Mirror symmetry, intersection of quadrics, and Hodge theory

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Outline

- joint with R. Donagi and C. Simpson
- HMS for the moduli of flat bundles on curves.
- Non-abelian Hodge theory as a tool for constructing objects in the Fukaya category (= quantum A-branes).
- Examples: Automorphic sheaves on intersections of quadrics.

HMS for moduli spaces (i)

Main characters: The moduli of flat bundles and the moduli of HIggs bundles on an algebraic curve.

Setup:

- C - a smooth compact curve of genus $g > 1$;
- G, ${}^{L}G$ - a pair of affine semisimple algebraic group over $\mathbb{C}$.

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To avoid special considerations

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Loc = moduli space of flat algebraic G bundles on C
= moduli of pairs $\mathbb{V} = (V, \nabla)$, with V a principal G bundle on C, ∇ an algebraic integrable connection on V.

Higgs = moduli space of algebraic Higgs G bundles on C
= moduli of pairs $\mathbb{E} = (E, \theta)$, with E a principal G bundle on C, $\theta \in H^0(C, \mathrm{ad}(E) \otimes \Omega^1_C)$.

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^{L}**Loc**, ^{L}**Higgs** - the analogous moduli for structure group ${}^LG$.

HMS for moduli spaces (ii)

> ^{L}**Loc** and **Higgs** are **mirror** Calabi-Yau (hyper-Kähler) spaces.

Explanation: SYZ Mirror Symmetry

- ^{L}**Loc** and ^{L}**Higgs** belong to the same twistor family and are related by a hyper-Kähler $\boxed{\text{rotation}}$.

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General fibers are holoLag abelian varieties

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- ^{L}**Higgs** and **Higgs** have Hitchin maps $^Lh : {}^L\mathbf{Higgs} \to {}^LB$ and $h : \mathbf{Higgs} \to B$ which are integrable systems.
- There is a natural identification $B \cong {}^LB$ under which $h : \mathbf{Higgs} \to B$ and $^Lh : {}^L\mathbf{Higgs} \to {}^LB$ become dual families of abelian varieties (cf [Donagi-P]).

HMS for moduli spaces (iii)

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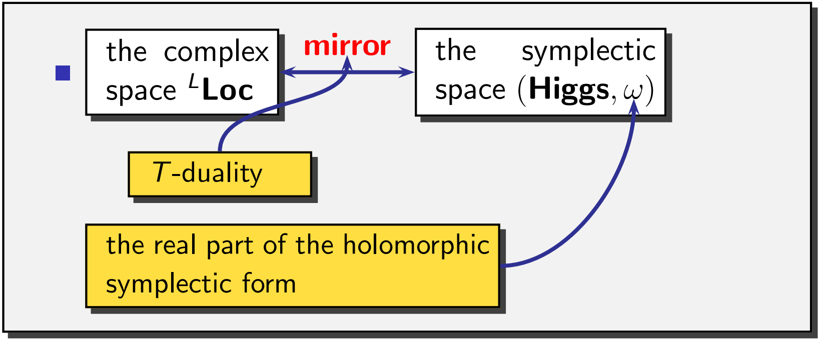

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- the complex space ${}^L\mathbf{Loc}$ **mirror** ⟷ the symplectic space $(\mathbf{Higgs}, \omega)$
- **Homological Mirror Symmetry** holds: there is a mirror equivalence of categories

$$\mathsf{hms} : D({}^L\mathbf{Loc}, \mathcal{O}) \xrightarrow{\cong} \mathbf{Fuk}^{\mathrm{wr}}(\mathbf{Higgs}, \omega).$$

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Goal: Understand the equivalence hms in geometric terms.

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Explanation:

- $\mathbf{Higgs} \cong T^{\vee}\, \mathbf{Bun}$ where $\mathbf{Bun}$ is the moduli of algebraic G-bundles on C. In particular each cotangent fiber $T^{\vee}_{E}\, \mathbf{Bun}$ is an object in $\mathbf{Fuk}^{\mathrm{wr}}(\mathbf{Higgs}, \omega)$

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Lucky break: The wrapped Fukaya category $\mathbf{Fuk}^{\mathrm{wr}}(\mathbf{Higgs}, \omega)$ admits an equivalent description in terms of D-modules.

Explanation:

- $\mathbf{Higgs} \cong T^\vee \mathbf{Bun}$ where $\mathbf{Bun}$ is the moduli of algebraic G-bundles on C.
- Floer theory (**Abouzaid, Fukaya-Seidel-Smith**) assigns a D-module on $\mathbf{Bun}$ to any $P \in \mathrm{ob}\,\mathbf{Fuk}^{\mathrm{wr}}(\mathbf{Higgs}, \omega)$:
  - P induces a stratification on $\mathbf{Bun}$

$$\mathsf{S}_k = \{E \in \mathbf{Bun} \mid \dim HF(P, T^\vee_E \mathbf{Bun}) = k\}.$$

  - Family Floer theory endows the bundle of Floer homologies on S_k with a flat connection.

