Extended topological field theory and the 2-dimensional Ising model

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Topological Quantum Field Theory

Edward Witten*
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Abstract. A twisted version of four dimensional supersymmetric gauge theory is formulated. The model, which refines a nonrelativistic treatment by Atiyah, appears to underlie many recent developments in topology of low dimensional manifolds; the Donaldson polynomial invariants of four manifolds and the Floer groups of three manifolds appear naturally. The model may also be interesting from a physical viewpoint; it is in a sense a generally covariant quantum field theory, albeit one in which general covariance is unbroken, there are no gravitons, and the only excitations are topological.

1. Introduction

One of the dramatic developments in mathematics in recent years has been the program initiated by Donaldson of studying the topology of low dimensional manifolds via nonlinear classical field theory [1, 2]. Donaldson's work uses heavily the self-dual Yang-Mills equations, which were first introduced by
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1. Introduction

One of the dramatic developments in mathematics in recent years has been the program of studying the topology of low dimensional manifolds; the Donaldson polynomial invariants of four manifolds and the Floer groups of three manifolds appear naturally. This picture has changed considerably because of the work of Floer on three manifolds. In this viewpoint, Floer theory can be seen as a generalization to infinite dimensional function space of the supersymmetric approach to Morse theory [9].

Topological Sigma Models

Edward Witten*
School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. A variant of the usual supersymmetric nonlinear sigma model is described, governing maps from a Riemann surface \( \Sigma \) to an arbitrary almost complex manifold \( M \). It possesses a fermionic BRST-like symmetry, conserved for arbitrary \( \Sigma \), and obeying \( Q^2 = 0 \). In a suitable version, the quantum ground states are the \( 1 + 1 \) dimensional Floer groups. The correlation functions of the BRST-invariant operators are invariants (depending only on the homotopy type of the almost complex structure of \( M \)) similar to those that have entered in recent work of Gromov on symplectic geometry. The model can be coupled to dynamical gravitational or gauge fields while preserving the fermionic symmetry; some observations by Atiyah suggest that the latter coupling may be related to the Jones polynomial of knot theory. From the point of view of string theory, the main novelty of this type of sigma model is that the graviton vertex operator is a BRST commutator. Thus, models of this type may correspond to a realization at the level of string theory of an unbroken phase of quantum gravity.

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One of the dramatic developments in mathematics in recent years has been the program initiated by Donaldson of studying the topology of low dimensional smooth four manifolds (as well as hard analysis of instanton moduli spaces related to physical ideas in an intimate way. However, such a relation has not been understood in a physically meaningful way.


topological

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Characteristic forms and geometric invariants

By Shing-shen Chern and James Simons*

1. Introduction

This work, originally announced in [4], grew out of an attempt to derive a purely combinatorial formula for the first Pontrjagin number of a 4-manifold. The hope was that by integrating the characteristic curvature form (with respect to some Riemannian metric) simplex by simplex, and replacing the integral over each interior by another on the boundary, one

\[ CS(A) = \int_M \langle A \wedge dA \rangle - \frac{1}{6} \langle A \wedge [A \wedge A] \rangle \pmod{1} \]
Two very distinct starting points—relationship between them?

Invariants of Knots and 3-Manifolds: A Puzzle

Ribbon Graphs and Their Invariants Derived from Quantum Groups

N. Yu. Reshetikhin and V. G. Turaev
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Abstract. The generalization of Jones polynomial of links to the case of graphs in \( R^3 \) is presented. It is constructed as the functor from the category of graphs to the category of representations of the quantum group.

1. Introduction

The present paper is intended to generalize the Jones polynomial of links and the related Jones-Conway and Kauffman polynomials to the case of graphs in \( R^3 \).

Originally the Jones polynomial was defined for links of circles in \( R^3 \) via an astonishing use of von Neumann algebras (see [Jo]). Later on it was understood that this and related polynomials may be constructed using the quantum \( R \)-matrices (see, for instance, [Tu,Jo]). This approach enables one to construct similar invariants for coloured links, i.e. links each of whose components is provided with a module over a fixed algebra (see [Re1]), where the role of the algebra is played by the quantized universal enveloping algebra \( U_q(G) \) of a semisimple Lie algebra \( G \).

\[
XY - YX = \frac{K^{r^2} - K^{-r^2}}{t^2 - t^{-2}}
\]

\[
XX = t^{-2} KX, \quad YK = t^2 KY, \quad KK^{-1} = K^{-1}K = 1
\]

\[
K^{4r} = 1, \quad X^r = Y^r = 0
\]

Invariants of 3-manifolds via link polynomials and quantum groups

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Oblatum 20-XII-1989 & 7-VII-1990

1. Introduction

The aim of this paper is to construct new topological invariants of compact oriented 3-manifolds and of framed links in such manifolds. Our invariant of (a link in) a closed oriented 3-manifold is a sequence of complex numbers parametrized by complex roots of 1. For a framed link in \( S^3 \) the terms of the sequence are equal to the values of the (suitably parametrized) Jones polynomial of
The definition of conformal field theory

Graeme Segal

I shall propose a definition of 2-dimensional conformal field theory which I believe is equivalent to that used by Fuchs et al. The idea arises from joint work with Quinn.

