Quantum curves, integrability and topological string partition functions

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The partition function of the topological string is of interest both for physics
(effective Sugra actions, Nekrasov partition functions,...)

and mathematics
(enumerative invariants: Gromov-Witten, Donaldson-Thomas, Gopakumar-Vafa,...)

There are various approaches to its computation
(Topological recursion, holomorphic anomaly, topological vertex,...)

Most of them are perturbative in one way or another, with some exceptions
(matrix model; cf. in particular Marino et. al.)
The problem

Let us consider CY of “class \( \Sigma \)”, local CY of the form

\[
x y - P(u, v) = 0, \quad \text{with} \quad P(u, v) = v^2 - Q_0(u),
\]

where \( Q_0 \) is a quadratic differential on a Riemann surface \( C = C_{g,n} \), for \( g = 0 \):

\[
Q_0 = \sum_{r=1}^{n} \left( \frac{\delta_r}{(u - z_r)^2} + \frac{E_r}{u - z_r} \right).
\]

CY of class \( \Sigma \) relevant for geometric engineering of 
\( d = 4, \mathcal{N} = 2 \) SUSY gauge theories of class \( \mathcal{S} \),

Seiberg-Witten curve: \( \Sigma = \{(u, v); P(u, v) = 0\} \subset T^*C \).

Problem: Define and compute topological string partition function \( Z_{\text{top}} \) for class \( \Sigma \).
Local mirror symmetry

A-model on $X$, Kähler moduli $t = t(m)$, $\Leftrightarrow$ B-model on local CY $Y$, cplx. structure moduli $m$

where complex structure moduli of $Y$: $m = (E, d, z)$,

$E = (E_1, E_2, \ldots)$, $d = (\delta_1, \delta_2, \ldots)$, $z = (z_1, z_2, \ldots)$,

and Kähler moduli $t$: Periods of canonical one form $vdu$ on $\Sigma$.

Regard $Z_{\text{top}}$ as function $Z_{\text{top}}(t; \lambda)$. 
A chain of dualities was discussed by Dijkgraaf-Hollands-Sulkowski-Vafa relating:

i) **Geometric (GW)** – Type IIB string theory on $TN \times Y$, where and $TN$ is the Taub-NUT space and $Y$ is the non-compact Calabi-Yau manifold $xy - P(u, v) = 0$.

ii) **D-branes (DT)** – Type IIA string theory on $\mathbb{R}^3 \times S^1 \times X$, where $X$ is the mirror of the Calabi-Yau $Y$ manifold in i) with a D6-brane wrapping $S^1 \times X$.

iii) **I-brane**: Type IIA string background with a D4 and a D6 intersecting along $\Sigma$.

It was argued that generating functions of BPS-states are related

$$Z_{GW} \sim Z_{DT} \sim Z_I,$$

where $Z_I = Z_{ff}$;

$Z_{ff}$: partition function of free fermions on $\Sigma$ (massless open strings between D4, D6)

Topological string coupling $\lambda \sim$ B-field along D6 $\rightsquigglyleftarrow$

$\rightsquigglyleftarrow$ non-commutative deformation of $\Sigma$, the “quantum curve”
Extracting the answer from free fermions?

More precisely, the prediction of Dijgraaf et. al. can be formulated as

\[ Z_{ff}(\xi, t; \lambda) = \sum_{p \in H^2(X, \mathbb{Z})} e^{p \cdot \xi} Z_{\text{top}}(t + \lambda p, \lambda). \]

This could give us an elegant non-perturbative definition of \( Z_{\text{top}}(t, \lambda) \) if we knew

a) exactly how to turn the curve \( \Sigma \) into a “quantum curve”,

b) how to associate a free fermion partition function to a “quantum curve”,

c) the relation between the variables \((\xi, t)\) and parameters of “quantum curve”.

This has been illustrated by some examples in the work of Dijkgraaf et. al..
Outline of the solution

Our goal: Turn this into a general and non-perturbative mathematical definition of the topological string partition functions for class $\Sigma$.

To explain the answer we need to address the following questions:

A) **How to quantize $\Sigma$ and turn it into a free fermion partition function?**
   - use meromorphic opers and theory of infinite Grassmannians / free fermions

B) **How to parameterise quantum curves in terms of $(\xi, t)$?**
   - use Riemann-Hilbert correspondence and Abelianisation

C) **Why is Abelianisation the right thing to use?**
   - exact WKB gives a canonical way to “quantize” the leading order result
A) From quantum curve to free fermion partition functions I

Quantum curve \( \sim \) Differential equation quantising the equation for \( \Sigma \):

\[
v^2 - Q_0(u) = 0 \quad \leadsto \quad (\lambda^2 \partial_u^2 + Q(u))\chi(u) = 0, \quad Q(u) = Q_0(u) + O(\lambda).
\]