HMS for moduli spaces (v)

Upshot: In this context HMS can be viewed as an equivalence

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Derived category of coherent
D-modules on **Bun**

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Note: This is precisely the setting of the Geometric Langlands correspondece (GLC) which predicts that there is a natural equivalence of categories:

$$\text{(GLC)} \qquad \boxed{\mathfrak{c} : D({}^L\mathbf{Loc}, \mathcal{O}) \xrightarrow{\cong} D(\mathbf{Bun}, \mathcal{D}),}$$

uniquely characterized by the property that $\mathfrak{c}$ intertwines the natural symmetries of the source (tensorization operators) and the target (Hecke operators).

HMS for moduli spaces (vi)

Recasting of the problem:

- The cotangent bundle structure of **Higgs** and family Floer theory convert hms into $\mathfrak{c}$.
- The mirrors of the B-branes (coherent sheaves) on ${}^L\mathbf{Loc}$ are naturally D-modules on **Bun**.

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Consistent with the Gukov-Witten big brane quantization procedure

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Note: The GLC map $\mathfrak{c}$ is uniquely characterized by the property that it sends the structure sheaves of points $\mathbb{V}$ in ${}^L\mathbf{Loc}$ to Hecke eigen D-modules $\mathfrak{c}(\mathcal{O}_{\mathbb{V}})$ on **Bun**:

$$H^{\mu}\left(\mathfrak{c}(\mathcal{O}_{\mathbb{V}})\right) = \mathfrak{c}(\mathcal{O}_{\mathbb{V}}) \boxtimes \rho^{\mu}(\mathbb{V}).$$

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Here μ is an appropriate character, and H^{μ} is the Hecke correspondence on **Bun** bounded by μ.

HMS for moduli spaces (vii)

Strategy:

- Use non-abelian Hodge theory (NAHT) to rewrite the D-module eigensheaf problem as an eigensheaf probem for (parabolic) Higgs sheaves.
- Use Fourier-Mukai duality (cf. **Hausel-Thaddeus, Donagi-P**) for the Hitchin systems $\mathbf{Higgs} \to B$ and ${}^{L}\mathbf{Higgs} \to {}^{L}B$ to construct a Higgs sheaf satisfying the NAHT and Hecke eigensheaf conditions.

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Remark: For any point $\mathbb{V} \in {}^{L}$**Loc** this gives: the corresponding Hecke eigensheaf on **Bun** or equivalently the object in the Fukaya category of **Higgs** which mirrors the skyscraper sheaf $\mathcal{O}_{\mathbb{V}}$.

Intersections of quadrics

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- $G = PSL_2$, ${}^LG = SL_2$ and C is a smooth curve of genus 2.

In the first case the connected components of **Bun** are related to the intersection of two quadrics in $\mathbb{P}^4$ while in the second case they are related to the intersection of two quadrics in $\mathbb{P}^5$.

Eigensheaves on del Pezzo surfaces (i)

Dictionary: Suppose Σ - an orbifold curve which is generically a variety with underlying curve C and divisor of orbifold points $D \subset C$. Then

$$\begin{pmatrix} \text{holomorphic} \\ \text{Higgs bundles } \Sigma \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{tamely ramified strongly} \\ \text{parabolic Higgs bundles} \\ \text{on } (C, D) \end{pmatrix}.$$

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In particular: Can use parabolic language on (C, D) to pose ans solve the Hecke eigensheaf problem on Σ.

Eigensheaves on del Pezzo surfaces (ii)

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Note:

- The moduli space of rank two parabolic bundles on $(C, \mathbf{Par}_C)$ depends on a set of numerical invariants - the degree of the level zero bundle in the parabolic family and the set of parabolic weights.

- The collection of weights has a chamber structure and the moduli space depends only on the chamber and not on the particular collection of weights in that chamber.

Eigensheaves on del Pezzo surfaces (ii)

Fix $C = \mathbb{P}^1$, and let $\mathbf{Par}_C = p_1 + p_2 + p_3 + p_4 + p_5$.

Theorem: [Donagi-P] There is a maximal chamber of parabolic weights such that:

- every semistable parabolic bundle is stable;
- the connected components of the moduli space corresponding to different degrees are canonically isomorphic to the dP_5 del Pezzo surface

$$X = \mathrm{Bl}_{\mathbf{Par}_C}(S^2 C).$$

Here $C \subset S^2 C$ diagonally, i.e. X is obtained by blowing up the 5 points $\{p_i\}_{i=1}^5$ on the conic $C \subset S^2 C \cong \mathbb{P}^2$.

Eigensheaves on del Pezzo surfaces (iii)

Equivalently:

- X can be described in its anticanonical model as the intersection of two quadrics in $\mathbb{P}^4$.
- The parameter space of the pencil of quadrics vanishing on X is naturally identified with C and the divisor $\mathbf{Par}_C$ corresponds to the locus of singular quadrics in the pencil.

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Theorem: [Donagi-P] The wobbly locus in X is the union of the 16 lines $L_I \subset X$.

Note: The 16 lines in $X \subset \mathbb{P}^4$ are naturally labeled by the subsets $I \subset \{1, 2, 3, 4, 5\}$ of odd cardinality.

