Topological Strings on Compact CY

and SCFT's

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Based on joint work with

H.Hayashi,P. Jefferson,H.-C. Kim,K. Ohmori

Calabi-Yau Singularities and SCFT's

Singularities of non-compact CY 3-folds: SCFT's.

Two cases:

Elliptic CY 3-folds → 6d SCFT's (1,0) susy
F-theory on elliptic CY 3-fold. The singularity could be the singularity of the base, or the elliptic fibration.

General CY 3-folds → 5d SCFT with N=1 susy
M-theory on CY 3-fold.

These cases have been studied intensively in the physics literature recently.

As a by product, methods have been devised to compute quantities of interest in these cases which end up being equivalent to computing topological string amplitudes on the corresponding non-compact CY.

Question: Can these methods be extended to the compact case? [BCOV (g=2), Klemm (g=51), ?]

In the compact case the topological string amplitudes get related to elliptic genus of certain strings in 6d and counting of black holes in 5d. So extending these to compact case is also of great physical interest.

Let us briefly review the connection with black holes.

5 dimensional black holes that were studied in mid 90's, were among the first black holes studied in the post-duality era. These black holes can be constructed in various ways. For concreteness we will focus on:

And, we consider the following duality equivalent brane configurations:

Type IIB: D3 branes wrapped on

M-theory: M2 brane wrapped on a 2-cycle.

These are dual to one another because:

Type IIB on Y x S^1 => M-theory on Y x T^2

Y=K3, or T^4

For concreteness let us focus on K3. Consider a genus g curve inside K3 and wrap a D3 brane on it. We get a sigma model on

These are dual to one another:

Type IIB on $Y \times S^1$ => M-theory on $Y \times T^2$

$Y = K3$, or T^4

For concreteness let us focus on K3.

Holographic statement of this:

$$AdS^3 \times S^3 \times K3 \qquad\qquad Sym^g(K3)$$

The corresponding string enjoys (4,4) supersymmetry

Moreover, the $SO(4)=SU(2)_L \times SU(2)_R$ rotation symmetry is realized as current algebras on the left- and right-moving parts of the conformal theory, except that on the R^4 part of the sigma model it does not quite split to left/right moving action.

Central charge $c_L= 6g + 6$.

If we consider an extra circle and string wrapped around it and give it a KK momentum n, for large n we get the growth of entropy given by

Which agrees with the prediction of Bekenstein-Hawking for large n and large g.

The 5d BPS black holes can be extended to spinning BPS black holes (BMPV) which further confirms this picture. The SU(2)_L captures the spinning BPS black hole.

F-theory and Spinning Black holes

In the context of M-theory we can reduce the amount of supersymmetry in the Bulk:

Again M2 branes wrapped on holomorphic curves give rise to spinning black hole. For general CY we do not know how to compute these—No strings!

We want to get a string in 6 dimensions. Can this appear? Yes, if the CY is elliptic:

Moreover D3 branes wrapped on

D3 branes wrapped around a curve in the base lead to strings with (0,4) supersymmetry.

It is not difficult to compute the central charge of this string and (ignoring CM) one finds that it is given by [V,1997]

Where c_1 is the first chern class of the base B.

This should lead to a holographic statement in the F-theory context [HMVV,2015 (see also CLMSW,2017)]

$$AdS^3 \times S^3 \times B$$

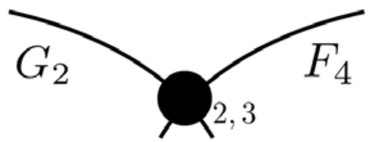
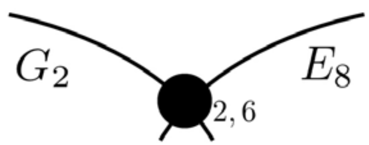
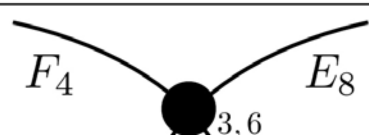
$$Ell(CFT) =$$

$$\mathbf{Z}^{top}(\boldsymbol{X}) =$$

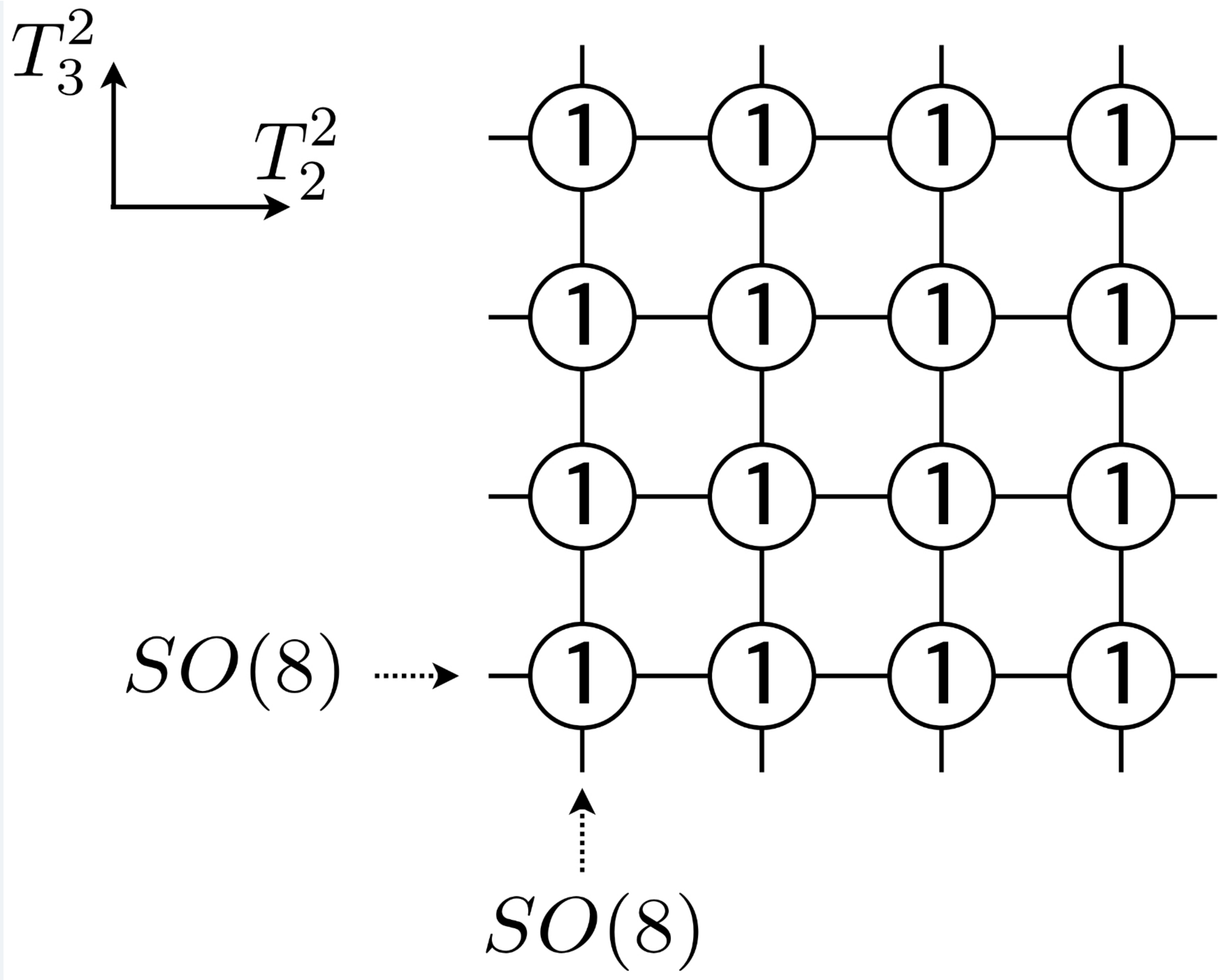
$$Z(BH)$$

How can we come up with the (0,4) 2d CFT?