§1 The definition

The category $\mathcal{C}$ is defined as follows. There is a sequence of objects $\{C_n\}_{n \geq 0}$, where $C_n$ is the disjoint union of a set of $n$
1988–90: New Avenues into QFT (Axioms)

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TOPOLOGICAL QUANTUM FIELD THEORIES
by Michael Atiyah

To René Thom on his 65th birthday.

1. Introduction

In recent years there has been a remarkable renaissance in the relation between Geometry and Physics. This relation involves the most advanced and sophisticated ideas on each side and appears to be extremely deep. The traditional links between the two subjects, as embodied for example in Einstein’s Theory of General Relativity or in Maxwell’s Equations for Electro-Magnetism are concerned essentially with classical fields of force, governed by differential equations, and their geometrical interpretation. The new feature of present developments is that links are being established between quantum physics and topology. It is no longer the purely local aspects that are involved but their global counterparts. In a very general sense this should not be too surprising. Both quantum theory and topology are characterized by discrete phenomena emerging from a continuous background. However, the realization that this vague philosophical viewpoint could be translated into reasonably precise and significant mathematical statements is mainly due to the efforts of Edward Witten who, in a variety of directions, has shown the insight that can be derived by examining the topological aspects of quantum field theories.

The best starting point is undoubtedly Witten’s paper [11] where he explained...

It will be clear how much this whole subject rests on the ideas of Witten. In formulating the axiomatic framework in § 2, I have also been following Graeme Segal who produced a very similar approach to conformal field theories [10]. Finally it seems appropriate to point out the major role that cobordism plays in these theories. Thus René Thom’s most celebrated contribution to geometry has now a new and deeper relevance.

We come now to the promised axioms. A topological quantum field theory (QFT), in dimension $d$ defined over a ground ring $A$, consists of the following data:

(A) A finitely generated $A$-module $Z(\Sigma)$ associated to each oriented closed smooth $d$-dimensional manifold $\Sigma$,

(B) An element $Z(M) \in Z(\partial M)$ associated to each oriented smooth $(d + 1)$-dimensional manifold (with boundary) $M$.

These data are subject to the following axioms, which we state briefly and expand upon below:

1. $Z$ is functorial with respect to orientation preserving diffeomorphisms of $\Sigma$ and $M$,

2. $Z$ is involutory, i.e. $Z(\Sigma^*) = Z(\Sigma)^*$ where $\Sigma^*$ is $\Sigma$ with opposite orientation and $Z(\Sigma)^*$ denotes the dual module (see below),

3. $Z$ is multiplicative.

We now elaborate on the precise meaning of the axioms. (1) means first that an orientation preserving diffeomorphism $f: \Sigma \rightarrow \Sigma'$ induces an isomorphism
3-Dimensional Finite Gauge Theory

$G$ 
finite group

$\text{Bord}_{(2,3)}$ 
(unoriented) bordism category

$\text{Vect}$ 
category of complex vector spaces

$\mathcal{C}_G : \text{Bord}_{(2,3)} \rightarrow \text{Vect}$ 
symmetric monoidal functor

finite path integral: 
$(P!M)_{2\Bun G(M)}$ 
fluctuating field

For $X^3$ closed sum the constant function 1: 
$G_{X} = \prod_{\alpha=0}^{\infty} \Bun G(X) = \# \text{Aut} \mathcal{P}$.

For $Y^2$ closed sum (= (co)limit) the constant $\text{Vect}$-valued function 
$G_Y = \text{Fun}_{\text{Bun} G(Y)}$.
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For \(X^3\) closed sum the constant function 1:

\[
\mathcal{G}_G(X) = \sum_{[P] \in \pi_0 \text{Bun}_G(X)} \frac{1}{\# \text{Aut } P}
\]
3-Dimensional Finite Gauge Theory

$G$ \hspace{1cm} \text{finite group}

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\textbf{Finite path integral:} $(P \rightarrow M) \in \text{Bun}_G(M)$ \hspace{1cm} \text{fluctuating field}

For $X^3$ closed sum the constant function 1:

$$\mathcal{G}_G(X) = \sum_{[P] \in \pi_0 \text{Bun}_G(X)} \frac{1}{\# \text{Aut} P}$$

For $Y^2$ closed have canonical quantization:

$$\mathcal{G}_G(Y) = \text{Fun}(\text{Bun}_G(Y))$$
Quantize (linearize) by push-pull:

\[ \mathcal{G}_G(X) = (r_1)_* \circ (r_0)^* : \mathcal{G}_G(Y_0) \to \mathcal{G}_G(Y_1) \]
$G_G(Y)$ is a finite path integral...of the constant function $C$:

$$G_G(Y) = \int_{\text{Bun}_G(Y)} C \simeq \bigoplus_{[P] \in \pi_0 \text{Bun}_G(Y)} C$$
$\mathcal{G}_G(Y)$ is a finite path integral... of the constant function $C$:

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\mathcal{G}_G(Y) \sim = \int_{\text{Bun}_G(Y)} C \cong \bigoplus_{[P] \in \pi_0 \text{Bun}_G(Y)} C
$$

Extend to lower dimensions: $\text{Bun}_G(S^1) \sim G//G$

$$
\mathcal{G}_G(S^1) \sim = \int_{\text{Bun}_G(Y)} \text{Vect}
$$

$G = \text{Sym}_3$
Extended Locality

\( \mathcal{G}_G(Y) \) is a finite path integral... of the constant function \( C \):

