ABELIANIZATION IN CLASSICAL CHERN-SIMONS

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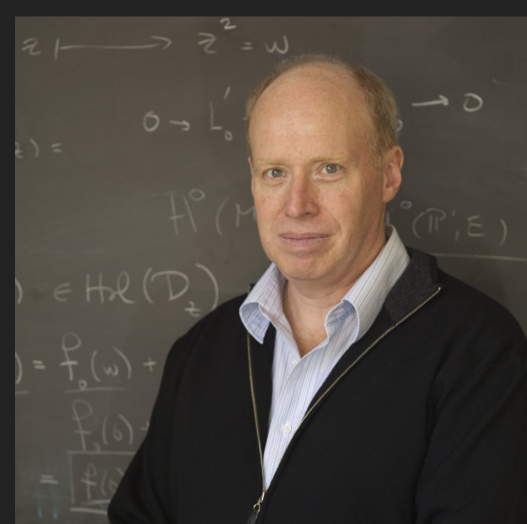
work in progress with Dan Freed

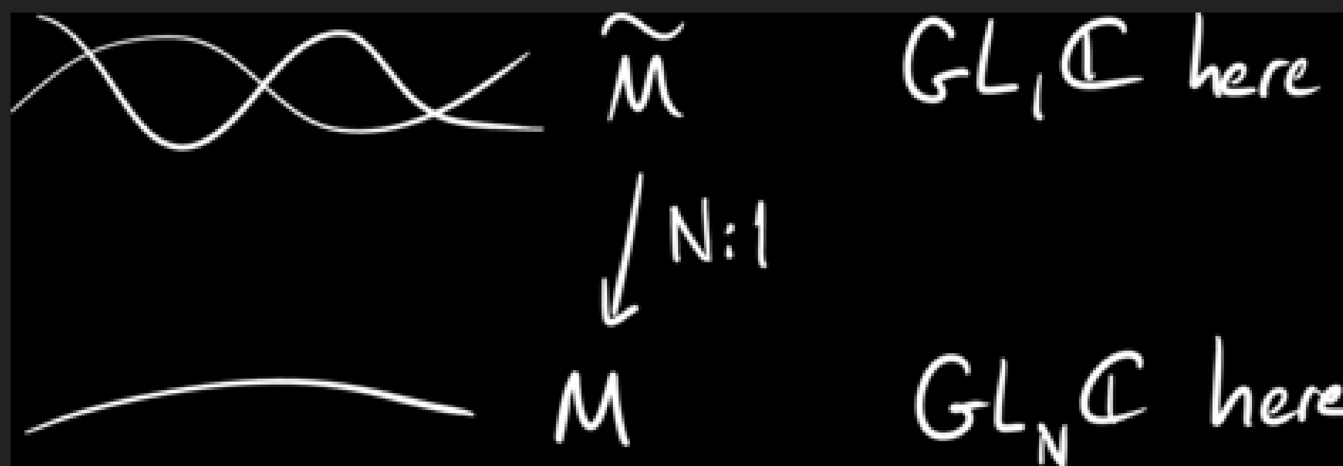


THE PLAN

By abelianization in classical Chern-Simons I mean a relation between

- classical $GL_N\mathbb{C}$ Chern-Simons theory on M
- classical $GL_1\mathbb{C}$ Chern-Simons theory with defects on $\widetilde{M}$

where $\widetilde{M}$ is an N-fold branched cover of M.



It is a classical version of a proposal of Cecotti-Cordova-Vafa. More generally, it is close to the circle of ideas around QFTs of "class R" and 3d-3d correspondence (Dimofte, Gaiotto, Gukov, Terashima, Yamazaki, Kim, Gang, Romo, ...).

It is not the same as the abelianization of Chern-Simons studied by Beasley-Witten or Blau-Thompson.

CLASSICAL $GL_N\mathbb{C}$ CHERN-SIMONS INVARIANT

Given a compact spin 3-manifold M, carrying a $GL_N\mathbb{C}$-connection ∇, there is a (level 1) classical Chern-Simons invariant (action):

$$CS(M;\nabla) \in \mathbb{C}^\times$$

If $\nabla = d + A$, $A \in \Omega^1(M;\mathfrak{gl}_N\mathbb{C})$, then

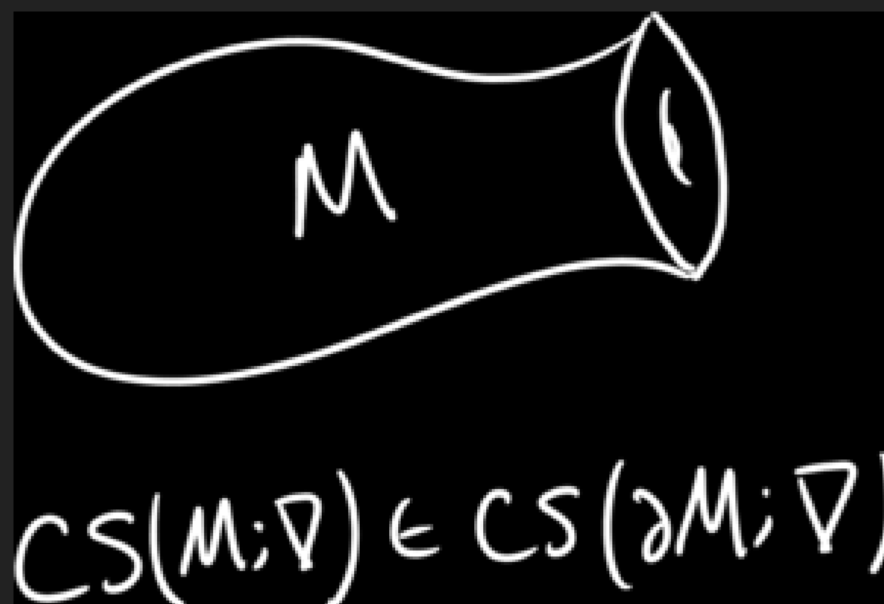
$$CS(M;\nabla) = \exp\left[\frac{1}{4\pi i}\int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right)\right]$$

I will focus on the invariant for flat connections (critical points of the action.)

CLASSICAL $GL_N\mathbb{C}$ CHERN-SIMONS LINE

When M is a manifold with boundary, $CS(M;\nabla)$ is not quite a number, because the integral is not gauge invariant.

Instead, it is an element of a line $CS(\partial M;\nabla)$, determined by $\nabla|_{\partial M}$.



If M' is a closed 2-manifold, we have a line $CS(M';\nabla)$ for each ∇ over M'; they fit together to Chern-Simons line bundle over the moduli space of flat $GL_N\mathbb{C}$-connections over M'.

CLASSICAL $GL_N\mathbb{C}$ CHERN-SIMONS TFT

The classical Chern-Simons invariant for a compact 3-manifold M is the top part of a 3-dimensional invertible spin TFT with $GL_N\mathbb{C}$ symmetry:

$\dim M$	$CS(M;\nabla)$
3	$\in \mathbb{C}^\times$
2	$\in \mathrm{Lines}$
$\cdots$	$\cdots$

We will focus just on the 3-2 part, given by a functor

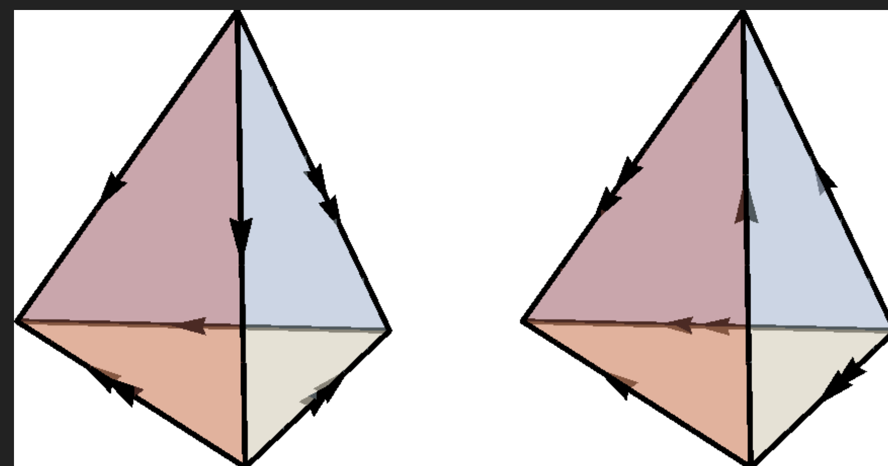
$$\mathrm{CS} : \mathrm{Bord}_{GL_N,spin} \to \mathrm{Lines}$$

SHAPE PARAMETERS

I'm almost done reviewing, but let me recall one other fact about classical Chern-Simons theory, on 3-manifolds.