Hint: For (1,0) 6d SCFT's the CFT for the the strings are known (0,4) quivers for some cases. Use these as local ingredient and `stitch them together' to get the CFT for the global model.

$\mathbb{Z}_2 \times \mathbb{Z}_2$:		$\bullet_{2,2}$: $\boxed{SO(8)}\ 1\ \boxed{SO(8)}$
$\mathbb{Z}_3 \times \mathbb{Z}_3$:		$\bullet_{3,3}$: $\boxed{E_6}\ 1\ 3\ 1\ \boxed{E_6}$
$\mathbb{Z}_4 \times \mathbb{Z}_4$:		$\bullet_{4,4}$: $\boxed{E_7}\ 1\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{so}_7}{3}\ \overset{\mathfrak{su}_2}{2}\ 1\ \boxed{E_7}$
$\mathbb{Z}_6 \times \mathbb{Z}_6$:		$\bullet_{6,6}$: $\boxed{E_8}\ 1\ 2\ \overset{\mathfrak{sp}_1}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{f}_4}{5}\ 1\ \overset{\mathfrak{g}_2}{3}\ \overset{\mathfrak{sp}_1}{2}\ 2\ 1\ \boxed{E_8}$
$\mathbb{Z}_2 \times \mathbb{Z}_3$:		$\bullet_{2,3}$: $\boxed{G_2}\ 1\ \boxed{F_4}$
$\mathbb{Z}_2 \times \mathbb{Z}_4$:		$\bullet_{2,4}$: $\boxed{SO(7)}\ \overset{\mathfrak{su}_2}{2}\ 1\ \boxed{E_7}$
$\mathbb{Z}_2 \times \mathbb{Z}_6$:		$\bullet_{2,6}$: $\boxed{G_2}\ 2\ 2\ 1\ \boxed{E_8}$
$\mathbb{Z}_3 \times \mathbb{Z}_6$:		$\bullet_{3,6}$: $\boxed{F_4}\ 1\ \overset{\mathfrak{g}_2}{3}\ \overset{\mathfrak{sp}_1}{2}\ 2\ 1\ \boxed{E_8}$

For each pair of fixed points we get a local picture as before, but now, the global symmetries get gauged.



T_3^2
T_2^2
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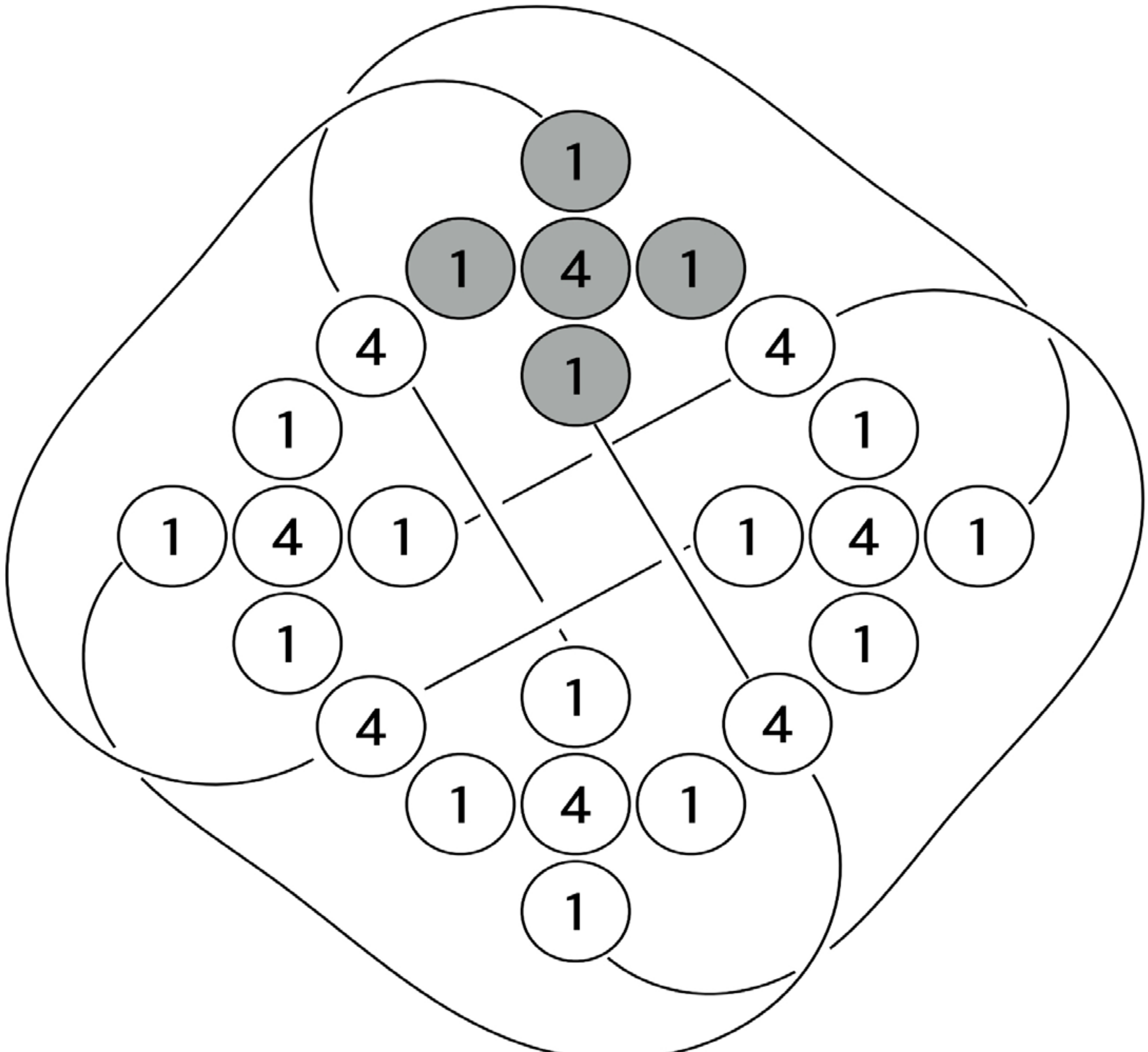
$SO(8)$
$SO(8)$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1 4 1
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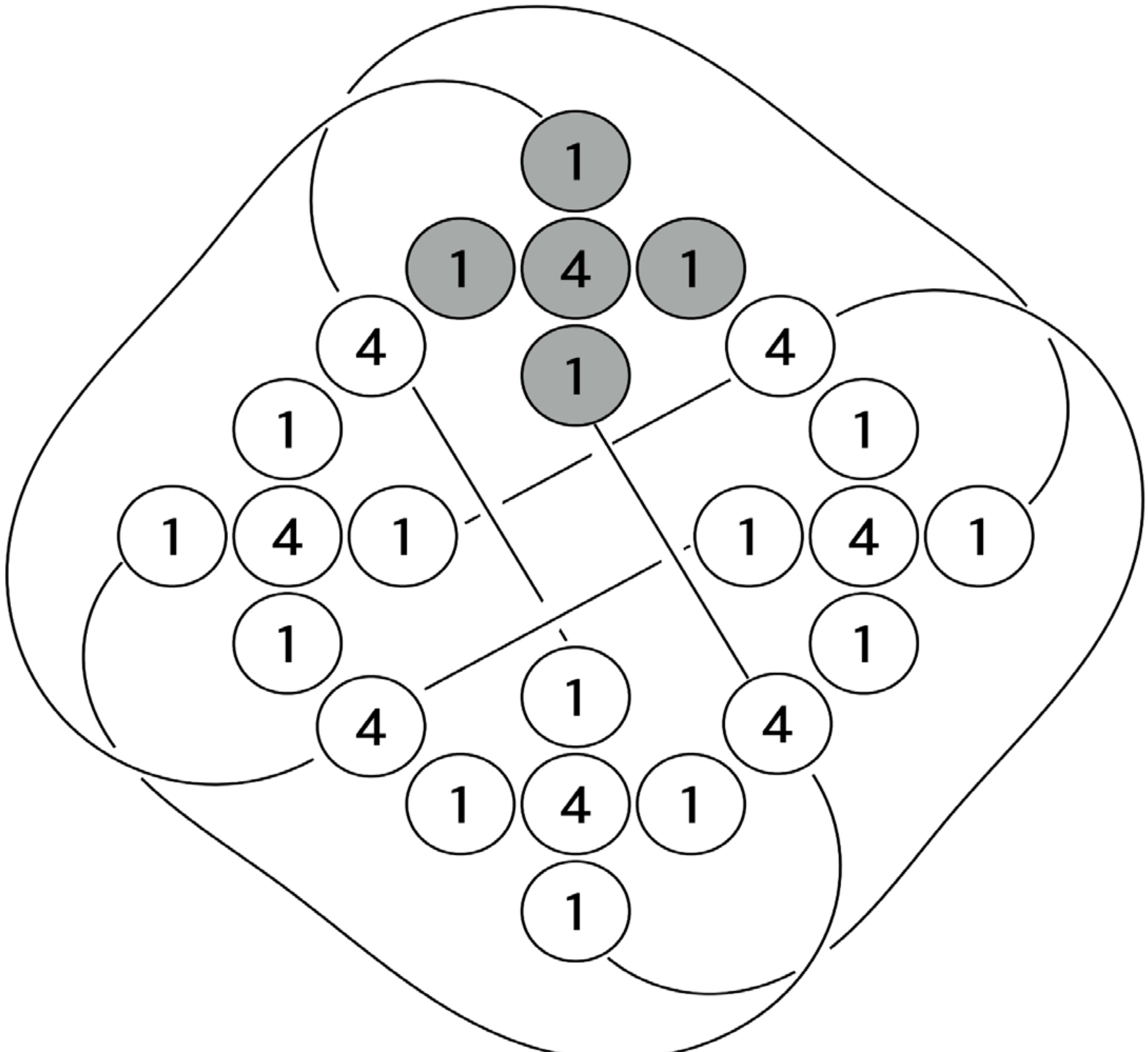
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1