Corresponding \( \mathcal{D} \)-module \( \sim \) flat connection having horizontal sections \( \Psi \),

\[
\nabla_\Sigma \Psi(u) \equiv \left[ \lambda \partial_u + \begin{pmatrix} 0 & Q \end{pmatrix} \right] \Psi(u) = 0.
\]

Fermionic state \( f_\Psi(Q) \) defined as

\[
f_\Psi(Q) = \exp \left( - \sum_{k>0} \sum_{l \geq 0} \psi_{-k} \cdot A_{kl} \cdot \bar{\psi}_{-l} \right) f_0 \quad \{\psi_{s,n}, \bar{\psi}_{t,m}\} = \delta_{s,t} \delta_{n,-m} \quad \{\psi_{s,n}, \psi_{t,m}\} = 0 = \{\bar{\psi}_{s,n}, \bar{\psi}_{t,m}\}
\]

\[
\frac{(\Psi(x))^{-1}\Psi(y)}{x - y} = \sum_{l \geq 0} y^{-l-1}w_l(x), \quad w_l(x) = -x^l + \sum_{k>0} x^{-k}A_{kl}
\]

Note that \( \{w_l(x), l = 0, 1, \ldots \} \) is a basis for the subspace \( W_\Psi \) in the Sato-Segal-Wilson Grassmannian associated to \( \Psi \).
A) From quantum curve to free fermion partition functions II

Proposal: Free fermion partition function = tau-function (Sato-Jimbo-Miwa-Segal-Wilson)

\[ Z_{\text{ff}}(\xi, t; \lambda) = \langle f_0, e^{H(\tau)} f_\Psi(Q) \rangle. \]

where \( H(\tau) = \sum_i H_i \tau_i \), \( H_i \): generators of an abelian sub-algebra \( \mathcal{A} \) of \( \mathcal{W}_{1+\infty} \),
\( \mathcal{W}_{1+\infty} \): Lie algebra generated by fermion bilinears.

Nice,

(+) relation to integrable hierarchies

but so far pretty useless, in general\(^*)\)

(−) don’t know which sub-algebra \( \mathcal{A} \) is “suitable” for our problem

(−) don’t know relation between \((\xi, t)\) and \((\tau, Q)\)

*) Exceptions: Examples investigated by Dijkgraaf et. al.
A) How to quantize the spectral curve I

Quantum curve receives \textit{quantum corrections}:

\[
Q_0(u) \to Q(u) = Q_0(u) + \lambda \sum_{k=1}^{d} \frac{v}{y - u_k} - \lambda^2 \sum_{k=1}^{d} \frac{3}{4(y - u_k)^2}.
\]

\[\lambda^2v_k^2 + Q_k = 0, \quad Q_k = \lim_{u \to u_k} \left( Q(u) - \lambda \frac{v}{u - u_k} + \lambda^2 \frac{3}{4(u_l - u_k)^2} \right)\]

Why?

• Only now we have \textit{enough} parameters in quantum curve \((m, u, v)\),
  \[u = (u_1, \ldots, u_{n-3}), \quad v = (v_1, \ldots, v_{n-3}),\] to account for both \(\xi\) and \(\tau\).

• The extra singularities are more \textit{apparent} than real, the \(\mathcal{D}\)-module associated to the quantum curve is \textit{non-singular} at \(u = u_k\),

\[
\lambda \partial_u + \begin{pmatrix} 0 & Q \\ 1 & 0 \end{pmatrix} \quad \text{gauge equivalent to} \quad \lambda \partial_u + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
\]

with \(A_{ij} = A_{ij}(u)\) non-singular at \(u = u_k\).
B) How to parameterise quantum curves in terms of $(\xi, t)$?

Main problem: Relation between $(\xi, t)$ and parameters of quantum curve.

Our proposal:

$$(\xi, t) \sim \text{very special coordinates for monodromy data}$$

made precise through

- **Riemann-Hilbert correspondence** – correspondence between monodromies (holonomies of flat connection) and $\mathcal{D}$-modules (quantum curves),

  and

- **Abelianisation**: Curve $\Sigma \mapsto$ very special coordinates for monodromy data.
B) Abelianisation (Hollands-Neitzke)

Fenchel-Nielsen (FN) network (black) decomposes surface $C$ into annular regions $A_i$.

- Connection can be diagonalised on each annular region $A_i$. Parallel transport $\leadsto$ collection of diagonal matrices $D_i, D_i', D_i''$, eigenvalues: simple functions of $e^{i\theta_r}$, $r = 1, 2, 3, 4$, $e^{i\sigma}$, and diagonal matrix $T$, eigenvalue $e^{i\tau}$.

