# Extended topological field theory and the 2-dimensional Ising model


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Joint work with Constantin Teleman



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# 1988–90: New Avenues into QFT (Examples)

## Topological Quantum Field Theory

Edward Witten*

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

**Abstract.** A twisted version of four dimensional supersymmetric gauge theory is formulated. The model, which refines a nonrelativistic treatment by Atiyah, appears to underlie many recent developments in topology of low dimensional manifolds; the Donaldson polynomial invariants of four manifolds and the Floer groups of three manifolds appear naturally. The model may also be interesting from a physical viewpoint; it is in a sense a generally covariant quantum field theory, albeit one in which general covariance is unbroken, there are no gravitons, and the only excitations are topological.

### 1. Introduction

One of the dramatic developments in mathematics in recent years has been the program initiated by Donaldson of studying the topology of low dimensional manifolds via nonlinear classical field theory [1, 2]. Donaldson's work uses heavily the self-dual Yang-Mills equations, which were first introduced by

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**Abstract.** A variant of the usual supersymmetric nonlinear sigma model is described, governing maps from a Riemann surface $\Sigma$ to an arbitrary almost complex manifold $M$. It possesses a fermionic BRST-like symmetry, conserved for arbitrary $\Sigma$, and obeying $Q^2 = 0$. In a suitable version, the quantum ground states are the $1 + 1$ dimensional Floer groups. The correlation functions of the BRST-invariant operators are invariants (depending only on the homotopy type of the almost complex structure of $M$) similar to those that have entered in recent work of Gromov on symplectic geometry. The model can be coupled to dynamical gravitational or gauge fields while preserving the fermionic symmetry; some observations by Atiyah suggest that the latter coupling may be related to the Jones polynomial of knot theory. From the point of view of string theory, the main novelty of this type of sigma model is that the graviton vertex operator is a BRST commutator. Thus, models of this type may correspond to a realization at the level of string theory of an unbroken phase of quantum gravity.

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In recent years, Yang–Mills instantons have played an important role in the study of four manifolds and three manifolds in the work of Donaldson [1] and Floer [2], respectively. More recently, Atiyah advocated an interpretation of Floer theory

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## TOPOLOGICAL GRAVITY

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### ABSTRACT

A version of conformal gravity is formulated with a local fermionic symmetry that is reminiscent of BRST invariance. It may have mathematical applications (gravitational counterpart of Donaldson theory) or physical ones (unbroken phase of general relativity).

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In a lecture at the Hermann Weyl Symposium last year [1], Michael Atiyah proposed two problems for quantum field theorists. The first problem was to give a physical interpretation to Donaldson theory. The second problem was to find an intrinsically three dimensional definition of the Jones polynomial of knot theory. These two problems might roughly be described as follows.

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## Characteristic forms and geometric invariants


By Shiing-shen Chern and James Simons*


### 1. Introduction

This work, originally announced in [4], grew out of an attempt to derive a purely combinatorial formula for the first Pontrjagin number of a 4-manifold. The hope was that by integrating the characteristic curvature form (with respect to some Riemannian metric) simplex by simplex, and replacing the integral over each interior by another on the boundary, one

$$CS(A) = \int_M \langle A \wedge dA \rangle - \frac{1}{6}\langle A \wedge [A \wedge A] \rangle \qquad (\mathrm{mod}\ 1)$$

# Invariants of Knots and 3-Manifolds: A Puzzle

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**Ribbon Graphs and Their Invariants Derived from Quantum Groups**

N. Yu. Reshetikhin and V. G. Turaev
L.O.M.I., Fontanka 27, SU-191011 Leningrad, USSR

**Abstract.** The generalization of Jones polynomial of links to the case of graphs in $R^3$ is presented. It is constructed as the functor from the category of graphs to the category of representations of the quantum group.

**1. Introduction**

The present paper is intended to generalize the Jones polynomial of links and the related Jones-Conway and Kauffman polynomials to the case of graphs in $R^3$.

Originally the Jones polynomial was defined for links of circles in $R^3$ via an astonishing use of von Neumann algebras (see [Jo]). Later on it was understood that this and related polynomials may be constructed using the quantum $R$-matrices (see, for instance, [$\mathrm{Tu}_1$]). This approach enables one to construct similar invariants for coloured links, i.e. links each of whose components is provided with a module over a fixed algebra (see [$\mathrm{Re}_1$], where the role of the algebra is played by the quantized universal enveloping algebra $U_q(G)$ of a semisimple Lie algebra $G$).