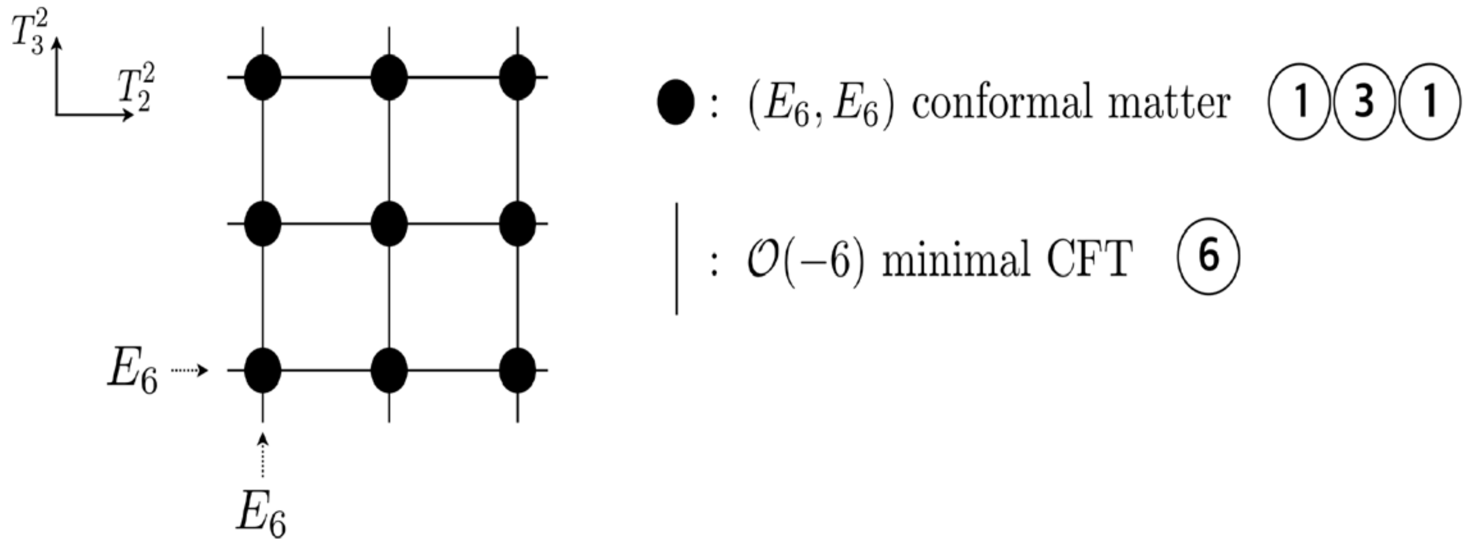
Gravitational anomaly cancels:

$$H - V + 29T - 273 = 4 - 8 \cdot 28 + 29 \cdot 17 - 273 = 0$$

Similarly one checks that mixed gravitational/gauge anomalies also cancel. One can also check that the 2-cycle lattice (modulo identifications) is self-dual, as it should.