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\( \mathcal{G}_G(S^1) = \text{Vect}_G(G) \)
Puzzle Solution

Cocycle for level $\lambda \in H^4(BG; \mathbb{Z})$ ... finite Chern-Simons theory

$$G_G(X) = \sum_{[P] \in \pi_0 \text{Bun}_G(X)} \frac{\lambda(P)}{\# \text{Aut } P}$$

$$G_G(Y) = \Gamma(L \to \text{Bun}_G(Y))$$

$$G_G(S^1) = \text{Vect}^\lambda_G(G)$$

Diagram:

- $L_{\xi'}^G \bigotimes W_{\bar{\xi}} \xrightarrow{\simeq} W_{\xi'}$
- $G/G$

Note: The diagram includes various arrows and symbols, indicating transformations and relationships within the context of the mathematical solution.
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Solution: classical Chern-Simons $\xrightarrow{\text{quantize on } S^1} \text{Hopf algebra}$

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Line operators

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\( S \subset X \) coframed 1d submanifold of \( X^3 \) closed

Link \( S^1 \) used to label \( S \) by objects of \( \mathcal{G}_G(S^1) = \text{Vect}_G(G) \)
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**Wilson loops**: \( \text{Rep}(G) \approx \) full subcategory of \( \text{Vect}_G(G) \) with support at \( e \in G \). Classical expression using holonomy with character \( \chi \):

\[
\mathcal{G}_G(X; (S, \chi)_W) = \sum_{[P] \in \pi_0 \text{Bun}_G(X)} \frac{h_{S,\chi}(P)}{\# \text{Aut } P}.
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‘t Hooft loops: Full subcategory of \( \text{Vect}_G(G) \) in which centralizers \( Z_x \) act trivially on fiber at \( x \in G \). Classical model sums bundles on \( X \setminus S \) with specified holonomy about \( S \).
\[ \mathcal{G}_G : \text{Bord}_{1,2,3} \to \text{Cat} \]

2-categories
\( \mathcal{G}_G : \text{Bord}_{\langle 1,2,3 \rangle} \rightarrow \text{Cat} \)

If \( \partial X \neq \emptyset \) there are line operators for neat 1d submanifolds \( S \subset X \). Evaluate by cutting out tubular neighborhood \( \nu_S \).

\[
\begin{align*}
S^1 \sqcup S^1 \xrightarrow{Y'} \downarrow X' \xrightarrow{\partial_0 \nu_S} \emptyset^1 \quad & \quad X' = X \setminus \nu_S \\
& \quad Y' = X' \cap \partial X
\end{align*}
\]

Can evaluate explicitly on Wilson (parallel transport) and ’t Hooft
Full Locality

\[ \mathcal{G}_G : \text{Bord}_{(0,1,2,3)} \rightarrow ??? \]

**Open Question:** Suitable codomain for general extended field theory?
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**Finite gauge theory:** \( \text{TensCat} \) (complex linear tensor categories)

\[ \mathcal{G}_G : \text{Bord}_3 \rightarrow \text{TensCat} \]
Full Locality

$$\mathcal{G}_G : \text{Bord}_{\langle 0,1,2,3 \rangle} \longrightarrow ???$$

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Finite gauge theory: TensCat (complex linear tensor categories)

$$\mathcal{G}_G : \text{Bord}_3 \longrightarrow \text{TensCat}$$

General theory: Etingof-Gelaki-Nikshych-Ostrik
3-categorical aspects: Douglas-{Schommer-Pries}-Snyder
**Full Locality**

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**Finite gauge theory:** \( \text{TensCat} \) (complex linear tensor categories)

\[ \mathcal{G}_G : \text{Bord}_3 \rightarrow \text{TensCat} \]

Quantize (finite path integral) \( \text{Bun}_G(pt) \simeq pt \sslash G \) to compute

\[ \mathcal{G}_G(pt) = \text{Vect}[G] \]
**Cobordism Hypothesis:** Evaluation on a point is an equivalence

\[ \text{TFT}(\mathcal{C}) \longrightarrow \left[ (\mathcal{C}^{\text{fd}})^{\sim} \right]^{O_3} \]

\[ \mathcal{F} \longmapsto \mathcal{F}(\text{pt}) \]

Baez-Dolan conjecture, Hopkins-Lurie in 2d, Lurie in general
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**Warning:** Need $O_3$-invariance data for unoriented theories
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New approaches by Ayala-Francis and others
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\]

\[
\mathcal{F} \longrightarrow \mathcal{F}(\text{pt})
\]

**Baez-Dolan conjecture, Hopkins-Lurie in 2d, Lurie in general**

New approaches by Ayala-Francis and others

**Construct** a theory \( \mathcal{R}_G : \text{Bord}_3 \rightarrow \text{TensCat} \) with

\[
\mathcal{R}_G(\text{pt}) = \text{Rep}(G)
\]

No classical model in general, **Turaev-Viro** state sum
Cobordism Hypothesis: Evaluation on a point is an equivalence

\[ \text{TFT}(C) \longrightarrow \left( (C^{\text{fd}})_{\sim} \right)^{O_3} \]

\[ \mathcal{F} \longmapsto \mathcal{F}(\text{pt}) \]