**Invariants of 3-manifolds via link polynomials and quantum groups**

**N. Reshetikhin**[1] **and V.G. Turaev**[2]
[1] Department of Mathematics, Harvard University, Cambridge, MA 02138, USA
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**1. Introduction**

The aim of this paper is to construct new topological invariants of compact oriented 3-manifolds and of framed links in such manifolds. Our invariant of (a link in) a closed oriented 3-manifold is a sequence of complex numbers parametrized by complex roots of 1. For a framed link in $S^3$ the terms of the sequence are equals to the values of the (suitably parametrized) Jones polynomial of

$$XY - YX = \frac{K^2 - K^{-2}}{t^2 - t^{-2}}$$

$$XK = t^{-2}KX, \quad YK = t^2KY, \quad KK^{-1} = K^{-1}K = 1$$

$$K^{4r} = 1 \qquad X^r = Y^r = 0$$

# 1988–90: New Avenues into QFT (Axioms)

## The definition of conformal field theory

Graeme Segal

I shall propose a definition of 2-dimensional conformal field theory which I believe is equivalent to that used by Friedan et al. The idea arises from joint work with Quillen.

### §1 The definition

The category $\mathcal{C}$ is defined as follows. There is a sequence of objects $\{C_n\}_{n\geq 0}$, where $C_n$ is the disjoint union of a set of $n$

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*Great theorems are awesome; great definitions are transformational*

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## TOPOLOGICAL QUANTUM FIELD THEORIES

*by* Michael ATIYAH

*To René Thom on his 65th birthday.*

### 1. Introduction

In recent years there has been a remarkable renaissance in the relation between Geometry and Physics. This relation involves the most advanced and sophisticated ideas on each side and appears to be extremely deep. The traditional links between the two subjects, as embodied for example in Einstein's Theory of General Relativity or in Maxwell's Equations for Electro-Magnetism are concerned essentially with classical fields of force, governed by differential equations, and their geometrical interpretation. The new feature of present developments is that links are being established between *quantum* physics and *topology*. It is no longer the purely *local* aspects that are involved but their *global* counterparts. In a very general sense this should not be too surprising. Both quantum theory and topology are characterized by discrete phenomena emerging from a continuous background. However, the realization that this vague philosophical view-point could be translated into reasonably precise and significant mathematical statements is mainly due to the efforts of Edward Witten who, in a variety of directions, has shown the insight that can be derived by examining the topological aspects of quantum field theories.

The best starting point is undoubtedly Witten's paper [11] where he explained

It will be clear how much this whole subject rests on the ideas of Witten. In formulating the axiomatic framework in § 2, I have also been following Graeme Segal who produced a very similar approach to conformal field theories [10]. Finally it seems appropriate to point out the major role that *cobordism* plays in these theories. Thus René Thom's most celebrated contribution to geometry has now a new and deeper relevance.

We come now to the promised axioms. A topological quantum field theory (QFT), in dimension $d$ defined over a ground ring $\Lambda$, consists of the following data:

(A) A finitely generated $\Lambda$-module $Z(\Sigma)$ associated to each oriented closed smooth $d$-dimensional manifold $\Sigma$,

(B) An element $Z(M) \in Z(\partial M)$ associated to each oriented smooth $(d+1)$-dimensional manifold (with boundary) $M$.

These data are subject to the following axioms, which we state briefly and expand upon below:

(1) $Z$ is *functorial* with respect to orientation preserving diffeomorphisms of $\Sigma$ and $M$,

(2) $Z$ is *involutory*, i.e. $Z(\Sigma^*) = Z(\Sigma)^*$ where $\Sigma^*$ is $\Sigma$ with opposite orientation and $Z(\Sigma)^*$ denotes the dual module (see below),

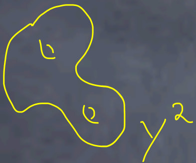
(3) $Z$ is *multiplicative*.

We now elaborate on the precise meaning of the axioms. (1) means first that an orientation preserving diffeomorphism $f: \Sigma \to \Sigma'$ induces an isomorphism

# 3-Dimensional Finite Gauge Theory

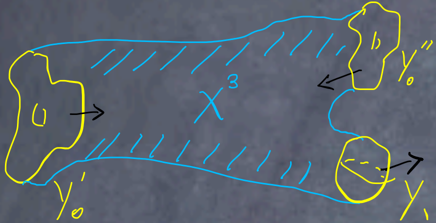
| | |
|---|---|
| $G$ | finite group |
| $\mathrm{Bord}_{\langle 2,3\rangle}$ | (unoriented) bordism category |
| $\mathrm{Vect}$ | category of complex vector spaces |
| $\mathscr{G}_G\colon \mathrm{Bord}_{\langle 2,3\rangle} \longrightarrow \mathrm{Vect}$ | symmetric monoidal functor |

objects :




morphisms :

$Y_0' \amalg Y_0'' \xrightarrow{X} Y_1$




arrow of time on each component of $\partial X$

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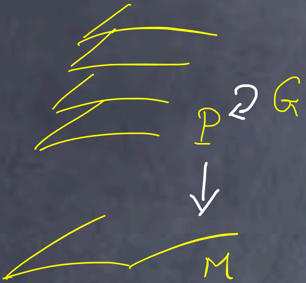
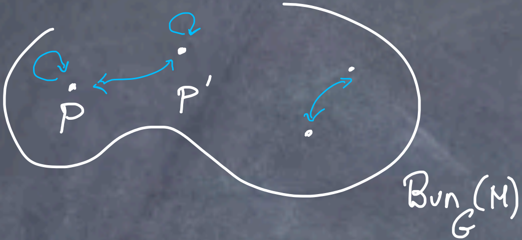
**Finite path integral:** $(P \to M) \in \mathrm{Bun}_G(M)$ fluctuating field