- Jump matrices $J_i, J_{ij}$ (non-diagonal!) representing non-abelian parallel transport across walls of FN network uniquely determined in terms of matrices $D_i, D_i', D_i''$ by consistency conditions.

Any closed path $\gamma$ on $C$ can be decomposed into segments contained in $A_i$ (blue), segments crossing walls (green), and a path traversing annulus between the two pairs of pants (grey) $\leadsto$ holonomies parameterised in terms of $\sigma, \tau, \theta_r$, $r = 1, 2, 3, 4$. 

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*Note: The diagram includes visual representations of the annular regions and connections, but the natural text is the primary source of information.*
B) Our proposal, finally

To given \( t \in \mathbb{R}^{3g-3+n} \) (Kähler parameters), \( \xi \) (twist parameters)

- find mirror curve \( \Sigma \), \( v^2 = Q_0(u) \) and canonical basis for \( H_1(\Sigma, \mathbb{Z}) \) such that parameters \( t \) are the a-cycle periods of \( \Sigma \)

- find Fenchel-Nielsen network defined by \( Q_0(u) \) for real \( t \)

- construct quantum curve \( \nabla_{\Sigma} \) associated to \((\xi, t)\) by Riemann-Hilbert, assuming

\[
\text{Dictionary:} \quad \sigma_r = t_r / \lambda, \quad i\tau_r = \xi_r, \quad \theta_r^2 = \delta_k / \lambda^2.
\]

- construct \( Z_{ff}(\xi, t; \lambda) \) as SJMSW tau-function associated to \( \nabla_{\Sigma} \)

- expand in \( e^{\xi \cdot p} \), extract \( Z_{\text{top}} \) using

\[
Z_{ff}(\xi, t; \lambda) = \sum_{p \in H^2(X, \mathbb{Z})} e^{p \cdot \xi} Z_{\text{top}}(t + \lambda p, \lambda)
\]

- Analytically continue in \( t \)
The proof for $C = C_{0,4}$:

Calculation of both sides, comparison

**Calculation of tau-functions:** Can be done using either

- Tau-functions are generalised conformal blocks of free fermion VOA
  (Moore; Palmer; J.T. ’17)

- $\rightsquigarrow$ can be factorised by gluing construction (Iorgov-Lisovyy-JT)

or, even better

- Factorisation of Riemann-Hilbert problems

- $\rightsquigarrow$ factorisation of tau-functions (Gavrylenko-Lisovyy, Cafasso-Gavrylenko-Lisovyy)

Either way $\rightsquigarrow$ explicit formulae (first conjectured by Gamayun-Iorgov-Lisovyy)

$$
T(\sigma, \tau; \theta; z) = \sum_{n \in \mathbb{Z}} \sum_{\xi, \zeta \in \mathbb{Y}} Z^{(n)}_{\xi, \zeta} + Z^{(n)}_{\xi, \zeta, -} = \sum_{n \in \mathbb{Z}} e^{in\tau} G(\sigma + n, \theta; z),
$$

where:
where $G(\sigma, \theta; z)$ can be factorised as

$$G(\sigma, \theta; z) = M(\sigma, \theta_4, \theta_3)M(\sigma, \theta_2, \theta_1)F(\sigma, \theta; z),$$

using the following notations:

- The functions $N(\theta_3, \theta_2, \theta_1)$ are defined as

$$M(\theta_3, \theta_2, \theta_1) = \frac{\prod_{\epsilon = \pm} G(1 + \theta_3 + \epsilon(\theta_2 + \theta_1))G(1 + \theta_3 + \epsilon(\theta_2 - \theta_1))}{G(1 + 2\theta_3)G(1 - 2\theta_2)G(1 - 2\theta_1)},$$

where $G(p)$ is the Barnes $G$-function that satisfies $G(p + 1) = \Gamma(p)G(p)$.

- $F(\sigma, \theta; z)$ can be represented by the following power series

$$F(\sigma, \theta; z) = z^{\sigma^2 - \theta_1^2 - \theta_2^2}(1 - z)^{2\theta_2\theta_3} \sum_{\xi, \zeta \in \mathbb{Y}} z^{||\xi|| + ||\zeta||} F_{\xi, \zeta}(\sigma, \theta),$$

with $\mathbb{Y}$ set of partitions, coefficients $F_{\xi, \zeta}(\sigma, \theta)$ explicitly given in

$$F_{\xi, \zeta}(\sigma, \theta) = \prod_{(i, j) \in \xi} \frac{((\theta_2 + \sigma + i - j)^2 - \theta_1^2)((\theta_3 + \sigma + i - j)^2 - \theta_4^2)}{(\xi_j' - i + \xi_i - j + 1)^2(\xi_j' - i + \xi_i - j + 1 + 2\sigma)^2} \prod_{(i, j) \in \zeta} \frac{((\theta_2 - \sigma + i - j)^2 - \theta_1^2)((\theta_3 - \sigma + i - j)^2 - \theta_4^2)}{(\zeta_j' - i + \zeta_i - j + 1)^2(\zeta_j' - i + \zeta_i - j + 1 - 2\sigma)^2},$$

$\zeta_i / \zeta_i'$ arm / leg length of $(i, j) \in \mathbb{Y}$. 