Baez-Dolan conjecture, Hopkins-Lurie in 2d, Lurie in general

New approaches by Ayala-Francis and others

Construct a theory \( \mathcal{R}_G : \text{Bord}_3 \rightarrow \text{TensCat} \) with

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No classical model in general, Turaev-Viro state sum

A finite abelian:

\[ \text{Rep}(A) \simeq \text{Vect}(A^\vee) \]

\[ \mathcal{R}_G \simeq G_{A^\vee} \]
Theorem: There is a Morita equivalence $\text{Vect}[G] \simeq \text{Rep}(G)$, and iso

$$\mathcal{F}: \mathcal{G}_G \xrightarrow{\simeq} \mathcal{R}_G$$

of field theories of oriented manifolds
(Nonabelian) Electromagnetic Duality

**Theorem:** There is a Morita equivalence $\text{Vect}[G] \simeq \text{Rep}(G)$, and iso

$$\mathcal{F}: \mathcal{G}_G \xrightarrow{\sim} \mathcal{R}_G$$

of field theories of *oriented* manifolds

$$\mathcal{F}: \mathcal{H}_A \xrightarrow{\sim} \mathcal{H}_A\mathbb{V}$$ is Fourier transform on states for $Y^2$ closed oriented:

$$\mathcal{F}: \text{Fun}(H^1(Y; A)) \xrightarrow{\sim} \text{Fun}(H^1(Y; A\mathbb{V}))$$
(Nonabelian) Electromagnetic Duality

**Theorem:** There is a Morita equivalence \( \text{Vect}[G] \cong \text{Rep}(G) \), and iso

\[
\mathcal{F}: \mathcal{G}_G \overset{\cong}{\longrightarrow} \mathcal{K}_G
\]

of field theories of *oriented* manifolds

\[
\mathcal{F}: \mathcal{F}_A \overset{\cong}{\longrightarrow} \mathcal{F}_{A^\vee}
\]

is Fourier transform on states for \( Y^2 \) closed oriented:

\[
\mathcal{F}: \text{Fun}(H^1(Y; A)) \overset{\cong}{\longrightarrow} \text{Fun}(H^1(Y; A^\vee))
\]

Line operators: \( \mathcal{G}_A(S^1) = \text{Vect}_A(A) \cong \text{Vect}(A \times A^\vee) \) duality \( A \leftrightarrow A^\vee \)
Extended field theory ideas appear in many places in geometry, topology, and quantum field theory.

Now onto a new application to lattice models (w/ Constantin Teleman)
Latticed 1- and 2-manifolds

Definition:

(i) A latticed 1-manifold $(S, \Pi)$ is a closed 1-manifold $S$ equipped with a finite subset; $\Pi \subset S$ is an embedded graph, each component of which is a polygon with $\geq 2$ sides.

(ii) A latticed 2-manifold $(Y, \Lambda)$ is a compact 2-manifold $Y$ equipped with a smoothly embedded finite graph $\Lambda \subset Y$ such that the closure of each face (component of $Y \setminus \Lambda$) is a smoothly embedded solid $n$-gon with $n \geq 2$. Furthermore, if $e$ is an edge of $\Lambda$, then either (a) $e \cap \partial Y = \emptyset$, (b) $e \cap \partial Y$ is a single boundary vertex of $e$, or (c) $e \subset \partial Y$. 
Latticed 1- and 2-manifolds

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(i) A latticed 1-manifold \((S, \Pi)\) is a closed 1-manifold \(S\) equipped with a finite subset; \(\Pi \subset S\) is an embedded graph, each component of which is a polygon with \(\geq 2\) sides.

(ii) A latticed 2-manifold \((Y, \Lambda)\) is a compact 2-manifold \(Y\) equipped with a smoothly embedded finite graph \(\Lambda \subset Y\) such that the closure of each face (component of \(Y \setminus \Lambda\)) is a smoothly embedded solid \(n\)-gon with \(n \geq 2\). Furthermore, if \(e\) is an edge of \(\Lambda\), then either (a) \(e \cap \partial Y = \emptyset\), (b) \(e \cap \partial Y\) is a single boundary vertex of \(e\), or (c) \(e \subset \partial Y\).

- No choice of embedding of \(n\)-gons
- Loops are disallowed by the conditions
- Faces may share multiple edges
**Ising model**

\[ A = \mu_2 = \{\pm 1\} \]
\[ \beta \in \mathbb{R}^{>0} \]
\[ \theta_\beta: A \rightarrow \mathbb{R}^{\geq 0} \]
\[ \pm 1 \mapsto e^{\pm \beta} \]

\[ A^{\text{Vert}(\Lambda)} = \text{Map}(\text{Vert}(\Lambda), A) \]

\[ g: A^{\text{Vert}(\Lambda)} \times \text{Edge}(\Lambda) \rightarrow A \]

abelian group of “spins”
inverse temperature
weight function
configuration space of spins
ratio of boundary spins

\[ g = +1 \]
\[ g = -1 \]
Ising model

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\[ I(Y, \Lambda) = \sum_{s \in A^{\text{Vert}(\Lambda)}} \prod_{e \in \text{Edge}(\Lambda)} \theta_\beta(g(s; e)) \]