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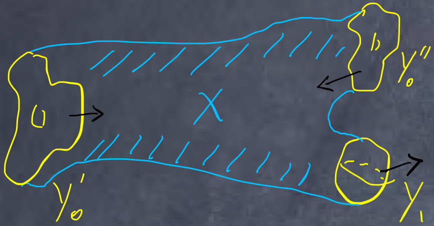
For $Y^2$ closed have canonical quantization:

$$\mathscr{G}_G(Y) = \mathrm{Fun}\big(\mathrm{Bun}_G(Y)\big)$$

$$
\begin{tikzcd}
 & \mathrm{Bun}_G(X) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\
\mathrm{Bun}_G(Y_0) & & \mathrm{Bun}_G(Y_1)
\end{tikzcd}
$$

Quantize (linearize) by push-pull:

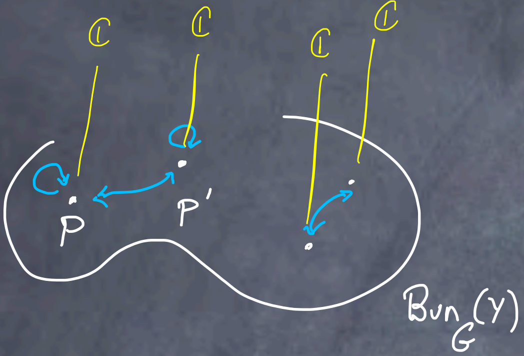
$$\mathscr{G}_G(X) = (r_1)_* \circ (r_0)^* \colon \mathscr{G}_G(Y_0) \longrightarrow \mathscr{G}_G(Y_1)$$



# Extended Locality

$\mathscr{G}_G(Y)$ is a finite path integral. . . of the constant function $\mathbb{C}$:

$$\mathscr{G}_G(Y) \text{“}=\text{”} \int_{\mathrm{Bun}_G(Y)} \mathbb{C} \quad \cong \bigoplus_{[P]\in\pi_0 \mathrm{Bun}_G(Y)} \mathbb{C}$$



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Extend to lower dimensions: $\mathrm{Bun}_G(S^1) \simeq G/\!/G$

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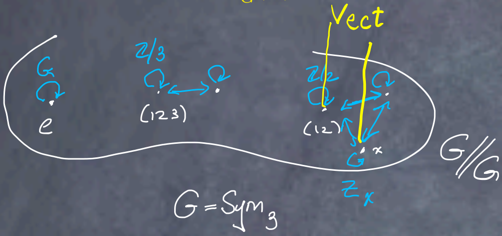

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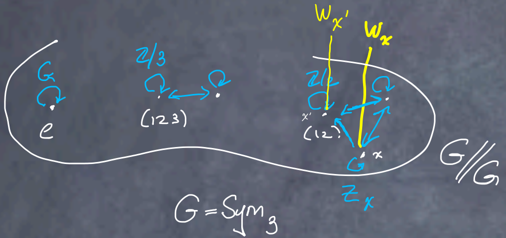
$W_x \xrightarrow{\sim} W_{x'}$



# Puzzle Solution

Cocycle for *level* $\lambda \in H^4(BG;\mathbb{Z})$ ... finite Chern-Simons theory

$$\mathscr{G}_G(X) = \sum_{[P]\in\pi_0 \operatorname{Bun}_G(X)} \frac{\lambda(P)}{\#\operatorname{Aut} P}$$

Dijkgraaf-Witten

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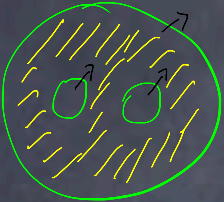
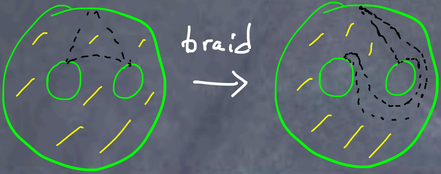


$\otimes$ structure



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**Solution:** classical Chern-Simons $\xrightarrow{\text{quantize on } S^1}$ Hopf algebra

**Quantum Field Theory and the Jones Polynomial** ⋆

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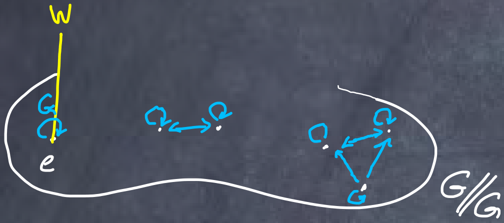
*Wilson loops*: $\mathrm{Rep}(G) \approx$ full subcategory of $\mathrm{Vect}_G(G)$ with support at $e \in G$. Classical expression using holonomy with character $\chi$:

$$\mathcal{G}_G\big(X; (S,\chi)_W\big) = \sum_{[P] \in \pi_0 \, \mathrm{Bun}_G(X)} \frac{h_{S,\chi}(P)}{\#\,\mathrm{Aut}\, P}.$$