Eigensheaves on del Pezzo surfaces (iv)

From the point of view of the anti-canonicla model the basic Hecke correspondence parametrizing the modifications of bundles at a single point can be compactified and resolved to the correspondence

$$\begin{array}{ccc} & H & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ X & & X \times C \end{array}$$

Eigensheaves on del Pezzo surfaces (iv)

$$\begin{array}{ccc} & H & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & & X \times C \end{array}$$

Here:

- $H = \mathrm{Bl}_{\coprod_I \widehat{L_I \times L_I}} \mathrm{Bl}_{\Delta}(X \times X)$;
- the two maps $H \to X$ correspond to the blow down map $H \to X \times X$ followed by the first or second projection;
- the map $H \to C$ is the resolution of the rational map $X \times X \dashrightarrow C$ which sends $(x, y) \in X \times X$ to the unique $\lambda \in C$ such that $Q_\lambda \subset \mathbb{P}^4$ contains the line through the two points $x, y \in \mathbb{P}^4$.

Eigensheaves on del Pezzo surfaces (v)

Note:

- By construction H is smooth. The general fibers of q are smooth rational curves (Hecke lines) and the general fibers of p are smooth dP_6 del Pezzo surfaces.
- All spaces in the Hecke diagram are naturally equipped with (normal crossings!) parabolic divisors:
  $\mathbf{Par}_C = \sum_{i=1}^{5} p_i$, $\mathbf{Par}_X = \sum_l L_l$
  $\mathbf{Par}_{X\times C} = \mathbf{Par}_X \times C + X \times \mathbf{Par}_C$,
  $\mathbf{Par}_H = p^*\mathbf{Par}_X + q^*\mathbf{Par}_{X\times C}$.
- This geometry provides the setup needed to formulate the parabolic version of the Hecke eigensheaf problem.

Eigensheaves on del Pezzo surfaces (v)

Theorem: [Donagi-P] Fix generic parabolic weights on $\mathbf{Par}_C$. Then

Hecke kernel: there exists a natural parabolic line bundle $\mathscr{I}_\bullet$ on $(H, \mathbf{Par}_H)$ with $\mathrm{parch}_1(\mathscr{I}_\bullet) = 0$.

Hecke eigensheaf: for any $(E_\bullet, \theta)$ there exists a unique $(F_\bullet, \varphi)$

stable strongly parabolic rank two Higgs bundle on C with $\mathrm{parch}_1 = 0$

stable strongly parabolic rank four Higgs bundle on X with $\mathrm{parch}_1, \mathrm{parch}_2 = 0$

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Hecke eigensheaf: for any $(E_\bullet, \theta)$ there exists a unique $(F_\bullet, \varphi)$ so that

$$q_*(p^*(F_\bullet, \varphi) \otimes (\mathscr{I}_\bullet, 0)) = p_X^*(F_\bullet, \phi) \otimes p_C^*(E_\bullet, \theta)$$

Eigensheaves on del Pezzo surfaces (vi)

Note:

- The theorem contains implicitly a theory of Grothendieck's six functors for parabolic Higgs bundles.
- Together with Donagi and Simpson we developed such a theory to ensure that NAHT converts the parabolic Hecke property in the theorem into the D-module Hecke property of the GLC.

Eigensheaves on del Pezzo surfaces (vi)

In particular we proved the following

Theorem: [Donagi-P-Simpson]

- There are explicit algebraic formulas for pushforward, pullback, and tensor product of semistable tame parabolic Higgs bundles with vanishing Chern classes.
- Under the NAH correspondence the constructions are compatible with the standard pushforwards, pullbacks, and tensor products of D-modules, and with L^2 pushforwards, pullbacks, and tensor products of harmonic bundles.

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Note: These algebraic formulas are crucial for the construction and the proof of the properties of $(F_\bullet, \varphi)$.

Eigensheaves on del Pezzo surfaces (vii)

Strategy of proof: Construct $(F_\bullet, \varphi)$ and check the Mochizuki and Hecke conditions by abelianization and higher dimensional versions of the spectral cover construction.

Eigensheaves on del Pezzo surfaces (vii)

Starting point: Understand the spectral data for $(E_\bullet, \theta)$.

- $(E_\bullet, \theta)$ is given by spectral data: a parabolic line bundle on the spectral cover $\widetilde{C}$ of C corresponding to θ.
- Genericity of $(E_\bullet, \theta))$ ensures that $\widetilde{C}$ is a smooth courve of genus two.
- Strong parabolicity implies that $\widetilde{C} \to C$ is branched at all five points of the parabolic divisor $\mathbf{Par}_C \subset C$ so specifying $\widetilde{C}$ is equivalent to specifying the sixth branch point $p_6 \in C$.
- The moduli space ${}^L\mathbf{Higgs}$ of strongly parabolic Higgs bundles on C is a 4-dimensional integrable system with Hitchin base $B = H^0(C, \mathcal{O}(1))$.

Eigensheaves on del Pezzo surfaces (viii)

Step one: Understand the spectral cover for $(F_\bullet, \varphi)$.