SHAPE PARAMETERS

Suppose M is an ideally triangulated 3-manifold.



Then (boundary-unipotent) flat $SL_N\mathbb{C}$-connections on M can be constructed by gluing, using

$$\frac{1}{6}(N^3 - N)$$

numbers $\mathcal{X}_i \in \mathbb{C}^\times$ per tetrahedron.

[W. Thurston, Neumann-Zagier, ... for $N = 2$]

[Dimofte-Gabella-Goncharov, Garoufalidis-D. Thurston-Goerner-Zickert for all N]

The $\mathcal{X}_i$ have to obey algebraic equations ("gluing equations") over $\mathbb{Z}$, determined by combinatorics of M.

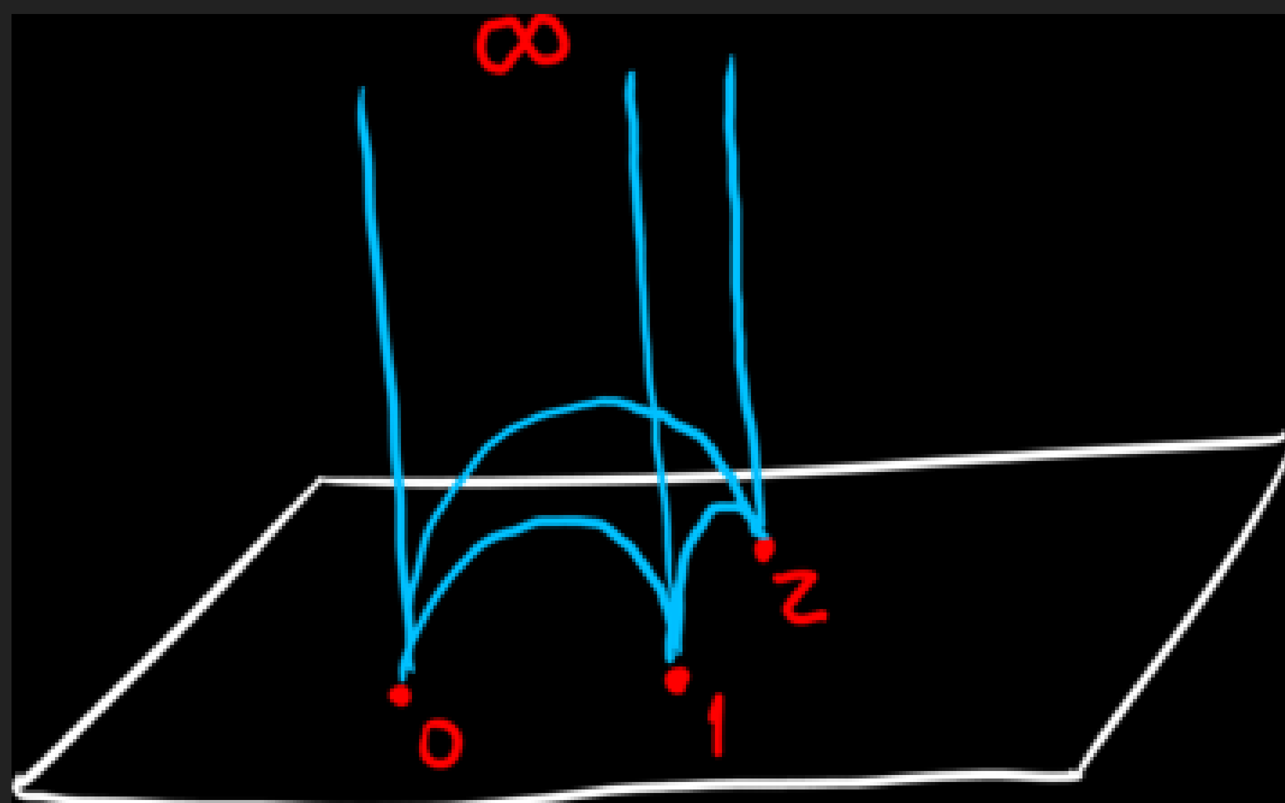
determined by combinatorics of M.

SHAPE PARAMETERS

There is one case where the shape parameters have a well known geometric meaning. [W. Thurston]

This is the case when ∇ is the $PSL(2,\mathbb{C})$ connection induced by a hyperbolic structure on the ideally triangulated M.

In this case, each tetrahedron of M is isometric to an ideal tetrahedron in the hyperbolic upper half-space.



The shape parameter is the cross-ratio of the 4 ideal vertices lying on the boundary $\mathbb{CP}^1$.

SHAPE PARAMETERS

When the $GL(N,\mathbb{C})$-connection ∇ has shape parameters $\mathcal{X}_i$, one has a formula of the sort

$$CS(M;\nabla) = \exp\left[\frac{1}{2\pi i}\sum_i \mathrm{Li}_2(\mathcal{X}_i)\right]$$

(Have to take care about the branch choices for Li_2.)

[W. Thurston, Goncharov, Dupont, Neumann, ..., Garoufalidis-D. Thurston-Zickert]

SHAPE PARAMETERS

One of the aims of this talk is to explain a different geometric picture of the shape parameters $\mathcal{X}_i$, and the dilogarithmic formulas for Chern-Simons invariants.

(Spoiler: the $\mathcal{X}_i \in \mathbb{C}^\times$ will turn out to be holonomies of a $GL_1\mathbb{C}$-connection.)

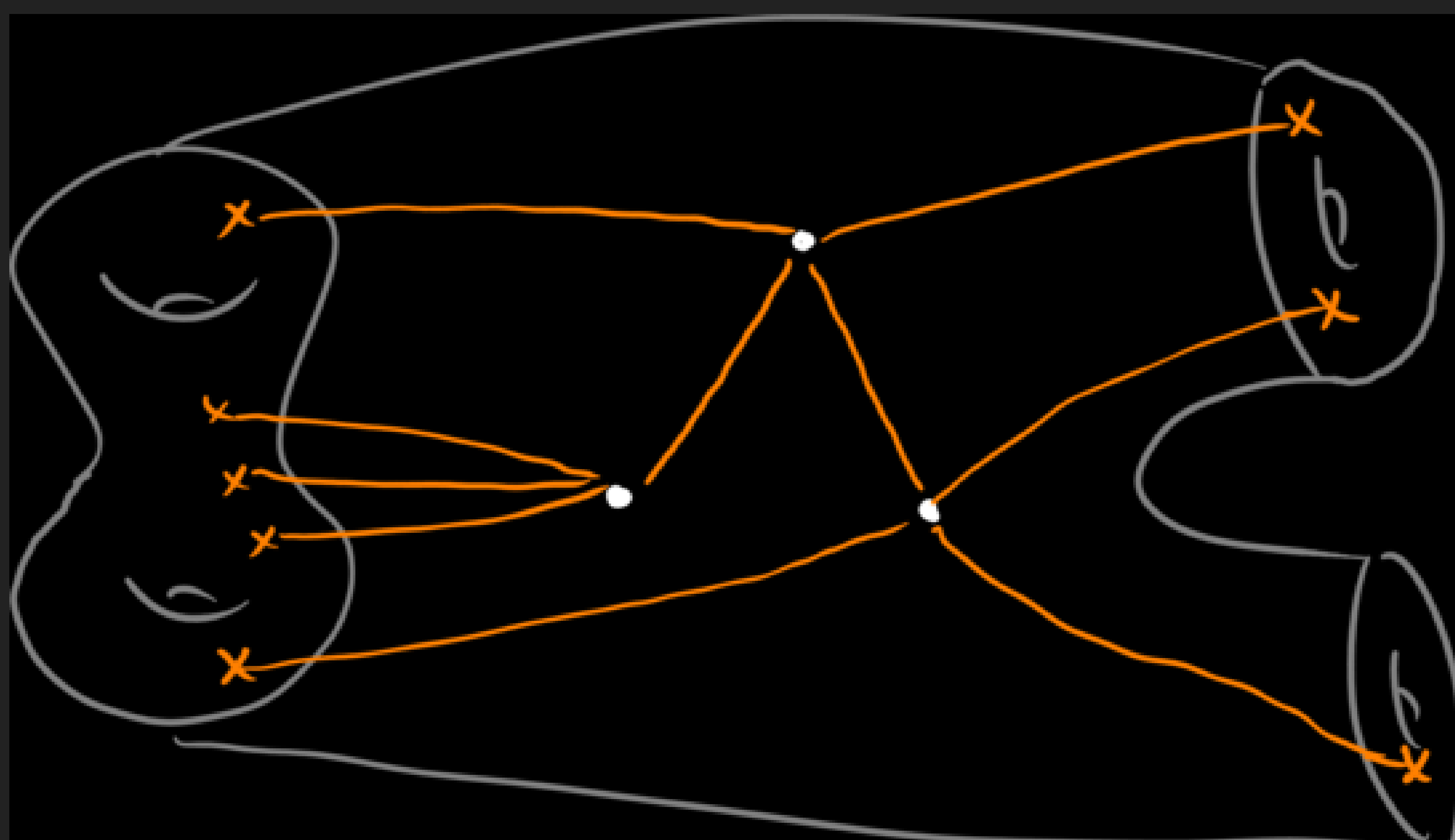
$GL_1\mathbb{C}$ CHERN-SIMONS WITH DEFECTS