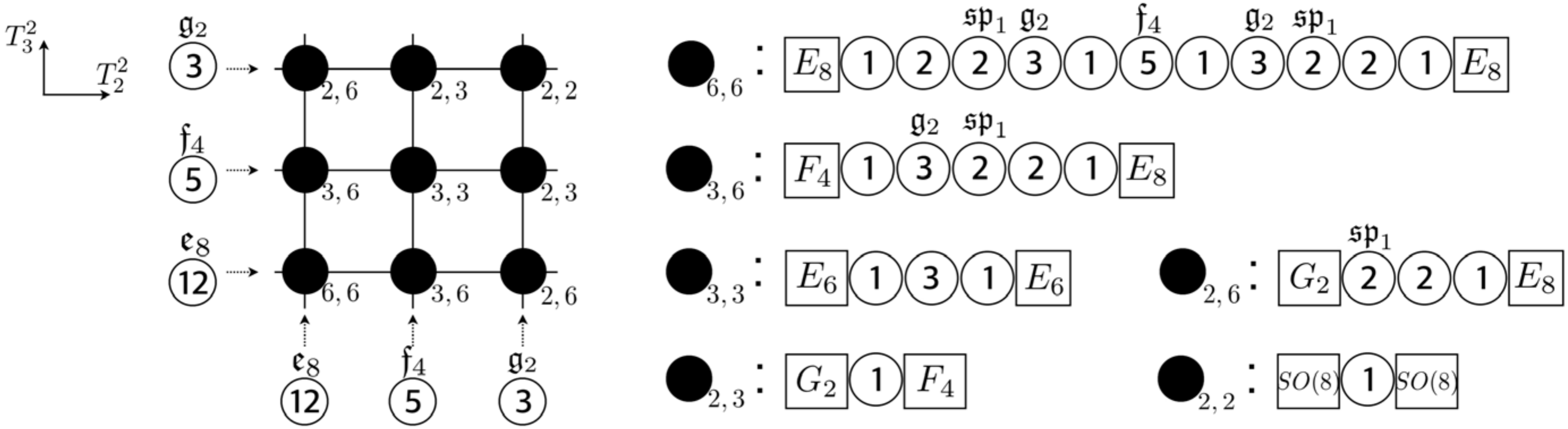
We seem to have a perfectly complete model.

Similarly for T^6/Z3xZ3 we get:

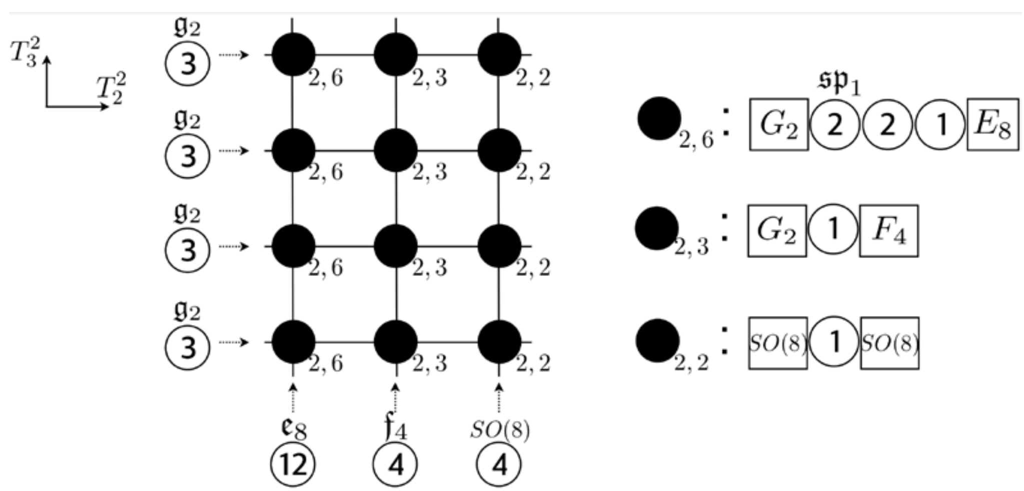


Gravitational anomalies cancel:

$$H - V + 29T - 273 = 1 - 540 + 29 \times 28 - 273 = 0$$

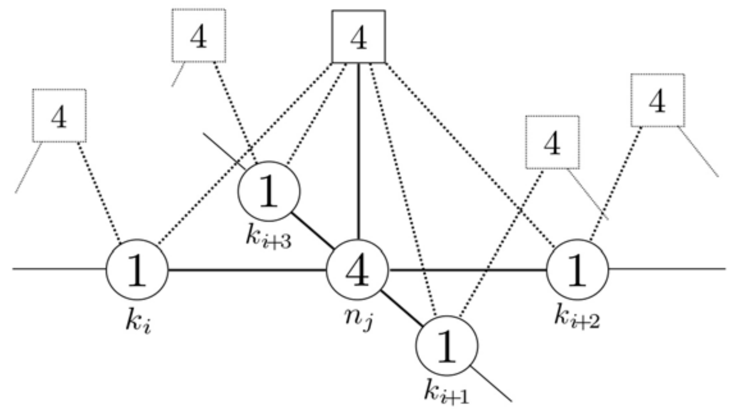


T_3^2
T_2^2
$\mathfrak{g}_2$ 3
$\mathfrak{f}_4$ 5
$\mathfrak{e}_8$ 12
2, 6 2, 3 2, 2
3, 6 3, 3 2, 3
6, 6 3, 6 2, 6
$\mathfrak{e}_8$ 12 $\mathfrak{f}_4$ 5 $\mathfrak{g}_2$ 3
6, 6 : E_8 1 2 2 3 1 5 1 3 2 2 1 E_8
$\mathfrak{sp}_1$ $\mathfrak{g}_2$ $\mathfrak{f}_4$ $\mathfrak{g}_2$ $\mathfrak{sp}_1$
3, 6 : F_4 1 3 2 2 1 E_8
$\mathfrak{g}_2$ $\mathfrak{sp}_1$
3, 3 : E_6 1 3 1 E_6
2, 6 : G_2 2 2 1 E_8
$\mathfrak{sp}_1$
2, 3 : G_2 1 F_4
2, 2 : $SO(8)$ 1 $SO(8)$



T_3^2
T_2^2
$\mathfrak{g}_2$ 3
$\mathfrak{g}_2$ 3
$\mathfrak{g}_2$ 3
$\mathfrak{g}_2$ 3
2, 6 2, 3 2, 2
$\mathfrak{e}_8$ 12 $\mathfrak{f}_4$ 4 $SO(8)$ 4
2, 6 : G_2 2 2 1 E_8
$\mathfrak{sp}_1$
2, 3 : G_2 1 F_4
2, 2 : $SO(8)$ 1 $SO(8)$

For the Z_2 x Z_2 model we have a concrete candidate quiver description of the strings



Quiver diagram near a -4 curve which intersects with four -1 curves. This is a subset of the full quiver diagram for 6d strings. Self-dual strings over -4 curve (denoted by a circle with 4) are described $Sp(n)$ gauge theory and strings over -1 curves (denoted by circles with 1) have $O(k)$ gauge theory description. A $Sp(n)$ gauge node includes a real antisymmetric hypermultiplet and 4 fundamental hypermultiplets denoted by a vertical solid line. A $O(k)$ gauge node includes a real symmetric hypermultiplet. A solid line between two gauge node denotes a 1/2 twisted hyper- and a fermi- multiplet in bifundamental representation. The dotted lines stand for fermi multiplets. When all other $\mathcal{O}(-4)$ tensor multiplets are decoupled (when the external lines in this diagram are removed), this diagram describes the little string theory of $T^4 \times \mathbb{C}/\mathbb{Z}_2 \times \mathbb{Z}_2$.