This is the Ising partition function. Note limits \( \beta \rightarrow \infty, \beta \rightarrow 0 \).
Ising model

\[ A = \mathbb{Z}_2 = \{\pm 1\} \]
\[ \beta \in \mathbb{R}^>0 \]
\[ \theta_\beta : A \rightarrow \mathbb{R}^{\geq 0} \]
\[ \pm 1 \mapsto e^{\pm \beta} \]
\[ A^{\text{Vert}(\Lambda)} = \text{Map}(\text{Vert}(\Lambda), A) \]
\[ g : A^{\text{Vert}(\Lambda)} \times \text{Edge}(\Lambda) \rightarrow A \]

The model can be defined for more general data:

\[ G \]
\[ \theta : G \rightarrow \mathbb{R}^{\geq 0} \]

finite group

admissible function
Ising model

\[ A = \mu_2 = \{ \pm 1 \} \]

\[ \beta \in \mathbb{R}^{>0} \]

\[ \theta_\beta : A \longrightarrow \mathbb{R}^{\geq 0} \]

\[ \pm 1 \longmapsto e^{\pm \beta} \]

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\[ g : A^{\text{Vert}(\Lambda)} \times \text{Edge}(\Lambda) \rightarrow A \]

abelian group of “spins”
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**Definition:** Let \( G \) be a finite group. A function \( \theta : G \rightarrow \mathbb{R} \) is admissible if (i) \( \theta(g) \geq 0 \) for all \( g \in G \); (ii) \( \theta(g^{-1}) = \theta(g) \) for all \( g \in G \); and (iii) \( \theta^{\vee}(\rho) \) is a nonnegative operator for each irreducible unitary representation \( \rho : G \rightarrow \text{Aut}(W) \).
Ising model

\[ A = \mathbb{H}_2 = \{\pm1\} \]
\[ \beta \in \mathbb{R}^{>0} \]
\[ \theta_\beta : A \rightarrow \mathbb{R}^{\geq0} \]
\[ \pm 1 \mapsto e^{\pm\beta} \]
\[ A^{\text{Vert}(\Lambda)} = \text{Map}(\text{Vert}(\Lambda), A) \]
\[ g : A^{\text{Vert}(\Lambda)} \times \text{Edge}(\Lambda) \rightarrow A \]

abelian group of “spins”
inverse temperature
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The group \( G \) acts by constant translation on \( A^{\text{Vert}(\Lambda)} \)

\[ s^h(v) = hs(v), \quad s \in A^{\text{Vert}(\Lambda)}, \quad h \in G, \quad v \in \text{Vert}(\Lambda) \]

preserving the function \( g \), so as a symmetry of the Ising model
Probabilistic interpretation:

\[ \delta_s = \frac{\prod_{e \in \text{Edge}(\Lambda)} \theta_\beta(g(s; e))}{I(Y, \Lambda)} \]

is a probability measure on \( A^{\text{Vert}(\Lambda)} \)
Probabilistic interpretation:

\[ \delta_s = \prod_{e \in \text{Edge}(\Lambda)} \theta_{\beta}(g(s; e)) \]

\[ \frac{I(Y, \Lambda)}{I(Y, \Lambda)} \]

is a probability measure on \( A^{\text{Vert}(\Lambda)} \)

\( \beta \to 0 \) uniform measure paramagnetic

\( \beta \to \infty \) support at 2 points ferromagnetic
Probabilistic interpretation:

\[
\delta_s = \frac{\prod_{e \in \text{Edge}(\Lambda)} \theta_\beta(g(s; e))}{I(Y, \Lambda)}
\]

is a probability measure on \(A^{\text{Vert}(\Lambda)}\)

\[\beta \to 0 \quad \text{uniform measure} \quad \text{paramagnetic}\]

\[\beta \to \infty \quad \text{support at 2 points} \quad \text{ferromagnetic}\]

Expectation value of a function

\[f : A^{\text{Vert}(\Lambda)} \to \mathbb{C}\]

such as \(f(s) = s(v_1)s(v_2)\) for vertices \(v_1, v_2\) (order operator):

\[
\langle f \rangle = \sum_{s \in A^{\text{Vert}(\Lambda)}} f(s)\delta_s
\]
Quantum mechanical interpretation (Wick-rotated time):

Construct a functor

\[ I: \text{Bord}^{\text{latticed}}_{(1,2)} \rightarrow \text{Vect}_\mathbb{C} \]
Quantum mechanical interpretation (Wick-rotated time):

Construct a functor

\[ I : \text{Bord}^{\text{latticed}}_{(1,2)} \rightarrow \text{Vect}_\mathbb{C} \]

**Objects:** closed latticed 1-manifold \((S, \Pi)\) maps to the vector space

\[ I(S, \Pi) = \text{Fun}(A^{\text{Vert}(\Pi)}) = \text{Map}(A^{\text{Vert}(\Pi)}, \mathbb{C}) \]
**Morphisms:** 2d latticed bordism \((Y, \Lambda) : (S_0, \Pi_0) \to (S_1, \Pi_1)\) gives a correspondence diagram of spin configuration spaces