We consider $GL_1\mathbb{C}$ Chern-Simons theory on spin manifolds $\widetilde{M}$ carrying connections $\widetilde{\nabla}$, with two unconventional defects added.

ie, a functor

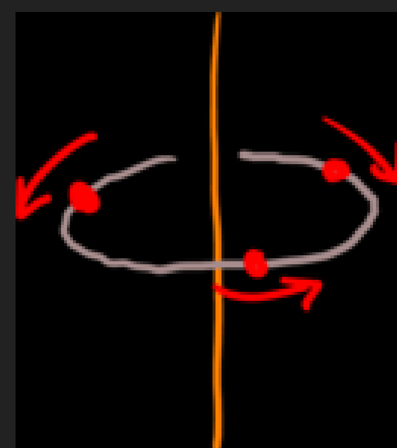
$$\widetilde{CS} : \widetilde{\mathrm{Bord}}_{GL_1,spin} \to \mathrm{Lines}$$

$\widetilde{\mathrm{Bord}}_{GL_1,spin}$ is a bordism category of spin manifolds $\widetilde{M}$, with $GL_1\mathbb{C}$-connections plus defects.



$GL_1\mathbb{C}$ CHERN-SIMONS WITH DEFECTS

The theory involves a codimension 2 defect. This defect needs extra "framing" structure on its linking circle: 3 marked points with arrows.

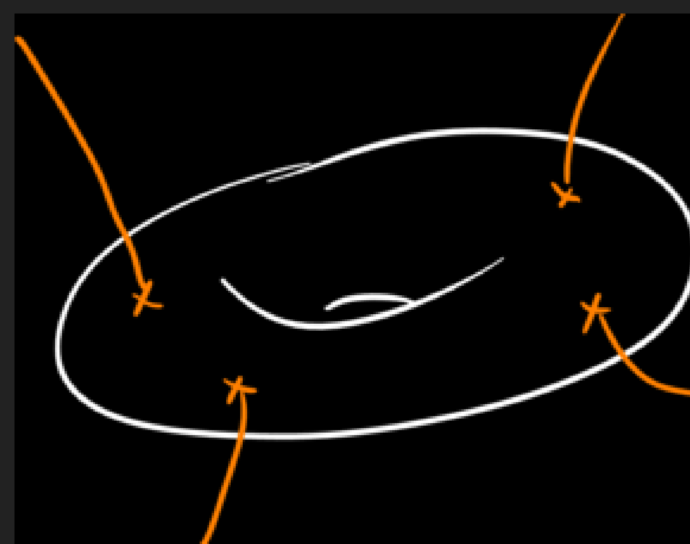


In expectation values, these codimension 2 defects contribute a cube root of unity for each $\frac{2\pi}{3}$ twist of the framing, in the case where the arrows are consistently oriented.

$GL_1\mathbb{C}$ CHERN-SIMONS WITH DEFECTS

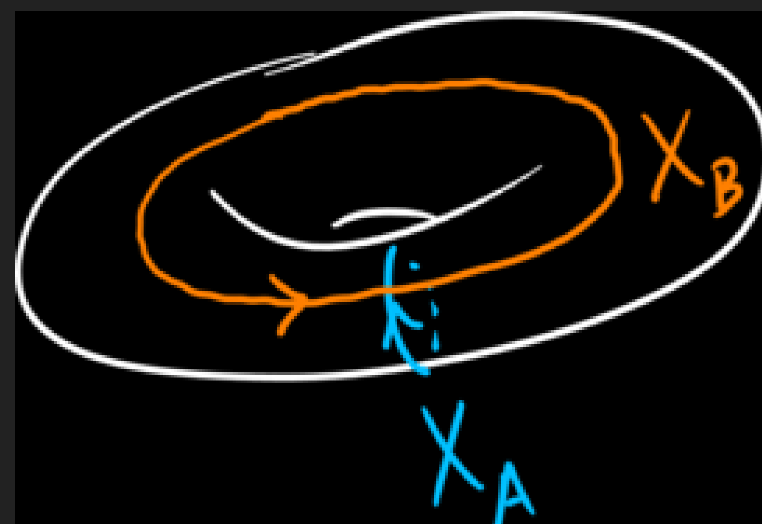
The theory also has a codimension 3 defect, which is a singularity of the manifold structure of $\widetilde{M}$: its link is a T^2 rather than S^2.

Each codimension 3 defect has 4 codimension 2 defects impinging.



$GL_1\mathbb{C}$ CHERN-SIMONS WITH DEFECTS

The connection $\widetilde{\nabla}$ does not extend over the codimension 3 defect: it has nontrivial holonomies over both cycles of the linking T^2.



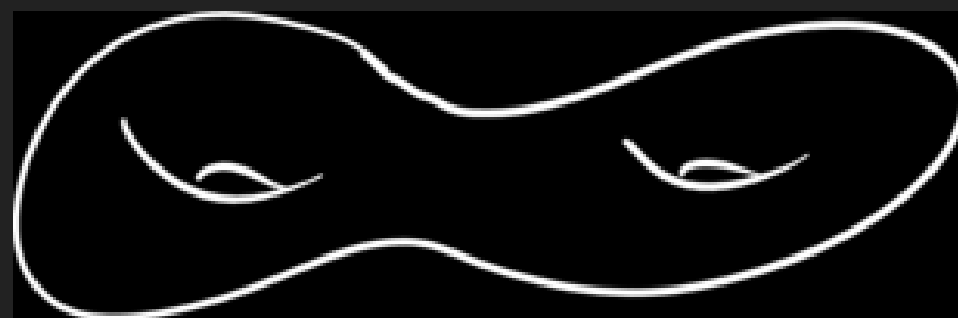
The holonomies obey a constraint:

$$\pm X_A \pm X_B = 1$$

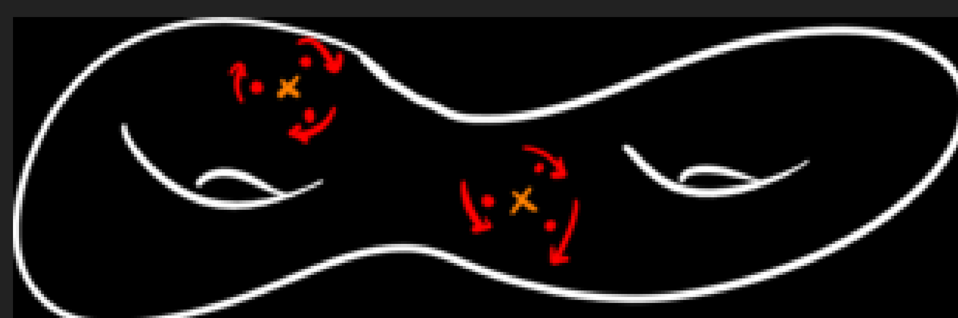
(The signs $\pm$ depend on the spin structure.)

$GL_1\mathbb{C}$ CHERN-SIMONS WITH DEFECTS

For a surface $\widetilde{M}$, with spin structure and flat $GL_1\mathbb{C}$-connection $\widetilde{\nabla}$, $\widetilde{CS}(\widetilde{M};\widetilde{\nabla})$ is the usual line of spin Chern-Simons theory.

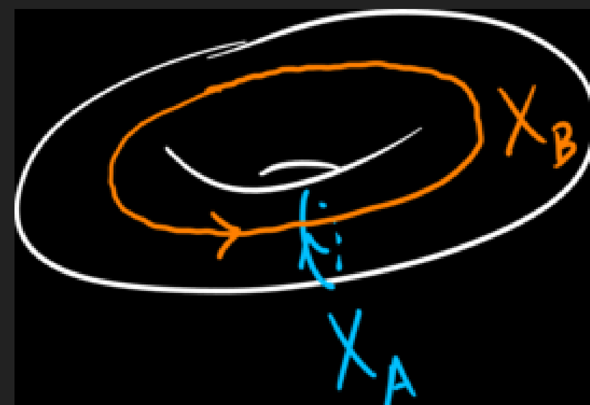


When codimension-2 defects impinge on $\widetilde{M}$, $\widetilde{CS}(\widetilde{M};\widetilde{\nabla})$ is the usual line of spin Chern-Simons, tensored with an extra line for each defect, depending only on the "framing".



