Quantum curves, integrability and topological string partition functions

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The partition function of the topological string is of interest both for physics

(effective Sugra actions, Nekrasov partition functions,...)

and mathematics

(enumerative invariants: Gromov-Witten, Donaldson-Thomas, Gopakumar-Vafa,...)

There are various approaches to its computation

(Topological recursion, holomorphic anomaly, topological vertex,...)

Most of them are perturbative in one way or another, with some exceptions

(matrix model; cf. in particular Marino et. al.)

The problem

Let us consider CY of "class Σ", local CY of the form

$$xy - P(u,v) = 0, \qquad \text{with} \quad P(u,v) = v^2 - Q_0(u),$$

where Q_0 is a quadratic differential on a Riemann surface $C = C_{g,n}$, for $g = 0$:

$$Q_0 = \sum_{r=1}^{n} \left(\frac{\delta_r}{(u-z_r)^2} + \frac{E_r}{u-z_r} \right).$$

CY of class Σ relevant for **geometric engineering** of $d=4$, $\mathcal{N}=2$ **SUSY gauge theories** of **class** $\mathcal{S}$,

Seiberg-Witten curve: $\quad \Sigma = \{(u,v); P(u,v) = 0\} \subset T^*C.$

Problem: Define and compute topological string partition function $\mathcal{Z}_{\text{top}}$ for class Σ

Local mirror symmetry

A-model on X, Kähler moduli $\mathbf{t} = \mathbf{t}(\mathbf{m})$,	$\leftrightarrow$	B-model on local CY Y, cplx. structure moduli $\mathbf{m}$

where **complex structure moduli** of Y: $\mathbf{m} = (\mathbf{E}, \mathbf{d}, \mathbf{z})$,

$$\mathbf{E} = (E_1, E_2, \dots), \quad \mathbf{d} = (\delta_1, \delta_2, \dots), \quad \mathbf{z} = (z_1, z_2, \dots),$$

and **Kähler moduli** $\mathbf{t}$: Periods of canonical one form vdu on Σ.

Regard $\mathcal{Z}_{\mathrm{top}}$ as function $Z_{\mathrm{top}}(\mathbf{t}; \lambda)$.

Predictions from string dualities

A chain of dualities was discussed by Dijkgraaf-Hollands-Sulkowski-Vafa relating:

i) **Geometric (GW)** – Type IIB string theory on $TN \times Y$, where and TN is the Taub-NUT space and Y is the non-compact Calabi-Yau manifold $xy - P(u,v) = 0$.

ii) **D-branes (DT)** – Type IIA string theory on $\mathbb{R}^3 \times S^1 \times X$, where X is the mirror of the Calabi-Yau Y manifold in i) with a D6-brane wrapping $S^1 \times X$.

iii) **I-brane:** Type IIA string background with a D4 and a D6 intersecting along Σ.

It was argued that generating functions of BPS-states are related

$$\mathcal{Z}_{\mathrm{GW}} \sim \mathcal{Z}_{\mathrm{DT}} \sim \mathcal{Z}_{\mathrm{I}}, \qquad \text{where} \qquad \mathcal{Z}_{\mathrm{I}} = \mathcal{Z}_{\mathrm{ff}},$$

$\mathcal{Z}_{\mathrm{ff}}$: partition function of free fermions on Σ (massless open strings between D4, D6)

Topological string coupling $\lambda \sim$ B-field along D6 $\rightsquigarrow$

$\rightsquigarrow$ non-commutative deformation of Σ, the "**quantum curve**"

Extracting the answer from free fermions?

More precisely, the prediction of Dijgraaf et. al. can be formulated as

$$Z_{\rm ff}(\xi, \mathbf{t}; \lambda) = \sum_{\mathbf{p} \in H^2(X, \mathbb{Z})} e^{\mathbf{p} \cdot \xi} Z_{\rm top}(\mathbf{t} + \lambda \mathbf{p}, \lambda).$$

This could give us an elegant non-perturbative definition of $Z_{\rm top}(\mathbf{t}, \lambda)$ if we knew

a) exactly how to turn the curve Σ into a “quantum curve”,

b) how to associate a free fermion partition function to a “quantum curve”,

c) the relation between the variables $(\xi, \mathbf{t})$ and parameters of “quantum curve”.

This has been illustrated by some examples in the work of Dijkgraaf et. al..

Outline of the solution

Our goal: Turn this into a **general** and **non-perturbative** mathematical definition of the topological string partition functions for class Σ.