We get a holographic statement:

Check: We can compute the anomalies for this quiver theory using:

$$c_R = 3\mathrm{Tr}(\gamma^3 R_{\mathrm{cft}}^2) \ , \quad c_R - c_L = \mathrm{Tr}(\gamma^3)$$

where R_{cft} is the right-moving R-charge in the IR superconformal algebra and γ^3 is the 2d chirality operator acting on a chiral fermion $\psi_\pm$ as $\gamma^3\psi_\pm = \mp\psi_\pm$.

There are two choices for R-symmetry. For one of them, the BH branch we get the expected answer:

$$c_R = 3C\cdot C + 3c_1(B)\cdot C \ , \quad c_L = 3C\cdot C + 9c_1(B)\cdot C + 2$$

It would be interesting to check this. However there are hints that this cannot be exactly right…but it is on the right track.

$$Z_{\vec{k},\vec{n}^H,\vec{n}^V} = \oint \prod_{i,j=1}^{4} Z_{k_{ij}}^{\mathcal{O}(-1)}(\varphi_{ij}, m_i^H, m_j^V) \times \prod_{i=1}^{4} Z_{n_i^H}^{\mathcal{O}(-4)}(\tilde{\varphi}_i^H, m_i^H) \times \prod_{j=1}^{4} Z_{n_j^V}^{\mathcal{O}(-4)}(\tilde{\varphi}_i^V, m_i^V)$$
$$\times \prod_{i,j=1}^{4} Z_{k_{ij},n_i^H}^{\mathcal{O}(-1)\times\mathcal{O}(-4)}(\varphi_{ij}, \tilde{\varphi}_i^H) \times Z_{k_{ij},n_j^V}^{\mathcal{O}(-1)\times\mathcal{O}(-4)}(\varphi_{ij}, \tilde{\varphi}_j^V)$$

$$Z_k^{\mathcal{O}(-1)}(\varphi, m) = \frac{1}{|W_k|} \prod_{I=1}^{r} \left(\frac{d\varphi_I}{2\pi i} \cdot \frac{\theta_1(2\epsilon_+)}{i\eta} \right) \prod_{e\in\text{root}} \frac{\theta_1(e(\varphi))\theta_1(2\epsilon_+ + e(\varphi))}{i^2\eta^2}$$
$$\times \prod_{\rho\in\text{sym}} \frac{i^2\eta^2}{\theta_1(\epsilon_{1,2} + \rho(\varphi))} \prod_{\rho\in\text{fund}} \prod_{a=1}^{8} \frac{\theta_1(m_a + \rho(\varphi))}{i\eta} \ ,$$

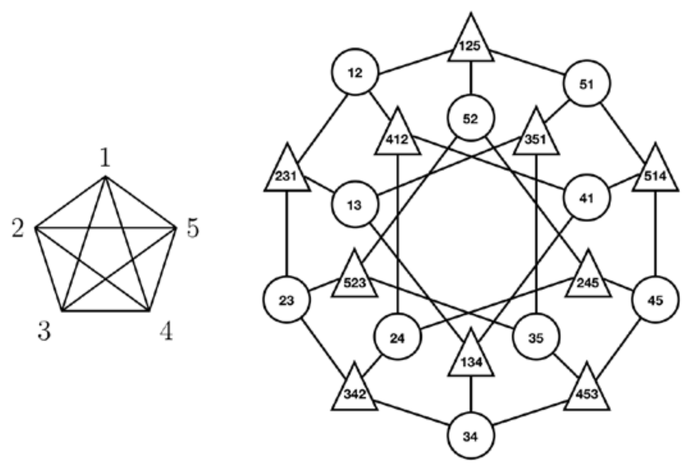
$$Z_n^{\mathcal{O}(-4)}(\tilde{\varphi}, m) = \frac{1}{|W_n|} \prod_{I=1}^{n} \left(\frac{d\tilde{\varphi}_I}{2\pi i} \cdot \frac{\theta_1(2\epsilon_+)}{i\eta} \right) \prod_{e\in\text{root}} \frac{\theta_1(e(\tilde{\varphi}))\theta_1(2\epsilon_+ + e(\tilde{\varphi}))}{i^2\eta^2}$$
$$\times \prod_{\rho\in\text{anti}} \frac{i^2\eta^2}{\theta_1(\epsilon_{1,2} + \rho(\tilde{\varphi}))} \prod_{\rho\in\text{fund}} \prod_{p=1}^{4} \frac{i\eta}{\theta_1(\epsilon_+ + \rho(\tilde{\varphi}) \pm m_p)}$$

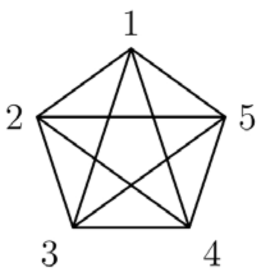
Extension to Non-elliptic Calabi-Yau threefold

The idea is to use 5d SCFTs. Consider the simple example of the mirror quintic:

(Quintic threefold/Z5xZ5xZ5)

There are point ($z_i=z_j=z_k=0$) and curve singularities where ($z_i=z_j=0$) with an A4 singularity: Geometry is:

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5
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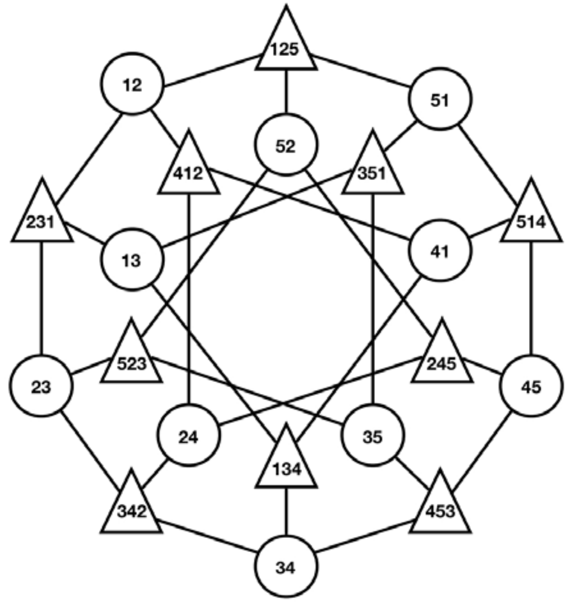
4