- The Hitchin fiber through $(E_\bullet, \theta)$ can be identified with the Jacobian J of $\widetilde{C}$.
- The natural rational map ${}^L\mathbf{Higgs} \dashrightarrow X$ restricts to a rational map $J \dashrightarrow X$ which is quasi finite of degree 4 and fails to be proper over the wobbly locus $\mathbf{Par}_X \subset X$.
- The map $J \dashrightarrow X$ is not defined at the 16 points of order two in J. Blowing these points up resolves the map to a $4:1$ finite cover $f: Y \to X$ -the modular spectral cover corresponding to $\widetilde{C}$.
- The map $f: Y \to X$ decomposes into two double covers: $Y \to \overline{Y}$ and $\overline{Y} \to X$ where $\overline{Y}$ is the Kummer K3 for the ableian surface J.

Eigensheaves on del Pezzo surfaces (ix)

Step two: Understand the spectral line bundle for $(F_\bullet, \varphi)$.

- The Fourier-Mukai transform of the skyscraper sheaf of $(E_\bullet, \theta) \in J$ is a degree zero line bundle on J which pulls back to a line bundle $\mathscr{L}_{(E_\bullet, \theta)}$ on Y.
- Choose undeterminate parabolic weights e along $\mathbf{Par}_Y = f^*\mathbf{Par}_X$ and define $F_\bullet$ to be the f-pushforward of the resulting parabolic line bundle:

$$F_\bullet = \mathscr{L}_{(E_\bullet, \theta)} \left(\mathrm{e} \cdot \mathbf{Par}_Y\right)_\bullet .$$

- The rational map $J \dashrightarrow T^\vee X$ resolves to a section $\alpha \in H^0(Y, f^*\Omega^1_X(\log \mathbf{Par}_X))$ and we define $\varphi = f_*(\alpha \otimes -)$.

Eigensheaves on del Pezzo surfaces (x)

- Use the fact that $\mathscr{L}_{(E_\bullet,\theta)}$ is an eigensheaf for the abelianized Hecke correspondece to rewrite the Mochizuki and Hecke conditions on $(F_\bullet, \varphi)$ as equations on the parabolic weights of $(F_\bullet, \varphi)$.
- Show that the numerical equations have a unique solution in terms of the parabolic weights for $(E_\bullet, \theta)$ - a higher dimensional version of the Aomoto map.

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- Show that the numerical equations have a unique solution in terms of the parabolic weights for $(E_\bullet, \theta)$ - a higher dimensional version of the Aomoto map.

Note: Carrying this out requires the algebraic formalism for computing pushforwards of Higgs bundles and computations with spectral covers of the abelianized Hecke correspondence.

Eigensheaves on quadric line complexes (i)

Fix C - a smooth curve of genus 2.

The moduli space of rank two bundles of fixed determinant on C has two interesting components:

$$\begin{aligned} X_0 &\cong \mathbb{P}^3 \\ X_1 &\cong \text{ a quadric line complex of } \mathbb{P}^3 \\ &= \text{ intersection of two quadrics in } \mathbb{P}^5 \end{aligned}$$

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$$= \text{ intersection of two quadrics in } \mathbb{P}^5$$

moduli of bundles with trivial determinant

moduli of bundles with determinant $\mathcal{O}_C(p)$

Eigensheaves on quadric line complexes (ii)

Theorem: [Pal-Pauly]

- The wobbly divisor in X_0 has 17 components. It consists of the quartic Kummer surface for the Jacobian of C and the 16 trope planes - the planes in $\mathbb{P}^3$ that are tangent to the Kummer surface along a conic.
- The wobbly divisor in X_1 is an irreducible surface.

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Note: There is a new feature in this case: the wobbly divisor in X_0 is not normal crossings in codimension two.

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Note: There is a new feature in this case: the wobbly divisor in X_0 is not normal crossings in codimension two. In fact the same holds for X_1.

Eigensheaves on quadric line complexes (ii)

Theorem: [Donagi-P-Simpson] Fix a Weierstrass point p of C and identify X_1 with the moduli of stable rank two bundles on C of determinant $\mathcal{O}_C(p)$. Let $\overline{C}$ be the 16-sheeted etale cover of C parametrizing degree zero line bundles L on C such that $L^{\otimes 2}(p)$ is effective.

- There is a natural embedding of the curve $\overline{C}$ in the quadric line complex X_1 and the wobbly divisor in X_1 is the union of all lines tangent to $\overline{C}$.
- The wobbly divisor in X_1 has a curve of cusps isomorphic to $\overline{C}$ and a curve of nodes isomorphic to the quotient of $\overline{C}$ by the lift of the hyperelliptic involution of C.

Eigensheaves on quadric line complexes (iii)

Again the basic Hecke correspondence can be compactifiied and resolved to a correspondence:

$$\begin{array}{ccc} & H & \\ {}^{p}\swarrow & & \searrow^{q} \\ X_1 & & X_0 \times C \end{array}$$

which is an incidence correspondence between points and planes in $\mathbb{P}^5$.