To explain the answer we need to address to following questions:

A) **How to quantize Σ and turn it into a free fermion partition function?**
– *use meromorphic opers and theory of infinite Grassmannians / free fermions*

B) **How to parameterise quantum curves in terms of (ξ, t)?**
– *use Riemann-Hilbert correspondence and Abelianisation*

C) **Why is Abelianisation the right thing to use?**
– *exact WKB gives a canonical way to "quantize" the leading order result*

A) From quantum curve to free fermion partition functions I

Quantum curve $\sim$ Differential equation quantising the equation for Σ:

$$v^2 - Q_0(u) = 0 \quad \rightsquigarrow \quad \boxed{(\lambda^2 \partial_u^2 + Q(u))\chi(u) = 0,} \quad Q(u) = Q_0(u) + \mathcal{O}(\lambda).$$

Corresponding $\mathcal{D}$-module $\sim$ flat connection having horizontal sections Ψ,

$$\nabla_\Sigma \Psi(u) \equiv \left[\lambda \partial_u + \begin{pmatrix} 0 & Q \\ 1 & 0 \end{pmatrix}\right] \Psi(u) = 0.$$

Fermionic state $\mathfrak{f}_\Psi(Q)$ defined as

$$\mathfrak{f}_\Psi(Q) = \exp\Big(-\sum_{k>0}\sum_{l\geq 0} \psi_{-k} \cdot A_{kl} \cdot \bar{\psi}_{-l} \Big)\mathfrak{f}_0 \qquad \begin{aligned} \{\psi_{s,n}, \bar{\psi}_{t,m}\} &= \delta_{s,t}\delta_{n,-m} \\ \{\psi_{s,n}, \psi_{t,m}\} &= 0 = \{\bar{\psi}_{s,n}, \bar{\psi}_{t,m}\} \end{aligned}$$

$$\frac{(\Psi(x))^{-1}\Psi(y)}{x-y} = \sum_{l\geq 0} y^{-l-1} w_l(x), \qquad w_l(x) = -x^l + \sum_{k>0} x^{-k} A_{kl}$$

Note that $\{w_l(x), l = 0, 1, \dots\}$ is a basis for the subspace W_Ψ in the Sato-Segal-Wilson Grassmannian associated to Ψ.

A) From quantum curve to free fermion partition functions II

Proposal: Free fermion partition function = tau-function (Sato-Jimbo-Miwa-Segal-Wilson)

$$Z_{\mathrm{ff}}(\xi, \mathbf{t}; \lambda) = \left\langle \, \mathfrak{f}_0 \, , \, e^{\mathsf{H}(\tau)} \mathfrak{f}_\Psi(Q) \right\rangle.$$

where $\mathsf{H}(\tau) = \sum_i \mathsf{H}_i \tau_i$, H_i: generators of an abelian sub-algebra $\mathcal{A}$ of $\mathcal{W}_{1+\infty}$,
$\mathcal{W}_{1+\infty}$: Lie algebra generated by fermion bilinears.

Nice,

(+) relation to integrable hierarchies

but so far pretty useless, in general*)

(–) don't know which sub-algebra $\mathcal{A}$ is "suitable" for our problem

(–) don't know relation between $(\xi, \mathbf{t})$ and (τ, Q)

*) Exceptions: Examples investigated by Dijkgraaf et. al.

A) How to quantize the spectral curve I

Quantum curve receives **quantum corrections**:

$$Q_0(u) \to Q(u) = Q_0(u) + \lambda \sum_{k=1}^{d} \frac{v}{y - u_k} - \lambda^2 \sum_{k=1}^{d} \frac{3}{4(y - u_k)^2}.$$

$$\lambda^2 v_k^2 + Q_k = 0, \qquad Q_k = \lim_{u \to u_k} \left(Q(u) - \lambda \frac{v}{u - u_k} + \lambda^2 \frac{3}{4(u_l - u_k)^2} \right)$$

Why?

- Only now we have **enough** parameters in quantum curve $(\mathbf{m}, \mathbf{u}, \mathbf{v})$, $\mathbf{u} = (u_1, \ldots, u_{n-3})$, $\mathbf{v} = (v_1, \ldots, v_{n-3})$, to account for both ξ and $\mathbf{t}$.

- The extra singularities are more **apparent** than real, the $\mathcal{D}$-module associated to the quantum curve is **non-singular** at $u = u_k$,

$$\lambda \partial_u + \begin{pmatrix} 0 & Q \\ 1 & 0 \end{pmatrix} \quad \text{gauge equivalent to} \quad \lambda \partial_u + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

  with $A_{ij} = A_{ij}(u)$ non-singular at $u = u_k$.