125
12
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231
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453
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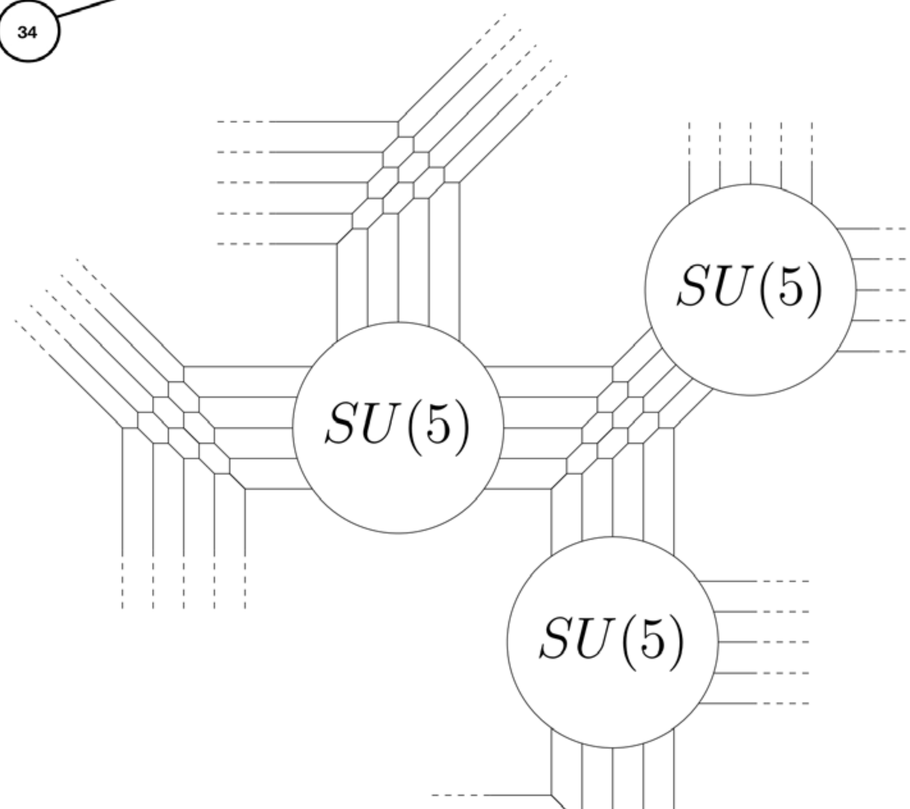

$SU(5)$
$SU(5)$
$SU(5)$

Standard Topological Vertex

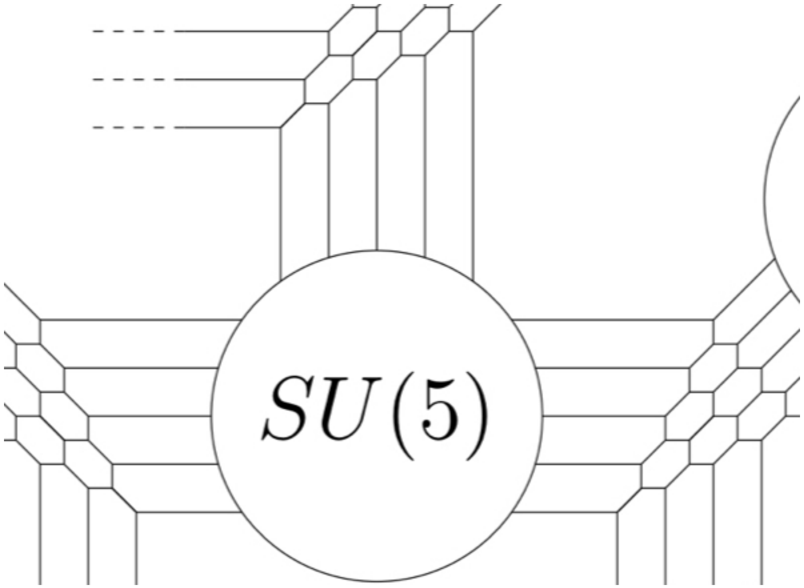
$$C_{\lambda\mu\nu}(q) = q^{-\frac{||\mu^t||^2}{2}+\frac{||\mu||^2}{2}+\frac{||\nu||^2}{2}}\tilde{Z}_\nu(q)\sum s_{\lambda^t/\eta}(q^{-\rho-\nu})s_{\mu/\eta}(q^{-\rho-\nu^t})$$

$s_{\mu/\nu}(x)$ is the skew Schur function

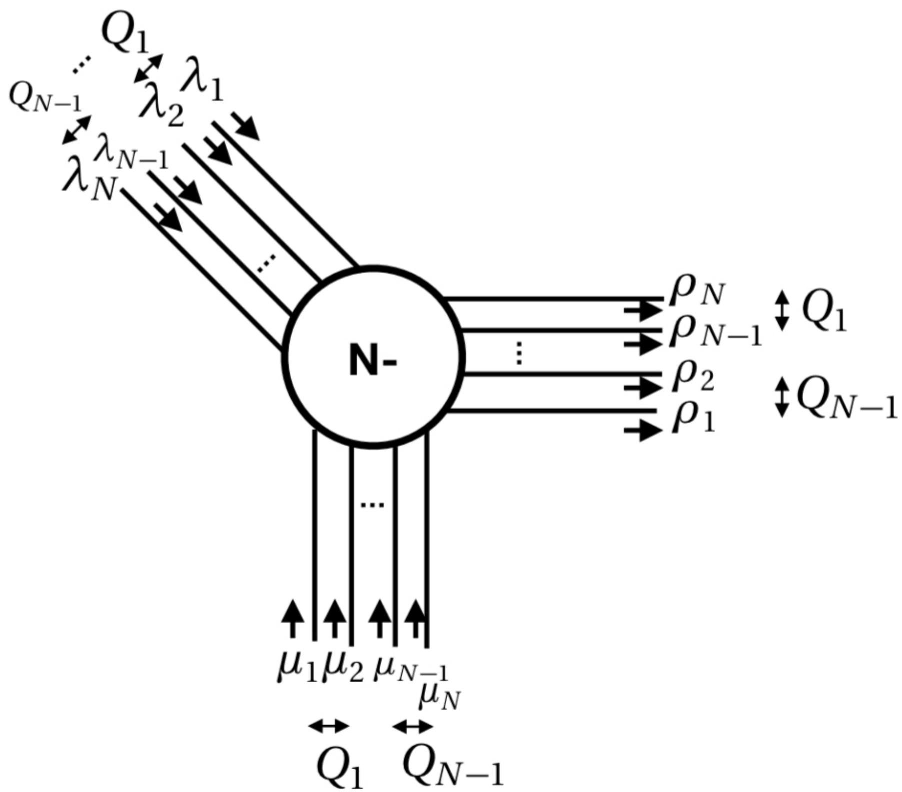
$$C^{(1^-_k)}_{\lambda,\mu,\rho}(q) = \frac{f_\lambda(q)^{-k}\delta_{\lambda,\mu,\rho}}{q^{\frac{1}{2}||\lambda||^2}\tilde{Z}_\lambda(q)}.$$

$$\tilde{Z}_\nu(q) = \prod_{(i,j)\in\nu} \left(1 - q^{\nu_i - j + \nu^t_j - i + 1}\right)^{-1}$$

$$f_\lambda(q) = (-1)^{|\lambda|} q^{\frac{||\lambda^t||^2 - ||\lambda||^2}{2}}$$


$SU(5)$

Following Hayashi and Ohmori one can extend the 1^- vertex:



$$C^{(5_0^-)}_{\vec{\lambda},\vec{\lambda}',\vec{\lambda}''}(\vec{A};y) = \frac{\prod_{a=1}^{5} f_{\lambda_a}(y)^{-3+a}\delta_{\lambda_a,\lambda'_a,\lambda''_a}}{Z^{\text{half vector}}_{SU(N),\vec{\lambda}}(\vec{A};y)}$$