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*'t Hooft loops*: Full subcategory of $\mathrm{Vect}_G(G)$ in which centralizers $Z_x$ act trivially on fiber at $x \in G$. Classical model sums bundles on $X \setminus S$ with specified holonomy about $S$.

$$\mathcal{G}_G\colon\ \mathrm{Bord}_{\langle 1,2,3\rangle} \longrightarrow \mathrm{Cat}$$

2-categories

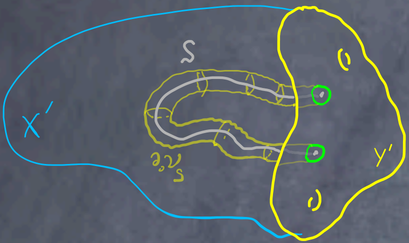
$$\mathscr{G}_G\colon\ \mathrm{Bord}_{\langle 1,2,3\rangle} \longrightarrow \mathrm{Cat}$$

If $\partial X \neq \emptyset$ there are line operators for neat 1d submanifolds $S \subset X$. Evaluate by cutting out tubular neighborhood $\nu_S$.

$$\begin{tikzcd} S^1 \amalg S^1 \arrow[rr, bend left, "Y'"] \arrow[rr, bend right, "\partial_0 \nu_S"'] \arrow[rr, phantom, "\Downarrow X'"] & & \emptyset^1 \end{tikzcd}$$

$$X' = X \setminus \nu_S$$
$$Y' = X' \cap \partial X$$

Can evaluate explicitly on Wilson (parallel transport) and 't Hooft



# Full Locality

$$\mathscr{G}_G : \ \mathrm{Bord}_{\langle 0,1,2,3\rangle} \longrightarrow ???$$

**Open Question:** Suitable codomain for general extended field theory?

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$$\mathscr{G}_G : \mathrm{Bord}_3 \longrightarrow \mathrm{TensCat}$$

General theory: Etingof-Gelaki-Nikshych-Ostrik
3-categorical aspects: Douglas-{Schommer-Pries}-Snyder

# Full Locality

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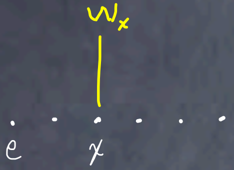
**Open Question:** Suitable codomain for general extended field theory?

**Finite gauge theory:** TensCat (complex linear tensor categories)

$$\mathscr{G}_G\colon\ \mathrm{Bord}_3 \longrightarrow \mathrm{TensCat}$$

Quantize (finite path integral) $\mathrm{Bun}_G(\mathrm{pt}) \simeq \mathrm{pt}\,/\!/G$ to compute

$$\mathscr{G}_G(\mathrm{pt}) = \mathrm{Vect}[G]$$




$$(W' * W'')_x = \bigoplus_{x'x''=x} W'_{x'} \otimes W''_{x''}$$

**Cobordism Hypothesis:** Evaluation on a point is an equivalence

$$\mathrm{TFT}(\mathcal{C}) \longrightarrow \left[(\mathcal{C}^{\mathrm{fd}})^{\sim}\right]^{O_3}$$

$$\mathscr{F} \longmapsto \mathscr{F}(\mathrm{pt})$$

Baez-Dolan conjecture, Hopkins-Lurie in 2d, Lurie in general

**On the Classification of Topological Field Theories**

Jacob Lurie


Our goal in this article is to give an expository account of some recent work on the classification of topological field theories. More specifically, we will outline the proof of a version of the *cobordism hypothesis* conjectured by Baez and Dolan in [**2**].


Contents

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Baez-Dolan conjecture, Hopkins-Lurie in 2d, Lurie in general

**Warning:** Need $O_3$-invariance data for unoriented theories

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No classical model in general, Turaev-Viro state sum

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Pontrjagin dual group

$A$ finite abelian: $\mathrm{Rep}(A) \simeq \mathrm{Vect}[A^\vee] \Longrightarrow \mathscr{R}_G \simeq \mathscr{G}_{A^\vee}$

## (Nonabelian) Electromagnetic Duality

**Theorem:** There is a Morita equivalence $\mathrm{Vect}[G] \simeq \mathrm{Rep}(G)$, and iso

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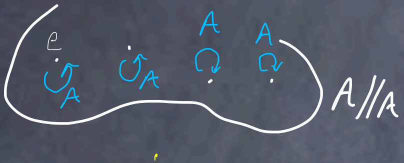
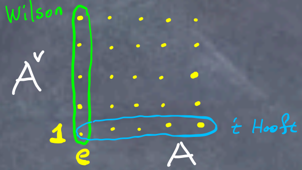
Line operators: $\mathscr{G}_A(S^1) = \mathrm{Vect}_A(A) \approx \mathrm{Vect}(A \times A^\vee)$ duality $A \longleftrightarrow A^\vee$



Extended field theory ideas appear in many places in geometry, topology, and quantum field theory

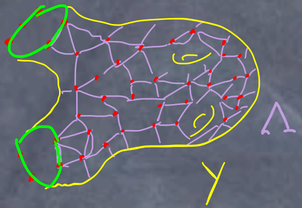
Now onto a new application to lattice models (w/Constantin Teleman)

## Latticed 1- and 2-manifolds

**Definition:**

**(i)** A *latticed 1-manifold* $(S, \Pi)$ is a closed 1-manifold $S$ equipped with a finite subset; $\Pi \subset S$ is an embedded graph, each component of which is a polygon with $\geq 2$ sides.

