

α, β are related to correlation functions for fields in the (0,2) QFT gotten from reducing 6d (0,2)

Class S: General Remarks

$$\Psi \sim \sum_{\lambda} e^{i\pi\lambda\cdot\xi} e^{-i\pi\bar{\tau}(\lambda_+, \lambda_+) - i\pi\tau(\lambda_- \cdot \lambda_-) + \dots}$$

$$\lambda \in \lambda_0 + \Gamma \otimes H^2(X; \mathbb{Z})$$

$$\xi \in \Gamma \otimes H^2(X; \mathbb{R})$$

$$\Gamma \subset H^1(\Sigma; \mathbb{Z})$$

Lagrangian
sublattice

If $\xi = \rho \otimes w_2(X) \bmod 2$ **then** WC from interior of \mathcal{B} will be cancelled by SW invariants

\Rightarrow No new four-manifold invariants...

Ψ comes from a “partition function” of a level 1 SD 3-form on $M_6 = \Sigma \times X$

Quantization: Choose a QRIF Ω on $H^3(M_6; \mathbb{Z})$

Natural choice: [Witten 96,99; Belov-Moore 2004]

$$\Omega(x) = \exp\left(i \pi \text{WCS}(\theta \cup x; S^1 \times M_6) \right)$$

Choice of weak-coupling duality frame + natural choice of $spin^c$ structure gives

$$\xi = \rho \otimes w_2(X)$$

Case Of $SU(2)$ $\mathcal{N} = 2^*$

Using the tail-wagging-dog argument, analogous formulae were worked out for $\mathcal{N} = 2^*$, by Moore-Witten and Labastida-Lozano in 1998, but only in the case when X is spin.

L&L checked S-duality for the case $b_2^+ > 1$

The generalization to X which is NOT spin is nontrivial: The standard expression from Moore-Witten and Labastida-Lozano is NOT single-valued on the u-plane.

This is not surprising: The presence of external $U(1)_{baryon}$ gauge field $F_{baryon} \sim c_1(\mathfrak{s})$ means there should be new interactions:

$$e^{\kappa_1(u)c_1(\mathfrak{s})^2 + \kappa_2(u)\lambda \cdot c_1(\mathfrak{s})} \quad \text{Shapere \& Tachikawa}$$

Holomorphy, 1-loop singularities,
single-valuedness forces:

$$(u - u_1) \frac{c_1(\mathfrak{s})^2}{8} e^{-i \frac{\partial a_D}{\partial m} \lambda \cdot c_1(\mathfrak{s})}$$

Surprise!!

It doesn't work!

Correct version appears to be non-holomorphic.

With Jan Manschot we have an alternative which is currently being checked.

Does the u-plane integral make sense for ANY family of Seiberg-Witten curves ?

どうもありがとうございます