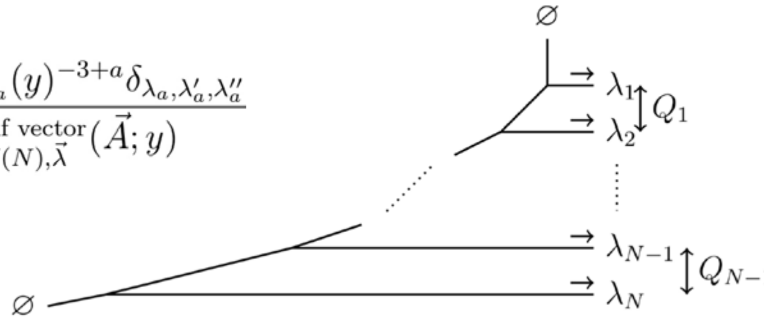
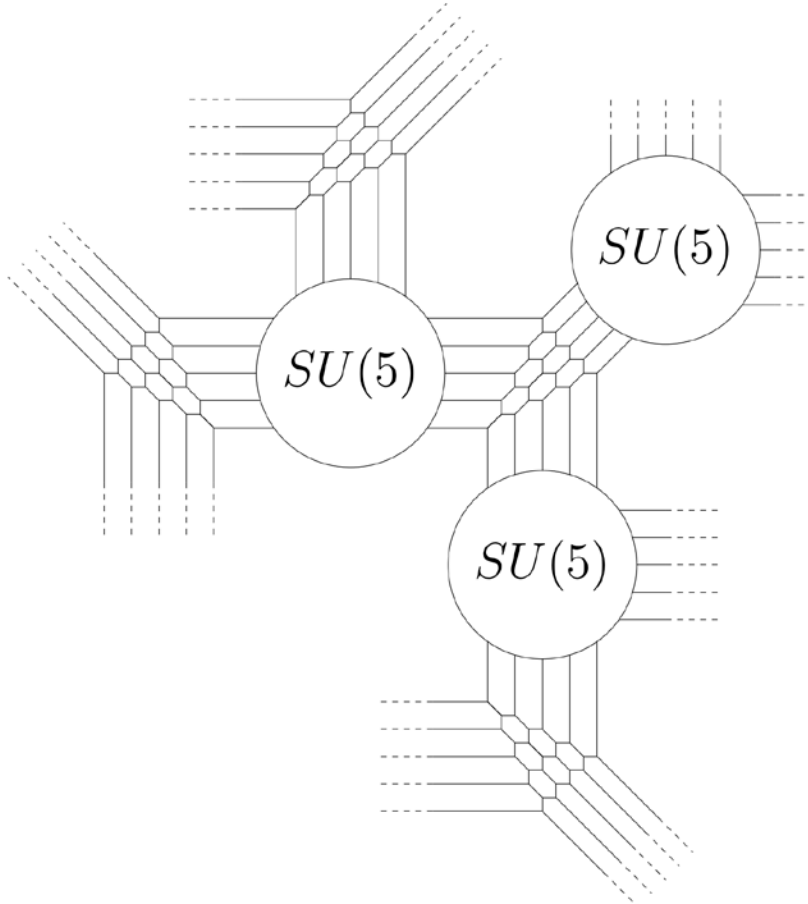


Figure 19. The web diagram that defines $Z^{\text{half vector}}_{SU(N),\vec{\lambda}}(\vec{Q};y)$. Each vertex represents the usual unrefined topological vertex, which is also called "+" vertex here.

$$\begin{aligned} Z^{\text{half vector}}_{SU(N),\vec{\lambda}}(\vec{Q};y) = {} & y^{\frac{1}{2}\sum_{a=1}^{N}||\lambda_a||^2}\prod_{a=1}^{N}\tilde{Z}_{\lambda_a}(y) \\ & \prod_{1\le a<b\le N-1}\Big[\prod_{i,j=1}^{\infty}\left(1-Q_aQ_{a+1}\cdots Q_b y^{i+j-1}\right)^{-1} \\ & \prod_{(i,j)\in\lambda_a}\left(1-Q_aQ_{a+1}\cdots Q_b y^{-\lambda_{a,i}+j-\lambda^t_{b+1,j}+i-1}\right)^{-1} \\ & \prod_{(i,j)\in\lambda_{b+1}}\left(1-Q_aQ_{a+1}\cdots Q_b y^{\lambda_{b+1,i}-j+\lambda^t_{a,j}-i+1}\right)^{-1}\Big]. \end{aligned} \tag{4.33}$$

$$Z_{SU(N)_k}(u_{\text{instanton}},\vec{Q};y) = \sum_{\vec{\lambda}}\left[\prod_{a=1}^{N}(-Q_{B_a})^{|\lambda_a|}(-1)^{|\lambda_a|}f_{\lambda_a}(y)^{N-k+2-2a}\right]Z^{\text{half vector}}_{SU(N),\vec{\lambda}}(\vec{Q};y)^2$$

$$u_{\text{instanton}} = Q_{B_1}\prod_{a=1}^{N-1}Q_a^{-2N+2k+2a\frac{N-k}{N}} \qquad f_\lambda(y) = (-1)^{|\lambda|}y^{\frac{||\lambda^t||^2-||\lambda||^2}{2}}$$


$SU(5)$
$SU(5)$
$SU(5)$

ℓ \ g	0	1	2	3	4	5	6	7
1	20	0	0	0	0	0	0	0
2	-50	0	0	0	0	0	0	0
3	280	-100	0	0	0	0	0	0
4	-2410	2785	-1200	175	0	0	0	0
5	25540	-63780	75160	-49740	18560	-3600	280	0

ℓ \ γ	0	1	2
0	$*$	300	-440
1	20	-60	140

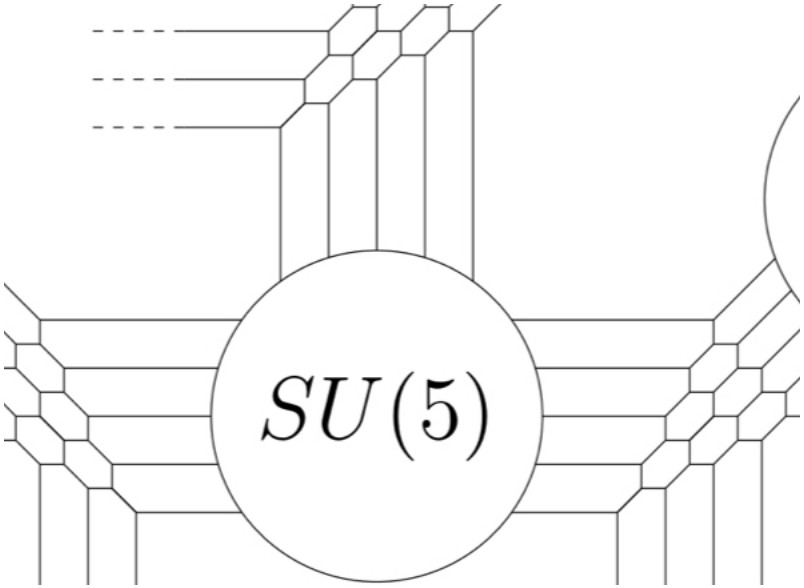
$\ell \backslash g$	0	1	2	3	4	5	6	7
1	20	0	0	0	0	0	0	0
2	-50	0	0	0	0	0	0	0
3	280	-100	0	0	0	0	0	0
4	-2410	2785	-1200	175	0	0	0	0
5	25540	-63780	75160	-49740	18560	-3600	280	0