**(ii)** A *latticed 2-manifold* $(Y, \Lambda)$ is a compact 2-manifold $Y$ equipped with a smoothly embedded finite graph $\Lambda \subset Y$ such that the closure of each face (component of $Y \setminus \Lambda$) is a smoothly embedded solid $n$-gon with $n \geq 2$. Furthermore, if $e$ is an edge of $\Lambda$, then either (a) $e \cap \partial Y = \emptyset$, (b) $e \cap \partial Y$ is a single boundary vertex of $e$, or (c) $e \subset \partial Y$ .

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- No choice of embedding of $n$-gons
- Loops are disallowed by the conditions
- Faces may share multiple edges

# Ising model

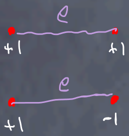
$A = \mu_2 = \{\pm 1\}$ abelian group of “spins”

$\beta \in \mathbb{R}^{>0}$ inverse temperature

$$\begin{aligned} \theta_\beta \colon A &\longrightarrow \mathbb{R}^{\geq 0} \\ \pm 1 &\longmapsto e^{\pm\beta} \end{aligned}$$ weight function

$A^{\mathrm{Vert}(\Lambda)} = \mathrm{Map}\big(\mathrm{Vert}(\Lambda), A\big)$ configuration space of spins

$g \colon A^{\mathrm{Vert}(\Lambda)} \times \mathrm{Edge}(\Lambda) \to A$ ratio of boundary spins




$g = +1$

$g = -1$

# Ising model

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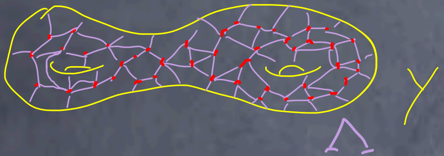


$Y$ closed:

$$I(Y, \Lambda) = \sum_{s \in A^{\mathrm{Vert}(\Lambda)}} \prod_{e \in \mathrm{Edge}(\Lambda)} \theta_\beta\big(g(s; e)\big)$$

This is the Ising *partition function*. Note limits $\beta \to \infty$, $\beta \to 0$.

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The model can be defined for more general data:

$G$ finite group

$\theta \colon G \longrightarrow \mathbb{R}^{\geq 0}$ admissible function

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**Definition:** Let $G$ be a finite group. A function $\theta \colon G \to \mathbb{R}$ is *admissible* if (i) $\theta(g) \geq 0$ for all $g \in G$; (ii) $\theta(g^{-1}) = \theta(g)$ for all $g \in G$; and (iii) $\theta^\vee(\rho)$ is a nonnegative operator for each irreducible unitary representation $\rho \colon G \to \mathrm{Aut}(W)$.

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The group $G$ acts by constant translation on $A^{\mathrm{Vert}(\Lambda)}$

$$s^h(v) = hs(v), \qquad s \in A^{\mathrm{Vert}(\Lambda)}, \quad h \in G, \quad v \in \mathrm{Vert}(\Lambda)$$

preserving the function $g$, so as a *symmetry* of the Ising model

Probabilistic interpretation:

$$\delta_s = \frac{\prod_{e \in \mathrm{Edge}(\Lambda)} \theta_\beta\big(g(s;e)\big)}{I(Y,\Lambda)}$$

is a probability measure on $A^{\mathrm{Vert}(\Lambda)}$

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Expectation value of a function

$$f\colon A^{\mathrm{Vert}(\Lambda)} \longrightarrow \mathbb{C}$$

such as $f(s) = s(v_1)s(v_2)$ for vertices $v_1, v_2$ (*order operator*):

$$\langle f \rangle = \sum_{s \in A^{\mathrm{Vert}(\Lambda)}} f(s)\delta_s$$

Quantum mechanical interpretation (Wick-rotated time):

Construct a functor

$$I\colon \ \mathrm{Bord}^{\mathrm{latticed}}_{\langle 1,2\rangle} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

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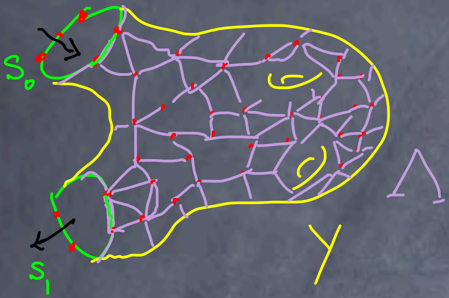
**Objects:** closed latticed 1-manifold $(S,\Pi)$ maps to the vector space

$$I(S,\Pi) = \mathrm{Fun}(A^{\mathrm{Vert}\,(\Pi)}) = \mathrm{Map}(A^{\mathrm{Vert}\,(\Pi)},\mathbb{C})$$