Define the linear map by push-pull

\[ I(Y, \Lambda) = (r_0) \Lambda K (r_1) : I(S_0, \Pi_0) \to I(S_1, \Pi_1) \]

where the "kernel" \(K\) is the weight function

\[ K(s) = Y e^\chi_g(s, e) \]

incoming or interior
**Morphisms:** 2d latticed bordism \((Y, \Lambda): (S_0, \Pi_0) \to (S_1, \Pi_1)\) gives a correspondence diagram of spin configuration spaces

\[
\begin{array}{ccc}
A_{\text{Vert}(\Lambda)} & \xrightarrow{r_1} & A_{\text{Vert}(\Pi_1)} \\
\downarrow & & \downarrow \\
A_{\text{Vert}(\Pi_0)} & \xleftarrow{r_0} & A_{\text{Vert}(\Pi_0)}
\end{array}
\]

Define the linear map by push-pull

\[I(Y, \Lambda) = (r_1)_* \circ K \circ (r_0)^* : I(S_0, \Pi_0) \longrightarrow I(S_1, \Pi_1)\]

where the "integral kernel" \(K\) is the weight function

\[K(s) = \prod_{e} \theta_\beta(g(s; e)), \quad e \text{ incoming or interior}\]
Wick-rotated discrete time evolution via product bordism (“prism”)

\[(Y, \Lambda) = [0, 1] \times (S, \Pi)\]

The resulting endomorphism of \(I(S, \Pi)\) is called the transfer matrix. We write it as \(e^{-H}\), where \(H\) is the Hamiltonian. Eigenvalues of \(H\) are energies (possibly infinite).
Fourier-Kramers-Wannier Duality

\((Y, \Lambda)\) closed latticed surface

\[
\begin{align*}
    C^0(\Lambda; A) &= A^{\text{Vert}(\Lambda)} \\
    C^1(\Lambda; A) &= A^{\text{Edge}(\Lambda)}
\end{align*}
\]

\[
C^0(\Lambda; A) \xrightarrow{\delta} C^1(\Lambda; A)
\]

\[
s \quad \mapsto \quad (e \mapsto g(s; e))
\]
Fourier-Kramers-Wannier Duality

$(Y, \Lambda)$ closed latticed surface

$$C^0(\Lambda; A) = A^{\text{Vert}(\Lambda)}$$

$$C^1(\Lambda; A) = A^{\text{Edge}(\Lambda)}$$

Two functions on $C^1(\Lambda; A)$:

$$\Theta = \prod_{e \in \text{Edge}(\Lambda)} \theta$$

$$\delta_* 1 = \# H^0(\Lambda; A) \Delta_{B^1(\Lambda; A)}$$
(Y, Λ) closed latticed surface

\[ C^0(Λ; A) = A^{\text{Vert}(Λ)} \]
\[ C^1(Λ; A) = A^{\text{Edge}(Λ)} \]

Two functions on \( C^1(Λ; A) \):
\[ \Theta = \prod_{e \in \text{Edge}(Λ)} \theta \]
\[ \delta_* 1 = \# H^0(Λ; A) \Delta_{B^1(Λ; A)} \]

Ising partition function:
\[ I(Y, Λ) = \sum_{s \in A^{\text{Vert}(Λ)}} \prod_{e \in \text{Edge}(Λ)} \theta_\beta(g(s; e)) \]
Fourier-Kramers-Wannier Duality

$(Y, \Lambda)$ closed latticed surface

$$C^0(\Lambda; A) = A^{\text{Vert}(\Lambda)}$$
$$C^1(\Lambda; A) = A^{\text{Edge}(\Lambda)}$$

Two functions on $C^1(\Lambda; A)$:

$$\Theta = \prod_{e \in \text{Edge}(\Lambda)} \theta$$

$$\delta_* 1 = \# H^0(\Lambda; A) \Delta_{B^1}(\Lambda; A)$$

Ising partition function as inner product of functions on $C^1(\Lambda; A)$:

$$I(Y, \Lambda) = \frac{1}{\# H^0(\Lambda; A)} \langle \Theta, \delta_* 1 \rangle = c \langle \Theta, \Delta_{B^1} \rangle$$
Pontrjagin dual groups and maps:

\[ C^0(\Lambda; A) \xrightarrow{\delta} C^1(\Lambda; A) \]
\[ C_0(\Lambda; A^\vee) \xleftarrow{\partial} C_1(\Lambda; A^\vee) \]
Pontrjagin dual groups and maps:

\[ C^0(\Lambda; A) \xrightarrow{\delta} C^1(\Lambda; A) \]

\[ C_0(\Lambda; A^\vee) \xleftarrow{\bar{\partial}} C_1(\Lambda; A^\vee) \]

Parsevel’s formula (Fourier transform \( f \mapsto f^\vee \) is an \( L^2 \)-isometry):

\[ I(Y, \Lambda) = c \langle \Theta, \Delta_{B^1} \rangle = c' \langle \Theta^\vee, \Delta_{B^1}^\vee \rangle \]
Pontrjagin dual groups and maps:

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Rewrite in terms of the dual triangulation \((Y \text{ oriented}) (Y, \Lambda^\vee)\) as an inner product of functions in \( C^1(\Lambda^\vee; A^\vee) \); then

\[ \Theta^\vee = \prod_{e^\vee \in \text{Edge}(\Lambda^\vee)} \theta^\vee \]
\[ \Delta_{B^1}(\Lambda; A) = c'' \Delta_{Z^1}(\Lambda^\vee; A^\vee) \]
Pontrjagin dual groups and maps:

\[ C^0(\Lambda; A) \xrightarrow{\delta} C^1(\Lambda; A) \]
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Parsevel’s formula (Fourier transform \( f \mapsto f^\vee \) is an \( L^2 \)-isometry):

\[ I(Y, \Lambda) = c \langle \Theta, \Delta_B \rangle = c' \langle \Theta^\vee, \Delta_B^\vee \rangle \]