Eigensheaves on quadric line complexes (iv)

Explanation:

- Viewing X_1 as the base locus of a pencil of quadrics in $\mathbb{P}^5$ we can identify C with the moduli of rulings by planes $\mathbb{P}^2$ of the quadrics in the pencil.
- Thus a point $q \in C$ determines a quadric Q in the pencil and a ruling R of Q.
- Viewing X_1 as a quadric line complex of X_0 identifies X_0 with a ruling of Q: a point $x \in X_0$ gives a plane in Q, i.e. the plane $A_x \subset Q \subset \mathbb{P}^5$ parametrizing all lines in X_0 passing through x.
- H consists of all triples $(\ell, x, q) \in X_1 \times X_0 \times C$ such that $\ell \in A_x$.

Eigensheaves on quadric line complexes (v)

Theorem: [Donagi-P-Simpson] Let (E, θ) be a stable rank two Higgs bundle on C with trivial determinant and a smooth spectral cover. Then there exist a unique rank 8 tame strongly parabolic Higgs bundle $(F_\bullet, \varphi)$ on $X = X_0 \coprod X_1$ so that

- The parabolic structure of $F_\bullet$ is along the wobbly divisor in X.
- $F_\bullet$ satisfies Mochizuki's conditions: it is stable and with vanishing parabolic Chern classes.
- (in progress) There exists a natural parabolic line bundle $\mathscr{I}_\bullet$ on H so that $(F_\bullet, \varphi)$ is a Hecke eigensheaf of eigenvalue (E, θ) for the Hecke kernel $(\mathscr{I}_\bullet, 0)$.

Eigensheaves on quadric line complexes (v)

Remark: The proof of this result requires tackling of several general difficulties that are not present in the del Pezzo case:

- One needs to resolve the wobbly divisors to be normal crossings in codimension two before Mochizuki's conditions can even be formulated. We handle this issue by going to a branched cover of the moduli space.
- In the construction of the Hecke eigensheaf one has to work with Prym varieties rather than Jacobians.
- One needs a conceptual way of resolving the indeterminacies of the rational maps from these Prym varieties to X. We give such a procedure based on successive blow ups in attracting sets for the $\mathbb{C}^{\times}$-action on **Higgs**.

NAHT (i)

Non Abelian Hodge theory (NAHT) **[Hitchin, Donaldson, Corlette, Simpson, Saito, Sabbah, Mochizuki, . . .]:** in a nutshell gives an equivalence

$$(\text{flat bundles}) \leftrightarrow (\text{Higgs bundles})$$

The equivalence is mediated by a richer object: **harmonic bundle** or **twistor D-module** which specializes to both flat bundles and Higgs bundles.

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The equivalence is mediated by a richer object: **harmonic bundle** or **twistor D-module** which specializes to both flat bundles and Higgs bundles.

A variant of Deligne's notion of a λ-connection: at $\lambda = 1$ we have a flat connection, while at $\lambda = 0$ we have a Higgs bundle.

NAHT (ii)

Note: For the application to GLC we need a ramified higher dimensional version of NAHT developed in a sequence of deep works by Biquard, Jost-Yang-Zuo, Sabbah, Saito, Mochizuki, and Simpson.

NAHT (ii)

Note: For the application to GLC we need a ramified higher dimensional version of NAHT. This theory has several special features:

- It deals with ramified objects - parabolic local systems and parabolic Higgs bundles.
- The objects involved must satisfy new subtler stability conditions discovered by Mochizuki.
- Application to GLC hinge on verification of Mochizuki's conditions. This requires a detailed analysis of instability loci in moduli spaces.

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- The objects involved must satisfy new subtler stability conditions discovered by Mochizuki.
- Application to GLC hinge on verification of Mochizuki's conditions. This requires a detailed analysis of instability loci in moduli spaces.

In NAHT we have to work with the moduli **spaces**, rather than the **stacks**. So stability is important.

Special loci

Important loci:

Unstable locus The locus in **Higgs** consisting of semistable Higgs bundles whose uderlying bundle is unstable.

Wobbly locus The locus in **Bun** consisting of non-very-stable bundles.

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A bundle E is **very stable** if the only nilpotent Higgs field θ on E is $\theta = 0$. ($\Leftrightarrow$ the cotangent fiber $T^{\vee}_{\{E\}}$**Bun** meets the Hitchin fiber over 0 only at the point $\theta = 0$.)

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A bundle E is **wobbly** if it is stable but not very stable. **Peon-Nieto-Pauly:** The wobbly locus is the 'image' of the unstable locus.

NAHT: the theorems (i)

Theorem: [Corlette-Simpson] Let $(X, \mathcal{O}_X(1))$ be a smooth complex projective variety. Then there is a natural equivalence of dg $\otimes$-categories:

$$\mathbf{nah}_X : \left(\begin{array}{l} \text{finite rank} \\ \text{flat bundles} \\ \text{on } X \end{array} \right) \longrightarrow \left(\begin{array}{l} \text{finite rank } \mathcal{O}_X(1)\text{-} \\ \text{semistable Higgs} \\ \text{bundles on } X \text{ with} \\ ch_1 = 0 \text{ and } ch_2 = 0 \end{array} \right)$$

NAHT: the theorems (ii)

Mochizuki proved a version of the NAH correspondence which allows for singularities of the objects involved.