**Morphisms:** 2d latticed bordism $(Y, \Lambda)\colon (S_0, \Pi_0) \to (S_1, \Pi_1)$ gives a correspondence diagram of spin configuration spaces

$$\begin{tikzcd} & A^{\mathrm{Vert}(\Lambda)} \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\ A^{\mathrm{Vert}(\Pi_0)} & & A^{\mathrm{Vert}(\Pi_1)} \end{tikzcd}$$



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Define the linear map by push-pull

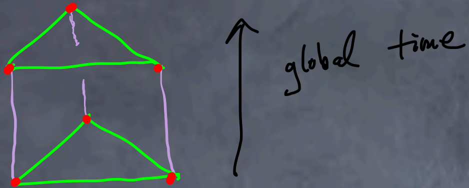
$$I(Y,\Lambda) = (r_1)_* \circ K \circ (r_0)^*\colon I(S_0,\Pi_0) \longrightarrow I(S_1,\Pi_1)$$

where the "integral kernel" $K$ is the weight function

$$K(s) = \prod_e \theta_\beta\big(g(s;e)\big), \qquad e \text{ incoming or interior}$$

Wick-rotated discrete time evolution via product bordism ("prism")

$$(Y, \Lambda) = [0,1] \times (S, \Pi)$$



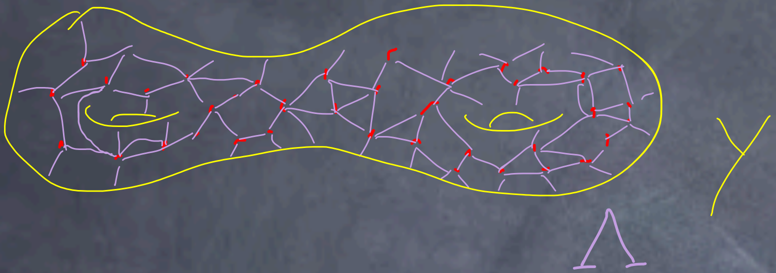

The resulting endomorphism of $I(S, \Pi)$ is called the *transfer matrix*. We write it as $e^{-H}$, where $H$ is the *Hamiltonian*. Eigenvalues of $H$ are energies (possibly infinite).

# Fourier-Kramers-Wannier Duality

$(Y, \Lambda)$ closed latticed surface

$$C^0(\Lambda; A) = A^{\mathrm{Vert}(\Lambda)}$$
$$C^1(\Lambda; A) = A^{\mathrm{Edge}(\Lambda)}$$

$$\begin{array}{ccc} C^0(\Lambda; A) & \xrightarrow{\delta} & C^1(\Lambda; A) \\ s & \longmapsto & \big(e \mapsto g(s; e)\big) \end{array}$$



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$$\delta_* 1 = \# H^0(\Lambda; A)\, \Delta_{B^1(\Lambda; A)}$$

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Ising partition function as inner product of functions on $C^1(\Lambda; A)$:

$$I(Y, \Lambda) = \frac{1}{\# H^0(\Lambda; A)} \langle \Theta, \delta_* 1 \rangle = c\, \langle \Theta, \Delta_{B^1} \rangle$$

Pontrjagin dual groups and maps:

$$C^0(\Lambda; A) \xrightarrow{\delta} C^1(\Lambda; A)$$

$$C_0(\Lambda; A^\vee) \xleftarrow{\partial} C_1(\Lambda; A^\vee)$$

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For $A = \mu_2$, Fourier transform: $\beta \leftrightarrow \beta^\vee$ where $\sinh(2\beta)\sinh(2\beta^\vee) = 1$

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**Key Idea:** Use the full strength of the *symmetry* group $G$

Settles these issues and much more:

- prediction for low energy behavior (all $G$)

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# Fibering over $BG$

If a group $G$ acts as a symmetry on mathematical object $M$ (condition), we can try to extend (data) to a fibering

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Equivariance $\longrightarrow$ Families

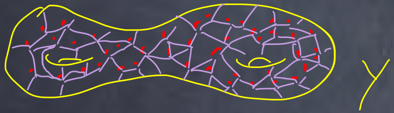
## ‘Fibering over $BG$’ in Ising Model

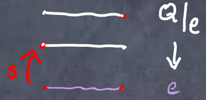
$G$-Ising model on $Y^2$: *background* lattice $\Lambda \subset Y$ and $G$-bundle $Q \to Y$
*fluctuating* field a “discrete gauged $\sigma$-model”

$$Q^{\mathrm{Vert}(\Lambda)} = \text{sections of } Q \to Y \text{ over } \mathrm{Vert}(\Lambda)$$