$GL_1\mathbb{C}$ CHERN-SIMONS WITH DEFECTS

At each codimension 3 defect we have a tiny torus boundary T.



To define the theory we need to specify an element

$$\Psi \in \widetilde{CS}(T; \widetilde{\nabla})$$

Luckily there is a natural candidate! Loosely

$$\Psi = c \exp\left(\frac{1}{2\pi i} R(X_A)\right)$$

where R is a variant of the Rogers dilogarithm,

$$R(z) = \mathrm{Li}_2(\pm z) - \frac{1}{2}\log(1 \pm z)$$

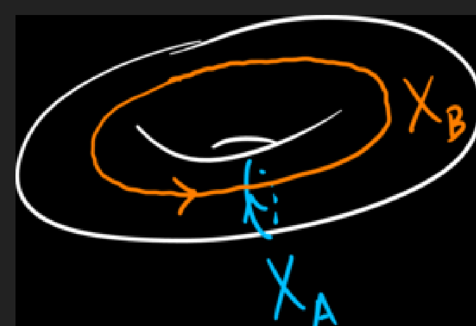
THE DILOG IN $GL_1\mathbb{C}$ CHERN-SIMONS

The equation

$$\Psi = c\exp\left(\frac{1}{2\pi i}R(X_A)\right)$$

has two problems on its face:

- The RHS is not a well defined function, because $R(z)$ is a multivalued function: requires a choice of branch.
- The LHS is not a well defined function, because it is an element of $\widetilde{CS}(T;\widetilde{\nabla})$: requires a choice of trivialization of the bundle underlying $\widetilde{\nabla}$.



These two problems cancel each other out; on both sides, we need to choose logarithms of $\pm X_A$ and $\pm X_B$, and then the transformation law of R matches the WZW cocycle for $\widetilde{CS}(T;\widetilde{\nabla})$.

THE DILOG IN $GL_1\mathbb{C}$ CHERN-SIMONS

This is a fun interpretation of the dilogarithm function:

It is a section of the spin Chern-Simons line bundle over the moduli of flat $GL_1\mathbb{C}$-connections on the torus, restricted to the locus

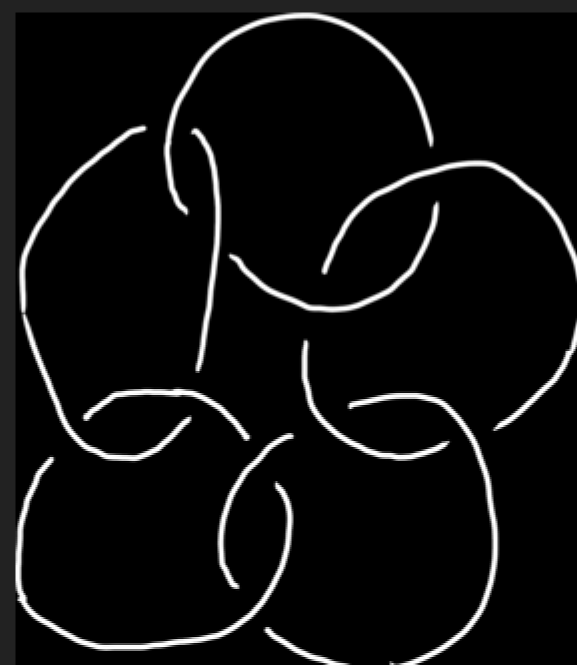
$$\mathcal{L} = \{\pm X_A \pm X_B = 1\} \subset (\mathbb{C}^\times)^2$$

(In fact, it is a flat section; this determines it up to overall normalization.)

THE DILOG IN $GL_1\mathbb{C}$ CHERN-SIMONS

From this point of view, 3-manifolds $\widetilde{M}$ where $\partial\widetilde{M}$ is a union of tori relate to dilog identities.

e.g. to get the five-term identity up to a constant, contemplate $M = S^3 \smallsetminus L$ for a link L:



There exist flat connections $\widetilde{\nabla}$ on $\widetilde{M}$ obeying $X_A + X_B = 1$ at all 5 torus boundaries. Spin Chern-Simons theory gives an element

$$CS(\widetilde{M}; \widetilde{\nabla}) \in \bigotimes_{i=1}^{5} CS(T_i; \widetilde{\nabla}|_{T_i})$$

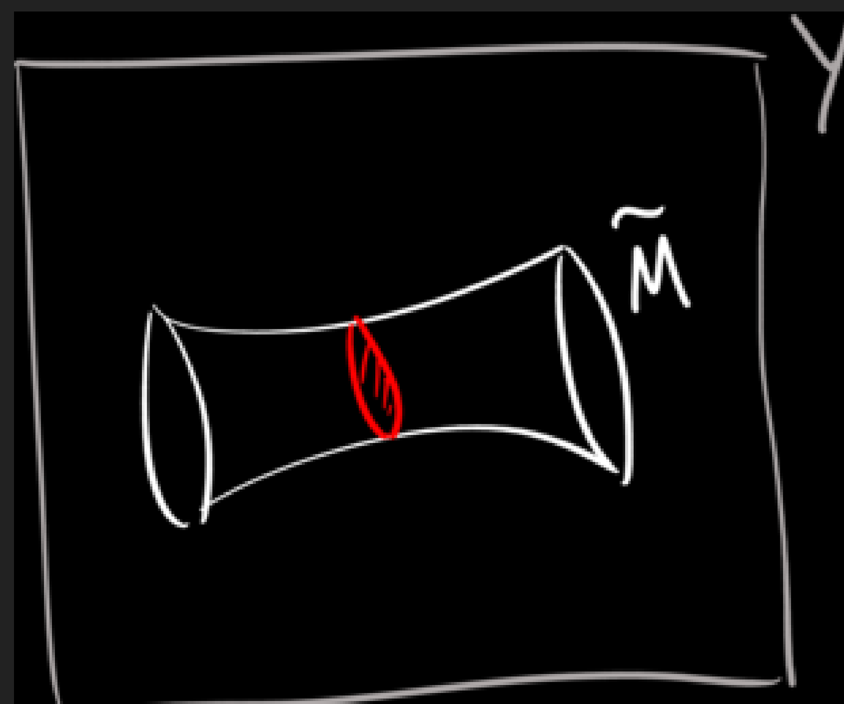
which is an abstract version of the dilog identity.

POINT DEFECTS IN $GL_1\mathbb{C}$ CHERN-SIMONS

These defects have appeared before: in the open topological A model.

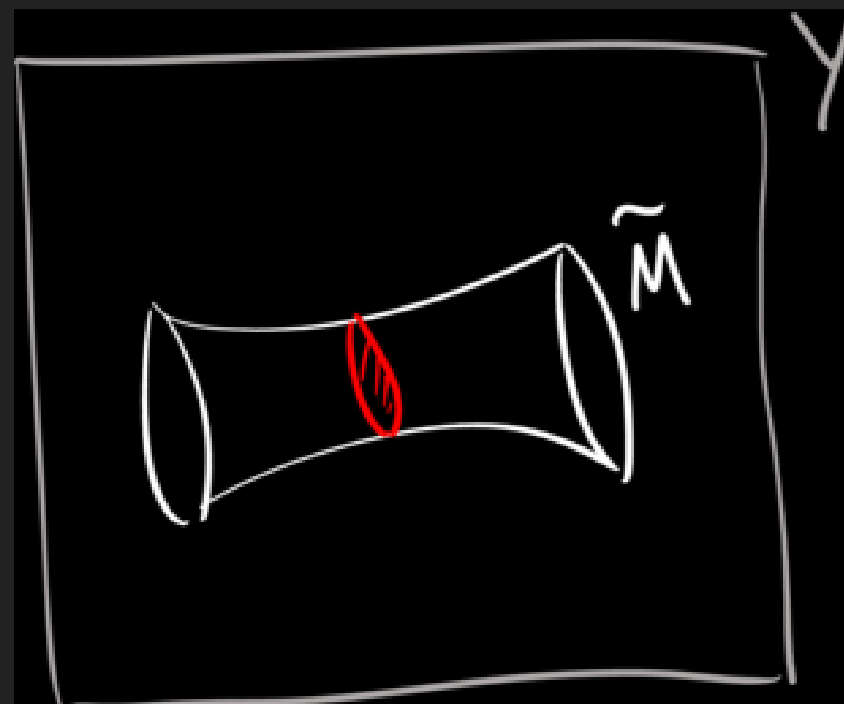
Suppose the 3-manifold $\widetilde{M}$ is a Lagrangian submanifold of a Calabi-Yau threefold Y.