NAHT: the theorems (ii)

Theorem: [Mochizuki] Let $(X, \mathcal{O}_X(1))$ be a polarized projective variety and let $D \subset X$ be an effective divisor. Suppose that we have a closed subvariety $Z \subset X$ of codimension $\geqslant 3$, such that $X - Z$ is smooth and $D - Z$ is a normal crossing divisor.
Then there is a canonical equivalence of dg $\otimes$-categories:

$$\left(\begin{array}{l}\text{finite rank tame} \\ \text{parabolic flat} \\ \text{bundles on} \\ (X, D)\end{array}\right) \xrightarrow{\mathbf{nah}_{X,D}} \left(\begin{array}{l}\text{finite rank locally abelian} \\ \text{tame parabolic Higgs} \\ \text{bundles on } (X, D) \\ \text{which are } \mathcal{O}_X(1)\text{-} \\ \text{semistable and satisfy} \\ parch_1 = 0 \text{ and } parch_2 = 0\end{array}\right)$$

NAHT: the theorems (iii)

Mochizuki requires three basic ingredients for this theorem:

(1) a good compactification, which is smooth and where the boundary is a divisor with normal crossings away from codimension 3;

(2) a local condition: tameness (the Higgs field is allowed to have at most logarithmic poles along D), and compatibility of filtrations (the parabolic structure is locally isomorphic to a direct sum of rank one objects);

(3) a global condition: vanishing of parabolic Chern classes.

NAHT: the theorems (iv)

Another important ingredient is Mochizuki's extension theorem

Theorem: [Mochizuki] Let U be a quasi-projective variety with two compactifications $\phi : U \to X$, and $\psi : U \to Y$ where:

- X, Y are projective and irreducible;
- X is smooth and $X - U$ is a normal crossing divisor away from codimension 3;

Then the restriction from X to U followed by the middle perversity extension from U to Y gives an equivalence of categories:

$$\phi_{*!} \circ \psi^* : \begin{pmatrix} \text{irreducible} & \text{tame} \\ \text{parabolic} & \text{flat} \\ \text{bundles on } (X, D) & \end{pmatrix} \longrightarrow \begin{pmatrix} \text{simple } \mathcal{D}\text{-modules on } Y \\ \text{which are smooth on } U \end{pmatrix}$$

The plan

The plan

$\boxed{{}^{L}G\text{-flat bundle } (V, \nabla) \text{ on } C}$

$$\boxed{{}^{L}G\text{-flat bundle } (V, \nabla) \text{ on } C}$$

$$\Downarrow (1)$$

$$\boxed{{}^{L}G\text{-Higgs bundle } (E, \theta) \text{ on } C}$$

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This is the Corlette-Simpson non-abelian Hodge correspondence $(E, \theta) = \mathbf{nah}_C(V, \nabla)$ on the smooth compact curve C.

^{L}G-flat bunle (V, ∇) on C

$$\Big\Downarrow (1)$$

^{L}G-Higgs bundle (E, θ) on C

^LG-flat bundle (V, ∇) on C

$\Downarrow$ (1)

^LG-Higgs bundle (E, θ) on C

$\Downarrow$ (2)

ab **Hecke**-eigensheaf on **Higgs**

The plan

^{L}G-flat bundle (V, ∇) on C

$\Downarrow$ (1)

^{L}G-Higgs bundle (E, θ) on C

$\Downarrow$ (2)

$_{\mathrm{ab}}$ **Hecke**-eigensheaf on **Higgs**

The functor (2) sends $(E, \theta) \in {}^{L}\mathbf{Higgs}$ to $\mathbf{FM}(\mathcal{O}_{(E,\theta)})$.

The plan

^{L}G-flat bundle (V, ∇) on C

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$\Downarrow$ (3)

parabolic Higgs sheaf on **Bun** satisfying MC (1)-(3)

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parabolic Higgs sheaf on **Bun** satisfying MC (1)-(3)

$\Downarrow$ (4)

parabolic flat bundle on **Bun** satisfying MC (1)-(3)

$^L G$-flat bundle (V, ∇) on C

$\Downarrow$ (1)

$^L G$-Higgs bundle (E, θ) on C

$\Downarrow$ (2)

$_{\text{ab}}$ **Hecke**-eigensheaf on **Higgs**

$\Downarrow$ (3)

parabolic Higgs sheaf on **Bun** satisfying MC (1)-(3)

$\Downarrow$ (4)

parabolic flat bundle on **Bun** satisfying MC (1)-(3)

The functor (4) is the parabolic non-abelian Hodge correspondence of Mochizuki.

${}^{L}G$-flat bundle (V, ∇) on C

$\Downarrow$ (1)

${}^{L}G$-Higgs bundle (E, θ) on C

$\Downarrow$ (2)

ab **Hecke**-eigensheaf on **Higgs**

$\Downarrow$ (3)

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parabolic Higgs sheaf on **Bun** satisfying MC (1)-(3)

$\Downarrow$ (4)

parabolic flat bundle on **Bun** satisfying MC (1)-(3)

$\Downarrow$ (5)

ordinary flat bundle on Zariski open in **Bun**

^{L}G-flat bundle (V, ∇) on C

⇓ (1)

^{L}G-Higgs bundle (E, θ) on C

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parabolic flat bundle on **Bun** satisfying MC (1)-(3)

⇓ (5)

ordinary flat bundle on Zariski open in **Bun**

Step 5 is just restriction.