The ratio of spins defined via parallel transport

$$g \colon Q^{\mathrm{Vert}(\Lambda)} \times \mathrm{Edge}(\Lambda) \longrightarrow G$$



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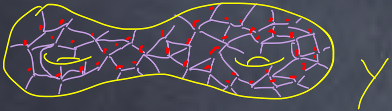




The partition function of $I = I_{(G,\theta)}$ is now a function of a $G$-bundle:

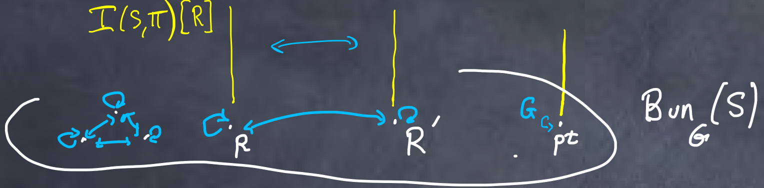
$$I(Y, \Lambda)\colon\ \mathrm{Bun}_G(Y) \longrightarrow \mathbb{C}$$

The old partition function is the value at the trivial bundle

To a latticed 1-manifold $(S, \Pi)$ we obtain a vector bundle

$$I(S, \Pi) \longrightarrow \mathrm{Bun}_G(S)$$

These are "twisted sectors"; the old state space is the fiber at $\mathrm{pt} \in BG$



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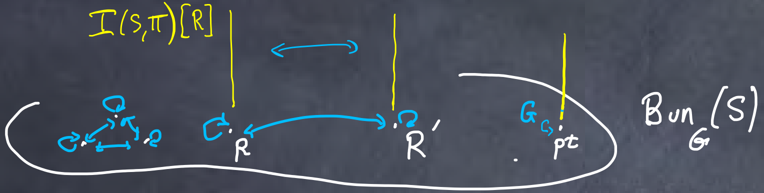


**Upshot:** $I$ is a *boundary theory* for $\mathscr{G}_G$:

$$I(Y,\Lambda) \in \mathscr{G}_G(Y)$$
$$I(S,\Pi) \in \mathscr{G}_G(S)$$

We learned recently that we were anticipated by Pavol Ševera (2002) in some of our pictures of the Ising model and topological field theory, though he works in a non-extended context: `arXiv:hep-th/0206162`

## Boundary theories

**Definition:** A *topological boundary theory* for $\mathscr{G}_G\colon \mathrm{Bord}_3 \to \mathrm{TensCat}$ is

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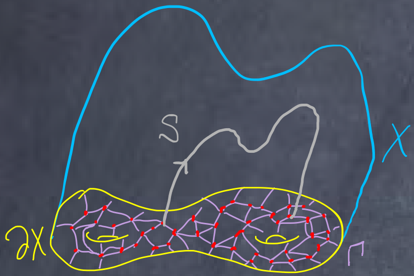
For $(Y,\Lambda)$ closed obtain a function on $\mathrm{Bun}_G(Y)$:


$$I(Y,\Lambda)[Q] = \sum_{s \in Q^{\mathrm{Vert}(\Lambda)}} \prod_{e \in \mathrm{Edge}(\Lambda)} \theta\big(g(s;e)\big)$$

Line operators for neat 1d submanifolds $S \subset X^3$ with $\partial S \subset (\partial X, \Gamma)$

**Wilson/order operators:** $\chi\colon G \to \mathbb{T}$ character, $S$ ends at vertices

$$(F, I)(X, \Gamma) = \sum_{[P] \in \pi_0 \operatorname{Bun}_G(X)} \frac{1}{\#\operatorname{Aut} P} \sum_{s \in \mathcal{S}_{(\partial X, \Gamma)}[\partial P]} h_{S,\chi}(P, s) \prod_{e \in \operatorname{Edges}(\Gamma)} \theta\big(g(s; e)\big)$$



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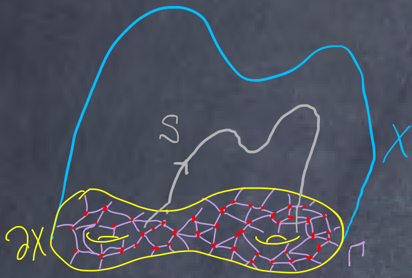
**'t Hooft/disorder operators:** conjugacy class in $G$, $S$ ends in faces



## Revisit **problems**

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## Low energy behavior; phase diagram

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| $\mathcal{M}$ | moduli space of quantum theories |
| $\Delta \subset \mathcal{M}$ | locus of phase transitions |
| $(\mathcal{M} \setminus \Delta)_{\text{gapped}} \subset \mathcal{M} \setminus \Delta$ | systems with spectral gap |
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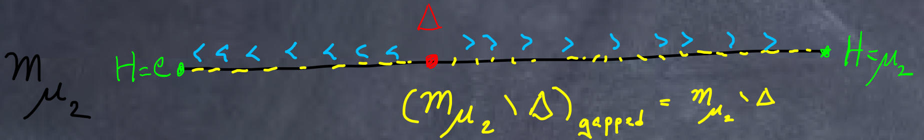
RG flow