We study the open topological A model on Y, with one D-brane on $\widetilde{M}$.



The open string field theory living on $\widetilde{M}$ is $GL_1\mathbb{C}$ Chern-Simons theory plus corrections from holomorphic curves in Y ending on $\widetilde{M}$. [Witten]

POINT DEFECTS IN $GL_1\mathbb{C}$ CHERN-SIMONS



The correction to the $GL_1\mathbb{C}$ Chern-Simons action that comes from an isolated holomorphic disc is $\mathrm{Li}_2(X)$, where X is the holonomy around the boundary. [Ooguri-Vafa]

So our defects appear naturally in this context: they are the boundaries of holomorphic discs (crushed to a point for technical convenience).

RELATING THE CHERN-SIMONS THEORIES

So far I described two versions of classical Chern-Simons theory, given as functors

$$\widetilde{\text{Bord}}_{GL_1,spin} \to \text{Lines}$$
$$\text{Bord}_{GL_N,spin} \to \text{Lines}$$

How are they related?

ABELIANIZED CONNECTIONS

There is a new bordism category Abel, fitting into a diagram:

$$\begin{array}{ccccc} & & \widetilde{\mathrm{Bord}}_{GL_1,spin} & & \\ & \nearrow & & \searrow & \\ \mathrm{Abel} & & & & \mathrm{Lines} \\ & \searrow & & \nearrow & \\ & & \mathrm{Bord}_{GL_N,spin} & & \end{array}$$

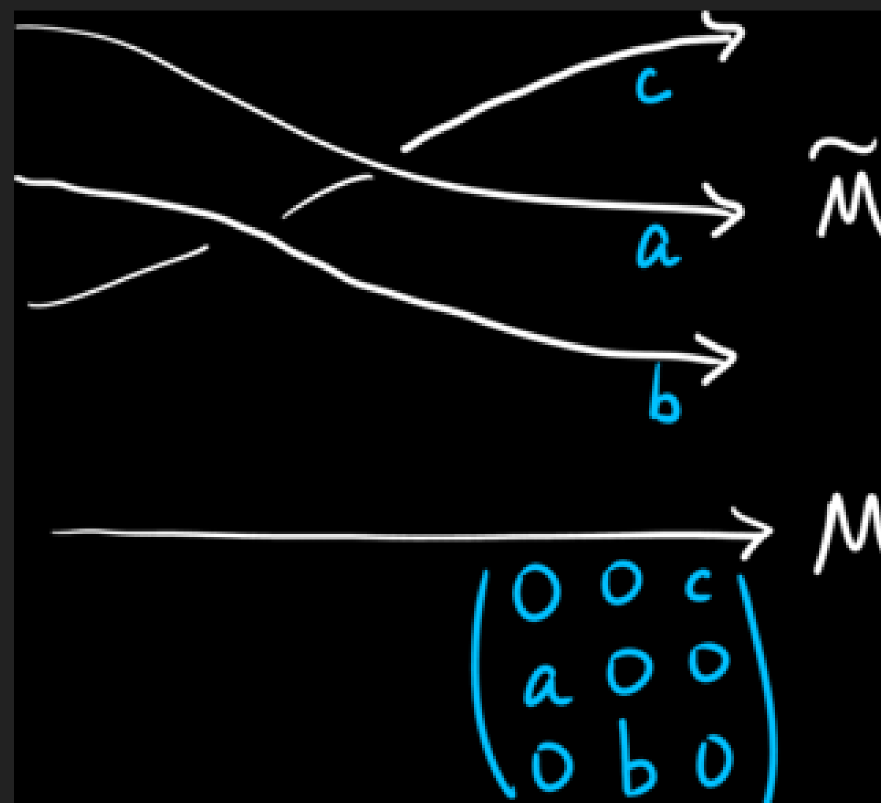
This diagram commutes, ie, the two functors $\mathrm{Abel} \to \mathrm{Lines}$ are naturally isomorphic.

A morphism or object of Abel is an abelianized connection.

This means a $GL_N\mathbb{C}$-connection ∇ over a manifold M, with a partial gauge fixing where ∇ looks "as simple as possible".

ABELIANIZED CONNECTIONS

In the bulk of M, parallel transports of ∇ are permutation-diagonal: they reduce to transports of a $GL_1\mathbb{C}$-connection $\widetilde{\nabla}$ over a cover $\widetilde{M}$.



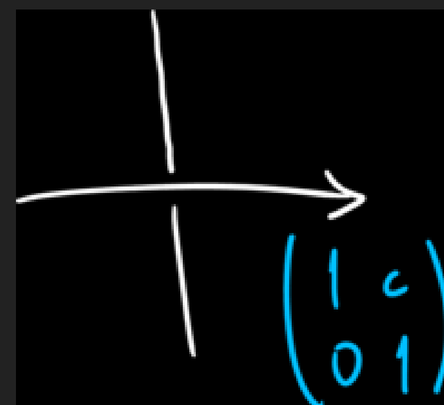
Usually this cannot be done globally on M.

(e.g. imagine M = once-punctured torus: can't simultaneously diagonalize monodromy on A and B cycles)

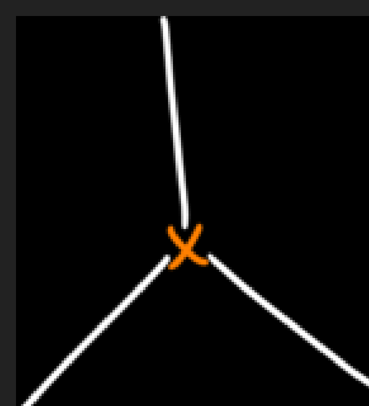
ABELIANIZED CONNECTIONS

We introduce a stratification of M (spectral network).

On codimension-1 strata (walls), we allow the gauge to jump by a unipotent matrix.



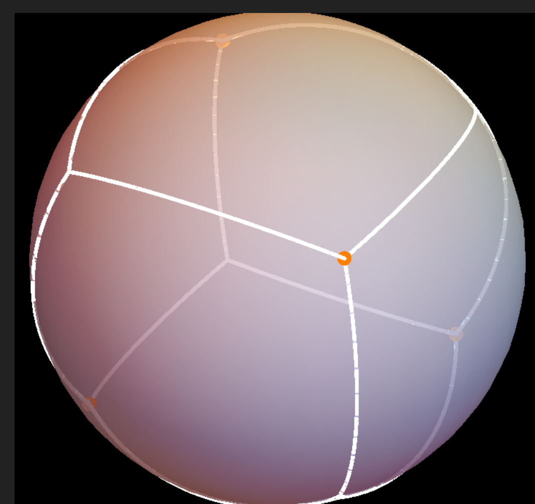
On codimension-2 strata, we allow a singularity around which $\widetilde{M} \to M$ is branched.



(Around this singularity, $\widetilde{\nabla}$ has holonomy -1, and pullback spin structure does not extend: need to make a $\mathbb{Z}_2$ twist of $\widetilde{\nabla}$ and the spin structure, to cancel this.)

ABELIANIZED CONNECTIONS

On codimension-3 (points), we allow a singularity; its linking sphere looks like:



There is no singularity of M or ∇ here, but in the double cover $\widetilde{M}$ and $\widetilde{\nabla}$ have a codimension-3 singularity.