^{L}G-flat bundle (V, ∇) on C

$\Downarrow$ (1)

^{L}G-Higgs bundle (E, θ) on C

$\Downarrow$ (2)

ab **Hecke**-eigensheaf on **Higgs**

$\Downarrow$ (3)

parabolic Higgs sheaf on **Bun** satisfying MC (1)-(3)

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ordinary flat bundle on Zariski open in **Bun**

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parabolic Higgs sheaf on **Bun** satisfying MC (1)-(3)

$\Downarrow$ (4)

parabolic flat bundle on **Bun** satisfying MC (1)-(3)

$\Downarrow$ (5)

ordinary flat bundle on Zariski open in **Bun**

$\Downarrow$ (6)

$\mathcal{D}$-module on **Bun**

Back

Parabolic sheaves

Fix a pair (X, D), where

- X - a compact complex manifold;
- $D \subset X$ - a divisor with simple normal crossings;
- $D = \cup_{i \in S} D_i$ - the irreducible decomposition of D.

Parabolic sheaves

Definition: A **torsion free parabolic sheaf** on (X, D) is a collection of torsion free coherent sheaves $\{\mathcal{E}_\alpha\}_{\boldsymbol{\alpha}\in\mathbb{R}^S}$ together with inclusions $\mathcal{E}_{\boldsymbol{\alpha}} \subset \mathcal{E}_{\boldsymbol{\beta}}$ of sheaves of $\mathcal{O}_X$-modules, specified for all $\alpha \leqslant \beta$, satisfying the conditions:

[semicontinuity] for every $\boldsymbol{\alpha} \in \mathbb{R}^S$, there exists a real number $c > 0$ so that $\mathcal{E}_{\boldsymbol{\alpha}+\boldsymbol{\varepsilon}} = \mathcal{E}_{\boldsymbol{\alpha}}$ for all functions $\boldsymbol{\varepsilon} : S \to [0, c]$.

[support] if $\boldsymbol{\delta}_i : S \to \mathbb{R}$ is the characteristic function of i, then for all $\boldsymbol{\alpha} \in \mathbb{R}^S$ we have $\mathcal{E}_{\boldsymbol{\alpha}+\boldsymbol{\delta}_i} = \mathcal{E}_{\boldsymbol{\alpha}}(D_i)$ (compatibly with the inclusion).

Flags and weights

Fix a parabolic torsion free sheaf $\mathbf{E}_\bullet$ on (X, D) and $\mathbf{c} \in \mathbb{R}^S$.

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For every $i \in S$ we get an induced filtration $\{{}^iF_a\}_{c_i-1<a\leqslant c_i}$ of the restricted sheaf $\mathbf{E}_{\mathbf{c}|D_i}$.

$$ {}^iF_a = \bigcup_{\substack{\boldsymbol{\alpha}\leqslant \mathbf{c} \\ \alpha_i \leqslant a}} \mathbf{E}_{\boldsymbol{\alpha}} $$

Flags and weights

Fix a parabolic torsion free sheaf $\mathbf{E}_\bullet$ on (X, D) and $\mathbf{c} \in \mathbb{R}^S$.
For every $i \in S$ we get an induced filtration $\{{}^iF_a\}_{c_i-1<a\leqslant c_i}$ of the restricted sheaf $\mathbf{E}_{\mathbf{c}|D_i}$.
Define ${}^i\mathrm{gr}_a\,\mathbf{E}_\mathbf{c} := {}^iF_a/{}^iF_{i_{F_{<a}}}$.
[semicontinuity] $\Rightarrow$ the **set of parabolic weights**

$$\mathrm{weights}(\mathbf{E}_\mathbf{c}, i) = \left\{a \in (c_i - 1, c_i] \;\middle|\; {}^i\mathrm{gr}_a \neq 0\right\}$$

is finite
Note: The sheaf $\mathbf{E}_\mathbf{c}$ together with the flags $\{{}^iF_a|\ i \in S, a \in \mathrm{weights}(\mathbf{E}_\mathbf{c}, i)\}$ reconstruct the parabolic sheaf $\mathbf{E}_\bullet$.

Locally abelian parabolic bundles (i)

Example: A **parabolic line bundle** is a parabolic sheaf $F_\bullet$ for which all sheaves F_α are invertible. If $\mathbf{a} \in \mathbb{R}^S$, then define a parabolic line bundle $\mathcal{O}_X\left(\sum_{i\in S} \mathbf{a}_i D_i\right)_\bullet$ by setting

$$\left(\mathcal{O}_X\left(\sum_{i\in S} \mathbf{a}_i D_i\right)\right)_{\boldsymbol{\alpha}} := \mathcal{O}_X\left(\sum_{i\in S} \lfloor \mathbf{a}_i + \boldsymbol{\alpha}_i \rfloor D_i\right)$$

Claim: Every parabolic line bundle $F_\bullet$ is isomorphic to $L \otimes \mathcal{O}_X\left(\sum_{i\in S} \mathbf{a}_i D_i\right)_\bullet$ for some $L \in \mathrm{Pic}(X)$, and some $\mathbf{a} \in \mathbb{R}^S$.