Rewrite in terms of the dual triangulation (\( Y \) oriented) \( (Y, \Lambda^\vee) \) as an inner product of functions in \( C^1(\Lambda^\vee; A^\vee) \); then

\[ \Theta^\vee = \prod_{e^\vee \in \text{Edge}(\Lambda^\vee)} \theta^\vee \]
\[ \Delta^\vee_B(\Lambda; A) = c'' \Delta_{Z^1(\Lambda^\vee; A^\vee)} \]

For \( A = \mu_2 \), Fourier transform: \( \beta \leftrightarrow \beta^\vee \) where \( \sinh(2\beta) \sinh(2\beta^\vee) = 1 \)
Features/Problems

1. Kramers-Wannier duality for $G = \text{Abelian}$ relates theories $I(A, \mathbf{x})$ and $I(A_\mathbf{c}, \mathbf{x})$ by homology groups.

2. Need to see how order operators map under duality; usual story with disorder operators not cleanly matching.


4. Mismatch in low energy states under duality.

Key Idea:
Use the full strength of the symmetry group $G$.

Settles these issues and much more:
- Prediction for low energy behavior (all $G$)
- More general classes of models
- Whole story in context of extended topological field theory
- Higher dimensional abelian models (stable homotopy theory)
Features/Problems

1. Kramers-Wannier duality for \( G = A \) abelian relates theories \( I_{(A,\theta)} \) and \( I_{(A^\vee,\theta^\vee)} \), but off by homology groups.
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Features/Problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\text{op},\theta^\text{op})}$, but off by homology groups.

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4. Mismatch in low energy states under duality.

**Key Idea:** Use the full strength of the symmetry group $G$.

Settles these issues and much more:

- prediction for low energy behavior (all $G$)
- more general classes of models
- whole story in context of extended topological field theory
- higher dimensional abelian models (stable homotopy theory)
If a group $G$ acts as a symmetry on mathematical object $M$ (condition), we can try to extend (data) to a fibering

\[ M \leftarrow \rightarrow M \]
\[ pt \leftarrow \rightarrow BG \]

The precise nature of $BG$ and 'fibering' vary in geometry/topology. $M$ is the Borel quotient. In general, there may be obstructions ('anomalies') which are important features of the symmetry. In any case, $M$ yields a richer picture.
Fibering over $BG$

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In geometry/topology $M$ is the Borel quotient
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Fiber over $BG$

If a group $G$ acts as a symmetry on mathematical object $M$ (condition), we can try to extend (data) to a fibering

$$
\begin{array}{ccc}
M & \overset{c}{\rightarrow} & M \\
\downarrow & & \downarrow \\
\text{pt} & \overset{c}{\rightarrow} & BG \\
\end{array}
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Equivariance $\rightarrow$ Families
‘Fibering over $BG$’ in Ising Model

$G$-Ising model on $Y^2$: background lattice $\Lambda \subset Y$ and $G$-bundle $Q \to Y$ fluctuating field a "discrete gauged $\sigma$-model"

$$Q^{\text{Vert}(\Lambda)} = \text{sections of } Q \to Y \text{ over } \text{Vert}(\Lambda)$$

The ratio of spins defined via parallel transport

$$g: Q^{\text{Vert}(\Lambda)} \times \text{Edge}(\Lambda) \to G$$
‘Fibering over $BG$’ in Ising Model

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The ratio of spins defined via parallel transport

$$g: Q^{\text{Vert}(\Lambda)} \times \text{Edge}(\Lambda) \longrightarrow G$$

The partition function of $I = I_{(G,\theta)}$ is now a function of a $G$-bundle:

$$I(Y, \Lambda): \text{Bun}_G(Y) \longrightarrow \mathbb{C}$$

The old partition function is the value at the trivial bundle
To a latticed 1-manifold \((S, \Pi)\) we obtain a vector bundle

\[ I(S, \Pi) \rightarrow \text{Bun}_G(S) \]

These are “twisted sectors”; the old state space is the fiber at \(\text{pt} \in BG\).
To a latticed 1-manifold $(S, \Pi)$ we obtain a vector bundle

$$I(S, \Pi) \to \text{Bun}_G(S)$$

These are “twisted sectors”; the old state space is the fiber at $\text{pt} \in BG$

**Upshot:** $I$ is a boundary theory for $\mathcal{G}_G$:

$$I(Y, \Lambda) \in \mathcal{G}_G(Y)$$

$$I(S, \Pi) \in \mathcal{G}_G(S)$$
We learned recently that we were anticipated by Pavol Ševera (2002) in some of our pictures of the Ising model and topological field theory, though he works in a non-extended context: arXiv:hep-th/0206162
Boundary theories

**Definition:** A topological boundary theory for $\mathcal{G}_G: \text{Bord}_3 \to \text{TensCat}$ is

$$B: 1 \longrightarrow \tau_{\leq 2}^{\mathcal{G}_G},$$

a map of functors $\text{Bord}_2 \to \text{TensCat}$

Ising is a boundary theory on latticed manifolds (finite path integral)
**Boundary theories**