ABELIANIZED CONNECTIONS

A "proof" of our equivalence, without boundaries:

When we have an abelianized connection ∇, we can use its abelian gauges to compute the Chern-Simons action.

$$CS(M;\nabla) = \exp\left[\frac{1}{4\pi i}\int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)\right]$$

In the bulk, just use

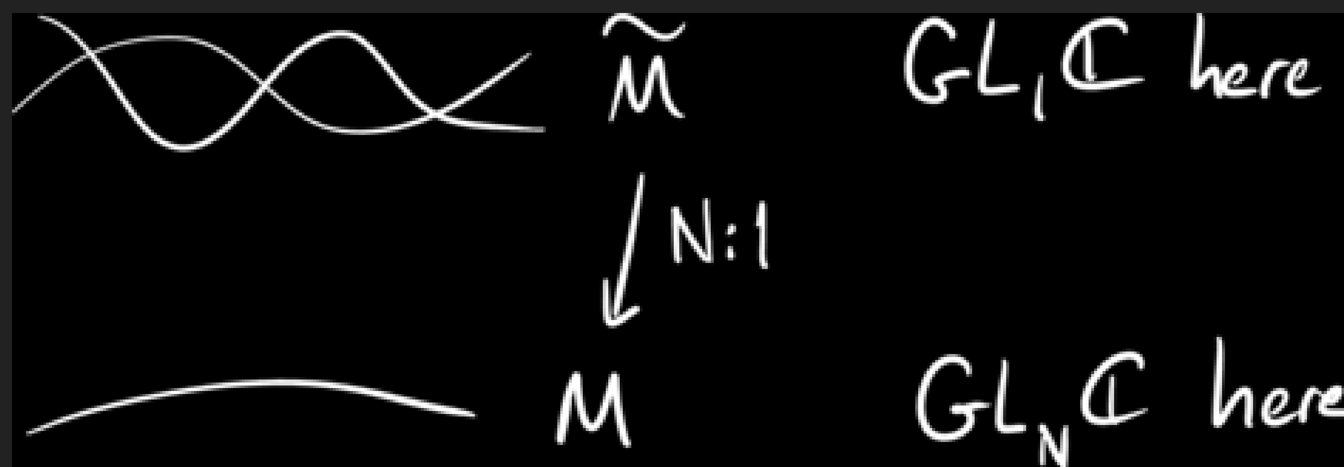
$$\mathrm{Tr}\,\mathrm{diag}(\alpha_1, \ldots, \alpha_N) = \alpha_1 + \cdots + \alpha_N$$

to reduce this to

$$CS(\widetilde{M};\widetilde{\nabla}) = \exp\left[\frac{1}{4\pi i}\int_{\widetilde{M}} \mathrm{Tr}\,(\alpha \wedge d\alpha)\right]$$

At the spectral network, this fails. Still, the walls do not contribute. Lower strata do contribute: they produce the codimension-2 and codimension-3 defects.

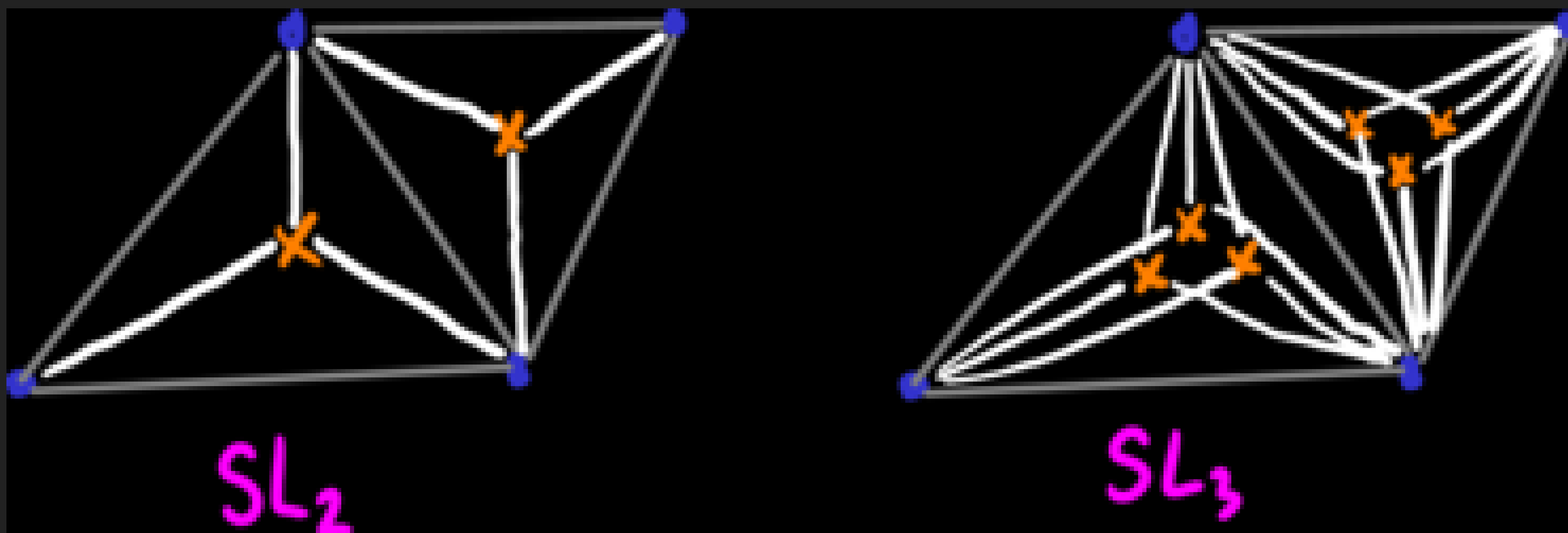
EXAMPLES



Time for examples.

TRIANGULATED 2-MANIFOLDS

Take a triangulated 2-manifold M, punctured at all the vertices. We can equip it with a spectral network and a covering $\widetilde{M} \to M$.



Using this network, a generic ∇ can be abelianized almost uniquely (in finitely many ways). The holonomies of $\widetilde{\nabla}$ around classes $\gamma \in H_1(\widetilde{M}, \mathbb{Z})$ then give cluster coordinates determining ∇.

[Fock-Goncharov, Gaiotto-Moore-N]

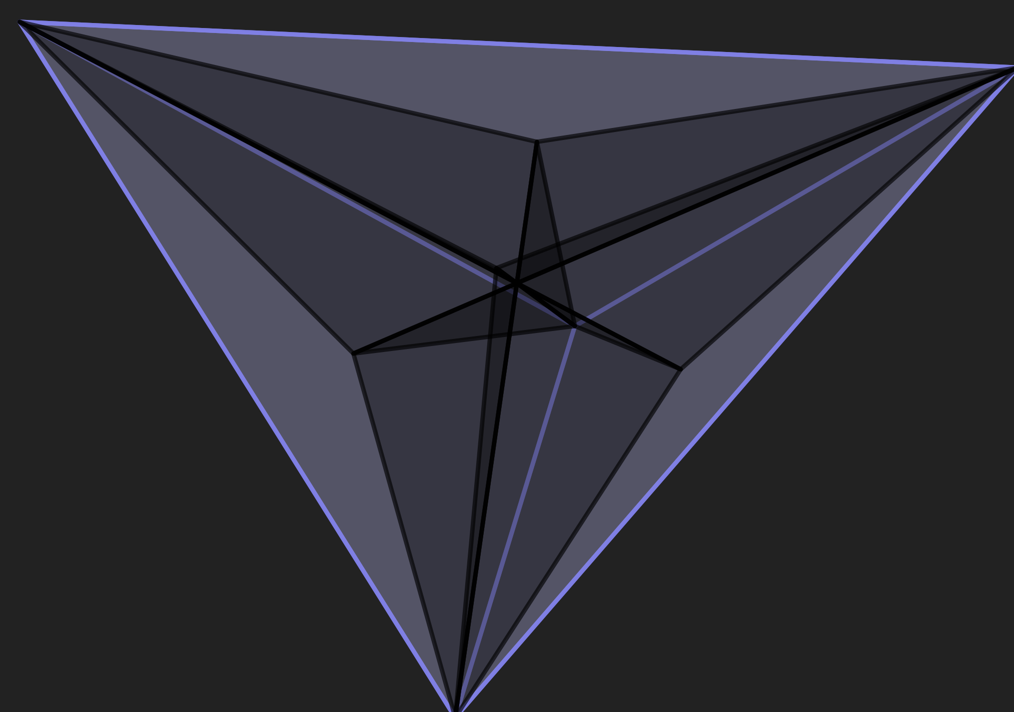
TRIANGULATED 3-MANIFOLDS

Now say M is a triangulated 3-manifold.

Again there is a natural double cover $\widetilde{M} \to M$ and spectral network.