Locally abelian parabolic bundles (ii)

Definition: A parabolic sheaf $F_\bullet$ is a **locally abelian bundle**, if in a Zariski neighborhood of any point $x \in X$ there is an isomorphism between $F_\bullet$ and a direct sum of parabolic line bundles.

Note: A parabolic bundle $\left(\mathbf{E_c}, \{{}^iF_\bullet\}_{i\in S}\right)$ is locally abelian iff on every intersection $D_{i_1} \cap \cdots \cap D_{i_k}$ the iterated graded ${}^{i_1}\mathrm{gr}_{a_1} \cdots {}^{i_k}\mathrm{gr}_{a_k} \mathbf{E_c}$ does not depend on the order of the components.

Variant: We can define similarly locally abelian parabolic local systems, Higgs bundles, or more generally locally abelian parabolic λ-connections.

Locally abelian parabolic λ-connections (i)

Let $\lambda \in \mathbb{C}$. A λ-connection with tame ramification along D, is a pair $(E, \mathbb{D}^\lambda)$, where:

- E is a holomorphic vector bundle on X;
- $\mathbb{D}^\lambda : E \to E \otimes \Omega^1_X(\log D)$, is a $\mathbb{C}$-linear map satisfying the λ-twisted Leibnitz rule

$$\mathbb{D}^\lambda(f \cdot s) = f\mathbb{D}^\lambda s + \lambda s \otimes df.$$

We say that $\mathbb{D}^\lambda$ is flat if $\mathbb{D}^\lambda \circ \mathbb{D}^\lambda = 0$.

Note:

(flat 1-connection) = (flat connection with regular singularities)

(flat 0-connection) = (Higgs bundle with logarithmic poles)

Locally abelian parabolic λ-connections (ii)

Definition: A **tame parabolic λ-connection** is a pair $(E_\bullet, \mathbb{D}^\lambda)$, where

- $E_\bullet$ is a prabolic bundle on (X, D);
- $\mathbb{D}^\lambda : E_\alpha \to E_\alpha \otimes \Omega^1_X(\log D)$ is a tame flat λ-connection specified for all $\boldsymbol{\alpha} \in \mathbb{R}^S$ (compatibly with the inclusions).

A tame parabolic λ-connection $(E_\bullet, \mathbb{D}^\lambda)$ is **locally abelian** if the underlying bundle $E_\bullet$ is locally abelian. It is **strongly parabolic** if the action of the residue of $\mathbb{D}^\lambda$ on the associated graded for the parabolic filtration is zero.

Back

Parabolic Chern classes (i)

Let $\mathcal{E}_\bullet$ be a parabolic torsion free sheaf on (X, D), then the parabolic Chern character of $\mathcal{E}_\bullet$ is given by the **Iyer-Simpson formula:**

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$$\operatorname{parch}(\mathcal{E}_\bullet) = \operatorname{parch}({}_{\mathbf{c}}E) = \frac{\prod_{i\in S}\int_{c_i}^{c_i+1} d\boldsymbol{\alpha}_i \left[ch\left(\mathcal{E}_{\boldsymbol{\alpha}_i}\right) e^{-\sum_{i\in S}\alpha_i D_i}\right]}{\prod_{i\in S}\int_0^1 d\boldsymbol{\alpha}_i e^{-\sum_{i\in S}\alpha_i D_i}}.$$

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$\mathbf{c} \in \mathbb{R}^S$ is any base point

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Note: Given $\mathbf{c} \in \mathbb{R}^S$ define the $\mathbf{c}$-truncation ${}_{\mathbf{c}}E$ of $\mathcal{E}_\bullet$ = the collection $\{\mathcal{E}_{\boldsymbol{\alpha}}\}_{\mathbf{c}<\boldsymbol{\alpha}\leqslant\mathbf{c}+\boldsymbol{\delta}}$, with $\boldsymbol{\delta} = \sum_{i\in S}\boldsymbol{\delta}_i$.

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[support] $\Rightarrow$ $\mathcal{E}_\bullet$ is effectively reconstructed by any truncation ${}_{\mathbf{c}}E$. In fact: the numerator of the Iyer-Simpson formula is independent of the choice of truncation.

Parabolic Chern classes (ii)

Example: The first parabolic Chern class of $\mathbf{E}_\bullet$ is given by:

Back

Parabolic Chern classes (ii)

Example: The first parabolic Chern class of $\mathbf{E}_\bullet$ is given by:

$$\mathrm{parc}_1 = c_1(\mathbf{E_c}) - \sum_{i \in S} \left(\sum_{a \in \mathrm{weights}(\mathbf{E_c}, i)} a \,\mathrm{rank}\, {}^i\mathrm{gr}_a \,\mathbf{E_c} \right) \cdot D_i$$

Back