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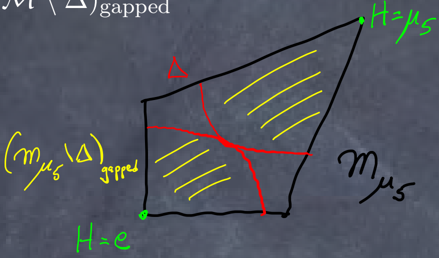

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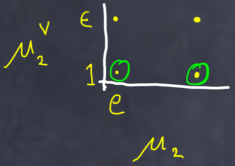


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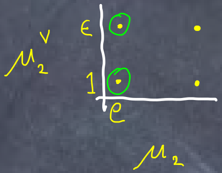


paramagnetic ($\beta \to 0$)

ferromagnetic ($\beta \to \infty$)

EM duality

$(G = \mu_2)$

## Topological construction; general theories

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Quartet of data:

| | |
|---|---|
| $\mathcal{T} = \mathrm{Vect}[G]$ | categorified group algebra |
| $\mathcal{B}_1 = \mathrm{Vect}(G)$ | Neumann boundary theory |
| $\mathcal{B}_2 = \mathrm{Vect}$ | Dirichlet boundary theory |
| $\delta$ | generator of $\mathrm{Hom}_{\mathcal{T}}(\mathcal{B}_1, \mathcal{B}_2)$ |

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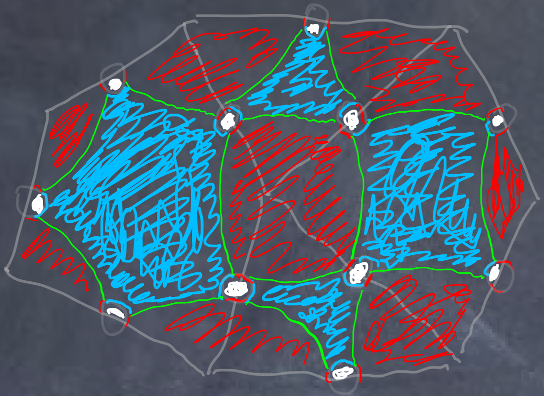

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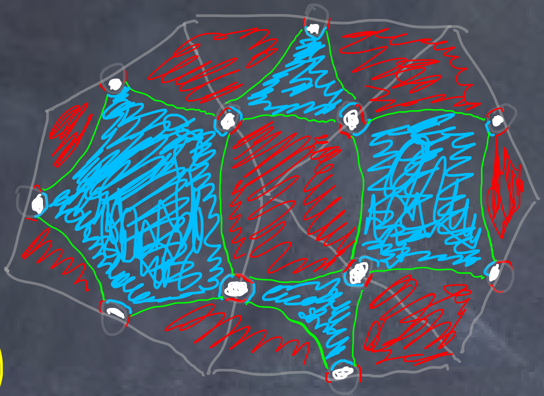

$$F_G(\bigcirc) = \mathrm{Fun}(G)$$

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More general theories: spherical fusion category and fiber functor
= Frobenius-Hopf algebra

Morita equivalence:

$$\begin{aligned} \mathrm{Vect}[G] &\longleftrightarrow \mathrm{Rep}(G) \qquad &\text{(tensor categories)} \\ \mathrm{Vect}(G/H) &\longleftrightarrow \mathrm{Rep}(H) \qquad &\text{(left modules)} \end{aligned}$$

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Cobordism hypothesis $\Longrightarrow$ Electromagnetic/Kramers-Wannier duality

**Theorem:** On oriented manifolds there is an equivalence of $G$-gauge theory and the Turaev-Viro $\mathrm{Rep}(G)$ theory which exchanges their lattice boundary theories, and exchanges Wilson/Order and 't Hooft/Disorder operators.

## Abelian duality in higher dimensions

$S$ pointed space, finite homotopy type

$\mathcal{F}_X$ $\mathrm{Map}(X_+, S)$

$n$-dimensional theory $F_S$ (finite path integral) with partition function

$$F_S(X) = \sum_{[\varphi]\in\pi_0\mathcal{F}_X} \frac{1}{\#\pi_1(\mathcal{F}_X,\varphi)}\,\frac{\#\pi_2(\mathcal{F}_X,\varphi)}{\#\pi_3(\mathcal{F}_X,\varphi)}\cdots$$

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$S$ pointed space, finite homotopy type

$\mathcal{F}_X$ $\quad \mathrm{Map}(X_+, S)$

$n$-dimensional theory $F_S$ (finite path integral) with partition function

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Canonical Dirichlet and Neumann boundary theories

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\end{tikzcd}$$

[unique domain wall]

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If $S$ is an $\infty$ loop space, the 0-space of a spectrum $\mathcal{T}$, then there is a (Pontrjagin) dual spectrum $\mathcal{T}^\vee$. Electromagnetic duality:

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The abelian Ising story is $n = 3$ and $\mathcal{T} = \Sigma HA$.