B) How to parameterise quantum curves in terms of (ξ, t)?

Main problem: Relation between (ξ, t) and parameters of quantum curve.

Our proposal:

(ξ, t) $\sim$ **very** special coordinates for monodromy data

made precise through

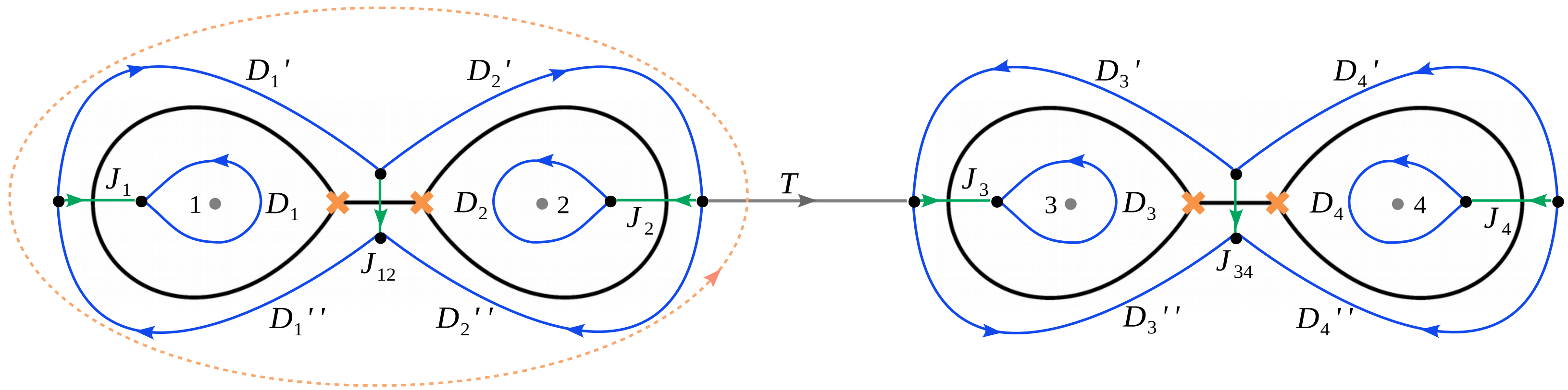
- **Riemann-Hilbert correspondence** – correspondence between monodromies (holonomies of flat connection) and $\mathcal{D}$-modules (quantum curves),

and

- **Abelianisation:** Curve $\Sigma \mapsto$ very special coordinates for monodromy data.

B) **Abelianisation** (Hollands-Neitzke)

Fenchel-Nielsen (FN) network (black) decomposes surface C into annular regions A_i.



- Connection can be diagonalised on each annular region A_i. Parallel transport $\rightsquigarrow$ collection of diagonal matrices D_i, D_i', D_i'', eigenvalues: simple functions of $e^{i\theta_r}$, $r = 1, 2, 3, 4$, $e^{i\sigma}$, and **diagonal matrix** T**, eigenvalue** $e^{i\tau}$.

- Jump matrices J_i, J_{ij} (non-diagonal!) representing **non-abelian** parallel transport across walls of FN network uniquely determined in terms of matrices D_i, D_i', D_i'' by consistency conditions.

Any closed path γ on C can be decomposed into segments contained in A_i (blue), segments crossing walls (green), and a path traversing annulus between the two pairs of pants (grey) $\rightsquigarrow$ **holonomies parameterised** in terms of σ, τ, θ_r, $r = 1, 2, 3, 4$.

B) Our proposal, finally

To given $\mathbf{t} \in \mathbb{R}^{3g-3+n}$ (Kähler parameters), ξ (twist parameters)

- find mirror curve Σ, $v^2 = Q_0(u)$ and canonical basis for $H_1(\Sigma, \mathbb{Z})$ such that parameters $\mathbf{t}$ are the a-cycle periods of Σ
- find Fenchel-Nielsen network defined by $Q_0(u)$ for real $\mathbf{t}$
- construct quantum curve ∇_Σ associated to $(\xi, \mathbf{t})$ by Riemann-Hilbert, assuming

Dictionary: $\boxed{\sigma_r = t_r/\lambda, \quad \mathrm{i}\tau_r = \xi_r, \quad \theta_r^2 = \delta_k/\lambda^2.}$