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a map of functors $\text{Bord}_2 \rightarrow \text{TensCat}$

Cobordism hypothesis: $\mathcal{B}$ determined by $\mathcal{B}(\text{pt})$, a left $\text{Vect}[G]$-module
Boundary theories

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a map of functors $\text{Bord}_2 \to \text{TensCat}$

Cobordism hypothesis: $\mathcal{B}$ determined by $\mathcal{B}(\text{pt})$, a left $\text{Vect}[G]$-module

Ising is a boundary theory on latticed manifolds (finite path integral)
Boundary theories

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Cobordism hypothesis: $\mathcal{B}$ determined by $\mathcal{B}(\text{pt})$, a left $\text{Vect}[G]$-module

Ising is a boundary theory on *latticed* manifolds (finite path integral)

For $(Y, \Lambda)$ closed obtain a function on $\text{Bun}_G(Y)$:

$$I(Y, \Lambda)[Q] = \sum_{s \in Q^{\text{Vert}(\Lambda)}} \prod_{e \in \text{Edge}(\Lambda)} \theta(g(s; e))$$
Line operators for neat 1d submanifolds $S \subset X^3$ with $\partial S \subset (\partial X, \Gamma)$

**Wilson/order operators:** $\chi : G \to \mathbb{T}$ character, $S$ ends at vertices

$$(F, I)(X, \Gamma) = \sum_{[P] \in \pi_0 B\text{un}_G(X)} \frac{1}{\# \text{Aut } P} \sum_{s \in S(\partial X, \Gamma)[\partial P]} h_{S, \chi}(P, s) \prod_{e \in \text{Edges}(\Gamma)} \theta(g(s; e))$$
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**Wilson/order operators:** $\chi: G \rightarrow \mathbb{T}$ character, $S$ ends at vertices

’t Hooft/disorder operators: conjugacy class in $G$, $S$ ends in faces
Revisit problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\vee,\theta^\vee)}$, but off by a sum over homology.

✓ Kramers-Wannier duality is part of electromagnetic duality.
Revisit problems

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4. Mismatch in low energy states under duality
   Discuss next
Low energy behavior; phase diagram

\[ \mathcal{M} \]
\[ \Delta \subset \mathcal{M} \]
\[ (\mathcal{M} \setminus \Delta)_{\text{gapped}} \subset \mathcal{M} \setminus \Delta \]
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moduli space of quantum theories
locus of phase transitions
systems with spectral gap
set of *phases*
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Uses twisted sectors—low energy states form a vector bundle

- $\mathbb{M}_2$ paramagnetic ($\beta \rightarrow 0$)
- $W \rightarrow G//G$
- $\mathbb{M}_2$ ferromagnetic ($\beta \rightarrow \infty$)
- EM duality ($G = \mu_2$)
Topological construction; general theories

Two canonical topological boundary theories: *Dirichlet* and *Neumann*.
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Quartet of data:

- $\mathcal{T} = \text{Vect}[G]$ categorified group algebra
- $\mathcal{B}_1 = \text{Vect}(G)$ Neumann boundary theory
- $\mathcal{B}_2 = \text{Vect}$ Dirichlet boundary theory
- $\delta$ generator of $\text{Hom}_G(\mathcal{B}_1, \mathcal{B}_2)$
Quartet: 3d TFT $\mathcal{G}_G$, boundary theories $\mathcal{B}_1, \mathcal{B}_2$, and domain wall $\mathcal{D}$.
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More general theories: spherical fusion category and fiber functor

$=\text{Frobenius-Hopf algebra}$
Morita equivalence:

\[ \text{Vect}[G] \leftrightarrow \text{Rep}(G) \quad \text{(tensor categories)} \]
\[ \text{Vect}(G/H) \leftrightarrow \text{Rep}(H) \quad \text{(left modules)} \]
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Exchanges theories specified by quartets \textbf{Dirichlet} \longleftrightarrow \textbf{Neumann}:

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Cobordism hypothesis \implies \text{Electromagnetic/Kramers-Wannier duality}

**Theorem:** On oriented manifolds there is an equivalence of $G$-gauge theory and the Turaev-Viro Rep($G$) theory which exchanges their lattice boundary theories, and exchanges Wilson/Order and ’t Hooft/Disorder operators.
Abelian duality in higher dimensions

\[ S \quad \text{pointed space, finite homotopy type} \]
\[ \mathcal{F}_X \quad \text{Map}(X_+, S) \]

\( n \)-dimensional theory \( F_S \) (finite path integral) with partition function

\[
F_S(X) = \sum_{[\varphi] \in \pi_0 \mathcal{F}_X} \frac{1}{\#\pi_1(\mathcal{F}_X, \varphi)} \frac{\#\pi_2(\mathcal{F}_X, \varphi)}{\#\pi_3(\mathcal{F}_X, \varphi)} \ldots
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Canonical Dirichlet and Neumann boundary theories

If \( S \) is an \( \infty \) loop space, the 0-space of a spectrum \( \mathcal{I} \), then there is a (Pontrjagin) dual spectrum \( \mathcal{I}^\vee \). Electromagnetic duality:

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F_\mathcal{I} \approx F_{\Sigma^{n-1} \mathcal{I}^\vee}
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The abelian Ising story is $n = 3$ and $\mathcal{I} = \Sigma HA$. 