The proof for $C = C_{0,4}$, II

Topological string partition function: Can be calculated using top. vertex

$$\text{Careful 4d limit } \sim \text{ match!}$$

Crucial is the precise formula for $M(\theta_3, \theta_2, \theta_1)$:

- Only for very special choices of $M(\theta_3, \theta_2, \theta_1)$ one gets Fourier-series of the form

$$\mathcal{T}(\sigma, \tau ; \theta ; z) := \sum_{n \in \mathbb{Z}} e^{i n \tau} \mathcal{G}(\sigma + n, \theta ; z),$$

Corollary: **Quantitative check of string dualities!**

- Only for very particular coordinate $\tau$ one gets right formula for $M(\theta_3, \theta_2, \theta_1)$. 

C) Why abelianisation is the right thing to use

Key-word: Exact WKB:

- Foliations defined by $Q_0$ for real periods $t$ decompose $C$ into annular regions.

- In each annular region there exist unique solutions of quantum curve equation with diagonal monodromy and leading asymptotics

$$\chi(u, \lambda) = \frac{\sqrt{\lambda}}{(Q_0(u))^{\frac{1}{4}}} \exp \left( \pm \int_{u_0}^{u} du \left( \frac{1}{\lambda} \sqrt{Q_0(u)} + \frac{Q_1(u)}{2\sqrt{Q_0(u)}} \right) \right) (1 + O(\lambda)),$$

defined through Borel-summation of $\lambda$-expansion.

- Analytic continuation across walls represented by jump matrices used in Abelianisation

- $\rightsquigarrow$ monodromy of Borel sums naturally parameterised by $\sigma, \tau$. 
Summary

We have presented a proposal for a non-perturbative\textsuperscript{*}) and computable definition of the topological string partition functions for class $\Sigma$.

\textsuperscript{*}) manifest in representation as a Fredholm determinant (Cafasso-Gavrylenko-Lisovyy)

Key elements of the proposal

A) How to quantize $\Sigma$ and turn it into a free fermion partition function?
   \begin{itemize}
   \item use meromorphic opers and theory of inf. Grassmannians / free fermions
   \end{itemize}

B) How to parameterise quantum curves in terms of $(\xi, t)$?
   \begin{itemize}
   \item use Riemann-Hilbert correspondence and Abelianisation
   \end{itemize}

C) Why is Abelianisation the right thing to do?
   \begin{itemize}
   \item exact WKB gives a canonical way to “quantise” the leading order result
   \end{itemize}
Relation to other approaches

This problem has previously been approached (in simple cases) by other methods

- **Integrable structures**: (Aganagic-Dijkgraaf-Klemm-Marino-Vafa,...,Okounkov)
  
  Our work makes integrability **effective** in complicated cases.

- **Topological recursion**: So far unclear which exact initial conditions to put. Can now be extracted from exact result (R. Belliard, J.T., in progress)

- **Quantisation of $H^3(Y, \mathbb{R})$, holomorphic anomaly.** The expansion

  $$Z_{ff}(\xi; t; \lambda) = \sum_{p \in H^2(X,\mathbb{Z})} e^{p \cdot \xi} Z_{\text{top}}(t + \lambda p, \lambda)$$

  has an interpretation as a Fourier-transformation relating natural representations for quantisation of $H^3(Y, \mathbb{R})$ (Iorgov-lisovyy-J.T., and work in progress)

- **Relation to Hitchin systems**: (cf. Diaconescu, Dijkgraaf, Donagi, Hofman, Pantev)

- **Matrix models**: Relation between contours and choices of coordinates ($\sigma, \tau$)
Outlook

- Toric CY: (Cf. Marino; Jimbo-Nagoya-Sakai)

- Higher genus, irregular singularities

- **Higher rank** (cf. Coman-Pomoni-J.T.'17, and Hollands-Neizke (to appear))

Crucial is the **interplay** between **two** integrable structures in this context:

- Integrable flows on moduli spaces –
  (integrable hierarches, Hitchin systems, isomonodromic deformations,....)

- Integrable structures on character varieties –
  best expressed in terms of coordinates of **Fenchel-Nielsen type**.