- construct $\mathcal{Z}_{\mathrm{ff}}(\xi, \mathbf{t}; \lambda)$ as SJMSW tau-function associated to ∇_Σ
- expand in $e^{\xi \cdot \mathbf{p}}$, extract $\mathcal{Z}_{\mathrm{top}}$ using

$$Z_{\mathrm{ff}}(\xi, \mathbf{t}; \lambda) = \sum_{\mathbf{p} \in H^2(X, \mathbb{Z})} e^{\mathbf{p} \cdot \xi} \mathcal{Z}_{\mathrm{top}}(\mathbf{t} + \lambda \mathbf{p}, \lambda)$$

- Analytically continue in $\mathbf{t}$

The proof for $C = C_{0,4}$:

Calculation of both sides, comparison

Calculation of tau-functions: Can be done using either

- Tau-functions are generalised conformal blocks of free fermion VOA (Moore; Palmer; J.T. '17)

- $\rightsquigarrow$ can be factorised by gluing construction (Iorgov-Lisovyy-JT)

or, even better

- Factorisation of Riemann-Hilbert problems

- $\rightsquigarrow$ factorisation of tau-functions (Gavrylenko-Lisovyy, Cafasso-Gavrylenko-Lisovyy)

Either way $\rightsquigarrow$ explicit formulae (first conjectured by Gamayun-Iorgov-Lisovyy)

$$\mathcal{T}(\sigma, \tau\,;\, \underline{\theta}\,;\, z) = \sum_{n\in\mathbb{Z}} \sum_{\xi,\zeta\in\mathbb{Y}} \mathcal{Z}^{(n)}_{\xi,\zeta,+}\, \mathcal{Z}^{(n)}_{\xi,\zeta,-} = \sum_{n\in\mathbb{Z}} e^{in\tau}\, \mathcal{G}(\,\sigma + n\,,\, \underline{\theta}\,;\, z\,),$$

where:

where $\mathcal{G}(\sigma , \underline{\theta} ; z)$ can be factorised as

$$\mathcal{G}(\sigma , \underline{\theta} ; z) = M(\sigma, \theta_4, \theta_3) M(\sigma, \theta_2, \theta_1) \mathcal{F}(\sigma , \underline{\theta} ; z),$$

using the following notations:

- The functions $N(\theta_3, \theta_2, \theta_1)$ are defined as

$$M(\theta_3, \theta_2, \theta_1) = \frac{\prod_{\epsilon=\pm} G(1+\theta_3+\epsilon(\theta_2+\theta_1))G(1+\theta_3+\epsilon(\theta_2-\theta_1))}{G(1+2\theta_3)G(1-2\theta_2)G(1-2\theta_1)},$$

  where $G(p)$ is the Barnes G-function that satisfies $G(p+1) = \Gamma(p)G(p)$.

- $\mathcal{F}(\sigma , \underline{\theta} ; z)$ can be represented by the following power series

$$\mathcal{F}(\sigma , \underline{\theta} ; z) = z^{\sigma^2-\theta_1^2-\theta_2^2}(1-z)^{2\theta_2\theta_3} \sum_{\xi,\zeta\in\mathbb{Y}} z^{|\xi|+|\zeta|} \mathcal{F}_{\xi,\zeta}(\sigma, \underline{\theta}),$$

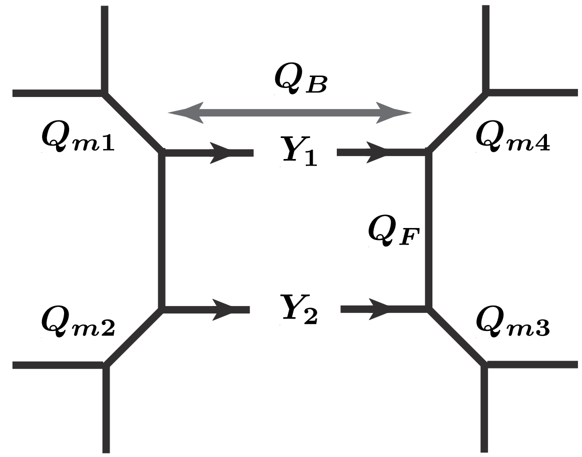
  with $\mathbb{Y}$ set of partitions, coefficients $\mathcal{F}_{\xi,\zeta}(\sigma, \underline{\theta})$ explicitly given in

$$\mathcal{F}_{\xi,\zeta}(\sigma, \underline{\theta}) = \prod_{(i,j)\in\xi} \frac{((\theta_2+\sigma+i-j)^2-\theta_1^2)((\theta_3+\sigma+i-j)^2-\theta_4^2)}{(\xi'_j-i+\xi_i-j+1)^2(\xi'_j-i+\zeta_i-j+1+2\sigma)^2}$$
$$\prod_{(i,j)\in\zeta} \frac{((\theta_2-\sigma+i-j)^2-\theta_1^2)((\theta_3-\sigma+i-j)^2-\theta_4^2)}{(\zeta'_j-i+\zeta_i-j+1)^2(\zeta'_j-i+\xi_i-j+1-2\sigma)^2}.$$

  ζ_i / ζ'_i arm / leg length of $(i,j) \in \mathbb{Y}$.