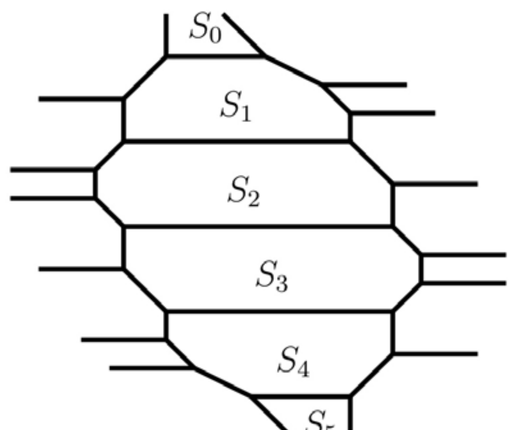
$\ell \backslash \gamma$	0	1	2
0	$*$	300	-440
1	20	-60	140

Katz and Morrison

$\ell \backslash \gamma$	0	1	2
0	$*$	300	-440
1	15	-60	155

Genus zero GV invariants of the mirror quintic


$SU(5)$


S_0
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S_3
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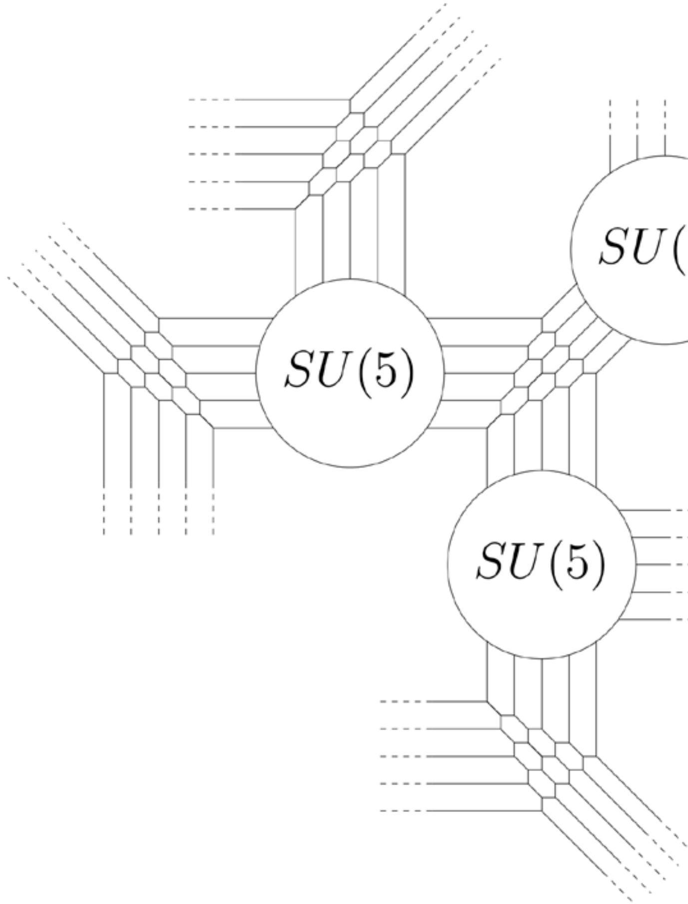
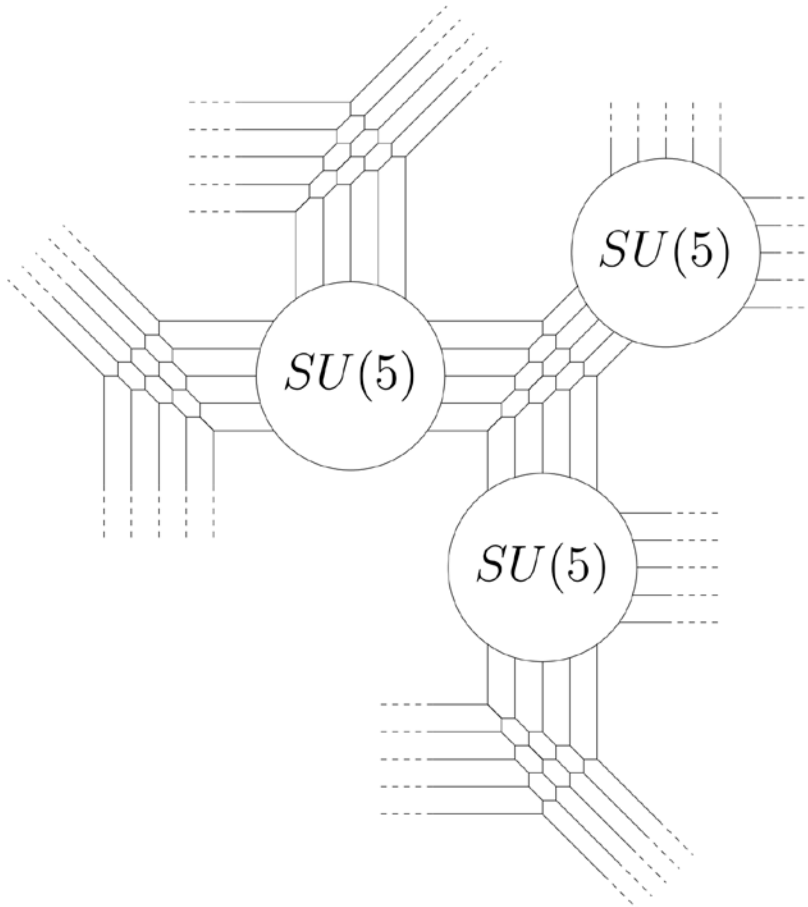
S_5



$\ell \backslash \gamma$	0	1	2
0	$*$	300	-440
1	20	-60	140

$\ell \backslash \gamma$	0	1	2
0	$*$	300	-440
1	15	-60	155

Genus zero GV invariants of the mirror quintic



$\ell \backslash \gamma$	0	1	2
0	$*$	300	-440
1	20	-60	140

$\ell \backslash \gamma$	0	1	2
0	$*$	300	-440
1	15	-60	155

Genus zero GV invariants of the mirror quintic

This agrees with the recent computation of Katz+Morrison

To complete the story for compact topological string on mirror quintic for all degrees we need to find more systematically how to correct the SU(5) gauging vertex to agree with the geometric picture.

Conclusion

We have seen that we can stitch local CFT's in 6 and 5 dimensions to have a complete description of non-gravitational sector of some compact CY 3-folds.

We can use these to propose:

Concrete holographic duality AdS3xS3XB with (0,4) CFT
Propose all genus answer for compact Calabi-Yau 3-fold.

Checks: The leading growth agrees with BH expectations
The local contributions are known to be correct
However it is also clear some corrections are needed to get the full correct answer.