The proof for $C = C_{0,4}$, II

Topological string partition function: Can be calculated using top. vertex



Careful 4d limit $\rightsquigarrow$ **match!**

Crucial is the precise formula for $M(\theta_3, \theta_2, \theta_1)$:

- Only for very special choices of $M(\theta_3, \theta_2, \theta_1)$ one gets Fourier-series of the form

$$\mathcal{T}\big(\sigma, \tau\,;\, \underline{\theta}\,;\, z\big) := \sum_{n\in\mathbb{Z}} e^{in\tau}\,\mathcal{G}(\,\sigma + n\,,\,\underline{\theta}\,;\, z\,),$$

Corollary: **Quantitative check of string dualities!**

- Only for very particular coordinate τ one gets right formula for $M(\theta_3, \theta_2, \theta_1)$.

C) Why abelianisation is the right thing to use

Key-word: **Exact WKB:**

- Foliations defined by Q_0 for real periods $\mathfrak{t}$ decompose C into annular regions.

- In each annular region there exist $unique$ solutions of quantum curve equation with diagonal monodromy and leading asymptotics

$$\chi(u,\lambda) = \frac{\sqrt{\lambda}}{(Q_0(u))^{\frac{1}{4}}} \exp\left(\pm \int^u du \left(\frac{1}{\lambda}\sqrt{Q_0(u)} + \frac{Q_1(u)}{2\sqrt{Q_0(u)}}\right)\right)(1+\mathcal{O}(\lambda)),$$

  defined through Borel-summation of λ-expansion.

- Analytic continuation across walls represented by jump matrices used in Abelianisation

- $\rightsquigarrow$ monodromy of Borel sums naturally parameterised by σ, τ.

Summary

We have presented a proposal for a **non-perturbative**$^{*)}$ and **computable** definition of the topological string partition functions for class Σ.

$^{*)}$ manifest in representation as a Fredholm determinant (Cafasso-Gavrylenko-Lisovyy)

Key elements of the proposal

A) *How to quantize* Σ *and turn it into a free fermion partition function?*
– **use meromorphic opers and theory of inf. Grassmannians / free fermions**

B) *How to parameterise quantum curves in terms of* $(\xi, \mathbf{t})$*?*
– **use Riemann-Hilbert correspondence and Abelianisation**

C) *Why is Abelianisation the right thing to do?*
– **exact WKB gives a canonical way to "quantise" the leading order result**

Relation to other approaches

This problem has previously been approached (in simple cases) by other methods

- **Integrable structures:** (Aganagic-Dijkgraaf-Klemm-Marino-Vafa,. . . ,Okounkov)

  Our work makes integrability **effective** in complicated cases.

- **Topological recursion:** So far unclear which exact initial conditions to put. Can now be extracted from exact result (R. Belliard, J.T., in progress)

- **Quantisation of $H^3(Y,\mathbb{R})$, holomorphic anomaly.** The expansion

$$Z_{\rm ff}(\xi,\mathbf{t};\lambda)=\sum_{\mathbf{p}\in H^2(X,\mathbb{Z})} e^{\mathbf{p}\cdot\xi}\mathcal{Z}_{\rm top}(\mathbf{t}+\lambda\mathbf{p},\lambda)$$

  has an interpretation as a Fourier-transformation relating natural representations for quantisation of $H^3(Y,\mathbb{R})$ (Iorgov-lisovyy-J.T., and work in progress)

- **Relation to Hitchin systems:** (cf. Diaconescu, Dijkgraaf, Donagi, Hofman, Pantev)

- **Matrix models:** Relation between contours and choices of coordinates (σ,τ)

Outlook

- Toric CY: (Cf. Marino; Jimbo-Nagoya-Sakai)

- Higher genus, irregular singularities

- **Higher rank** (cf. Coman-Pomoni-J.T.'17, and Hollands-Neizke (to appear))

Crucial is the **interplay** between **two** integrable structures in this context:

- Integrable flows on moduli spaces –
  (integrable hierarches, Hitchin systems, isomonodromic deformations,....)

- Integrable structures on character varieties –
  best expressed in terms of coordinates of **Fenchel-Nielsen type**.