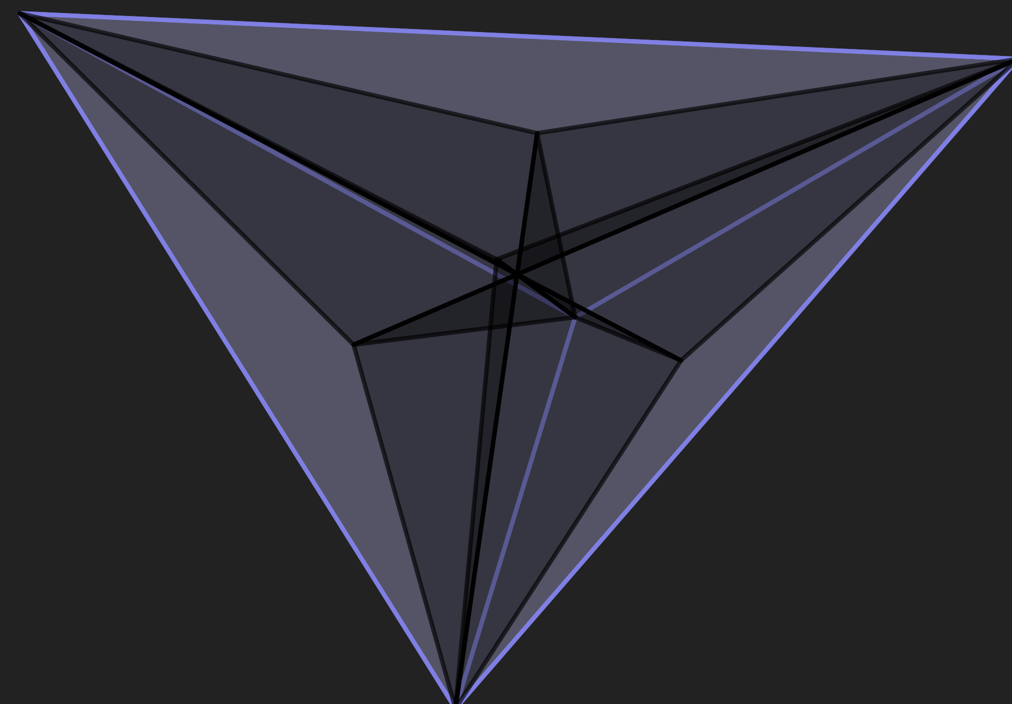
[Cecotti-Cordova-Vafa]

The walls of the spectral network form the dual spine of the triangulation.



There is one codimension-3 defect in the center of each tetrahedron.

TRIANGULATED 3-MANIFOLDS



We can reinterpret the construction of flat $SL_2\mathbb{C}$-connections on M using Thurston's shape parameters $\mathcal{X}_i$ obeying gluing equations.

First, we construct a $GL_1\mathbb{C}$-connection $\widetilde{\nabla}$ over $\widetilde{M}$. The $\mathcal{X}_i$ are its holonomies.

Then, we build the (unique) corresponding ∇ over M.

DILOGARITHM FORMULAS ON TRIANGULATED 3-MANIFOLDS

Our equivalence of Chern-Simons theories says

$$CS(M; \nabla) = \widetilde{CS}(\widetilde{M}; \widetilde{\nabla})$$

So, to get $CS(M; \nabla)$, we can compute in the $GL_1\mathbb{C}$ theory on $\widetilde{M}$.

The line bundle underlying $\widetilde{\nabla}$ turns out to be globally trivial; choose a trivialization. Then,

- the bulk of $\widetilde{M}$ contributes trivially, $\exp\left[\frac{1}{4\pi i} \int_{\widetilde{M}} A \wedge dA\right] = 1$,
- each codim-3 defect contributes a dilogarithm $c_i \exp\left[\frac{1}{2\pi i} R(X_i)\right]$,
- codim-2 defects can contribute third roots of 1.

This recovers the dilogarithm formulas I reviewed before, for $SL_2\mathbb{C}$ (actually a slight generalization: we don't need an "orderable" triangulation).

ABELIANIZATION IN NATURE

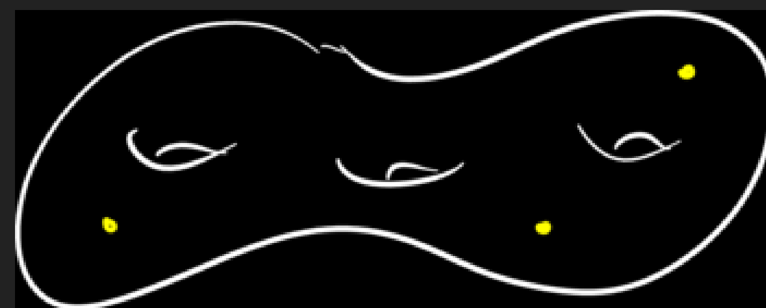
I explained that any time we have an abelianized connection we get an equivalence of Chern-Simons theories.

This statement is most interesting when we have abelianized connections arising in some natural way.

Natural examples are better understood in the 2-dimensional case than the 3-dimensional case, so let me start there.

ABELIANIZATION VIA WKB

Suppose M is a punctured Riemann surface.



On M we can consider meromorphic Schrodinger equations, aka SL_2-opers, locally of the shape

$$\left[\partial_z^2 - \hbar^{-2} P(z)\right] \psi(z) = 0$$

Also higher-order analogues like SL_3-opers,

$$\left[\partial_z^3 - \hbar^{-2} P_2(z) \partial_z - \frac{1}{2} \hbar^{-2} P_2'(z) + \hbar^{-3} P_3(z)\right] \psi(z) = 0$$

These equations give flat connections $\nabla_\hbar$ over M (with singularities at punctures).

ABELIANIZATION VIA WKB

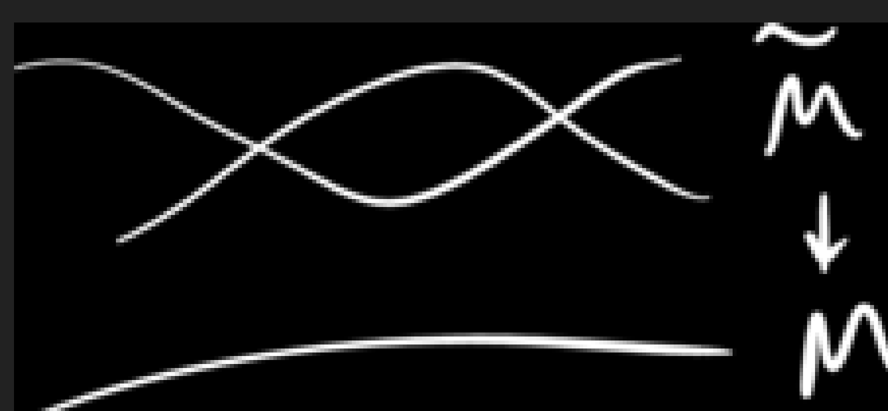
In the exact WKB method for Schrodinger operators, one studies opers ∇ by building local WKB solutions, of the form: [Voros, ..., Koike-Schafke]

$$\psi(z) = \exp\left[\hbar^{-1} \int_{z_0}^{z} \lambda(\hbar)\, dz\right]$$

where $\lambda(\hbar)$ has the WKB asymptotic expansion (for $\mathrm{Re}\,\hbar > 0$)

$$\lambda(\hbar) = \sqrt{-P(z)} + \hbar\lambda_1 + \hbar^2\lambda_2 + \cdots$$

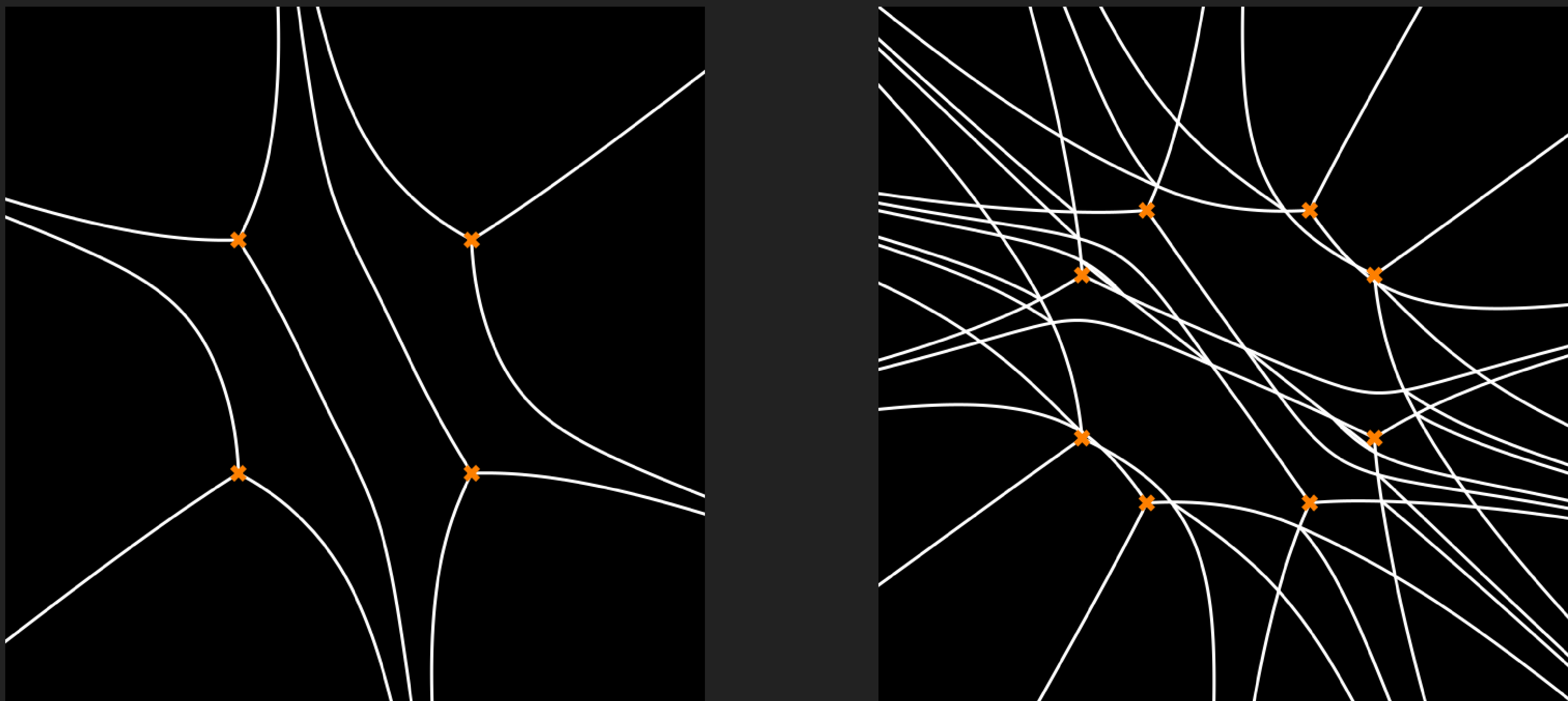
The local solutions $\psi(z)$ reduce ∇ to a $GL_1\mathbb{C}$-connection $\widetilde{\nabla}$ over the spectral curve, a double cover of M, branched at turning points:



$$\widetilde{M} = \{y^2 + P(z) = 0\} \subset T^*M$$

ABELIANIZATION VIA WKB

These local solutions exist in domains of M separated by Stokes curves; these form a spectral network.

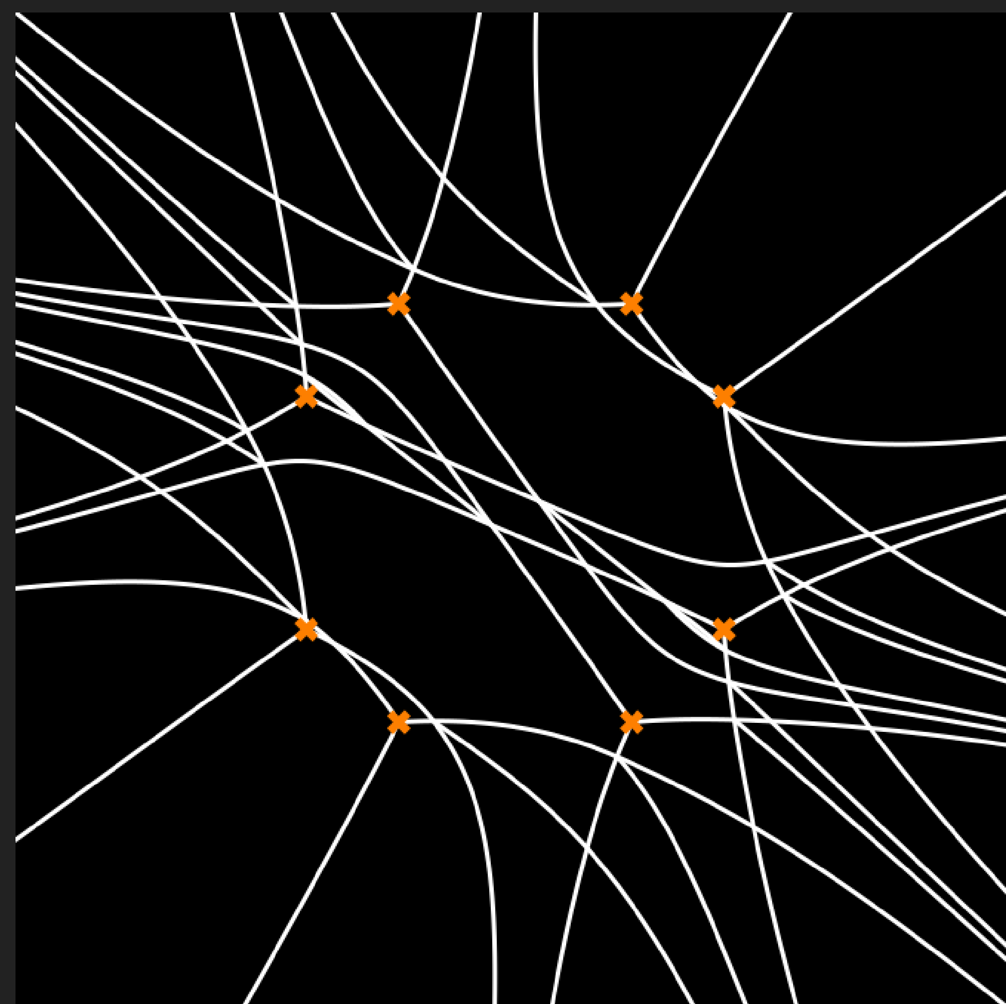


Crossing Stokes curves mixes the local solutions by unipotent changes of basis (WKB connection formula).

So, the structure that comes automatically from WKB analysis of an oper is that of an abelianized connection.

ABELIANIZATION VIA WKB

The combinatorics of such Stokes graphs are generally rather complicated.



Except for $G = SL_2\mathbb{C}$, they are not just captured by ideal triangulations.

ABELIANIZATION IN CLASS S

A variation of this story appeared in quantum field theories of class S. These are 4-dimensional $\mathcal{N} = 2$ theories, obtained by compactification of six-dimensional $(2, 0)$ SCFTs on a punctured Riemann surface M.

Such a theory has a canonical surface defect preserving $\mathcal{N} = (2, 2)$ SUSY, whose space of couplings is M.

This defect has a flat connection ∇ in its vacuum bundle. (like tt^*) This connection is abelianized to a $GL_1\mathbb{C}$-connection $\widetilde{\nabla}$ over the Seiberg-Witten curve $\widetilde{M}$. (UV-IR map). This is a powerful tool for studying the IR theory, BPS states.

[Gaiotto-Moore-N]

ABELIANIZATION IN CLASS R?

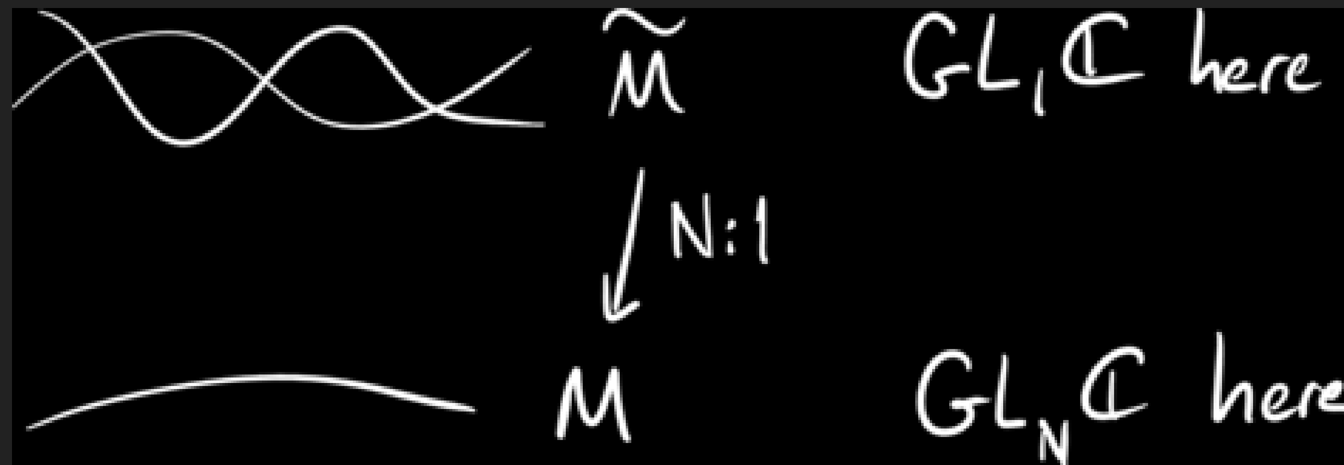
There are 3-dimensional quantum field theories of class R, associated to 3-manifolds M instead of 2-manifolds.

[Dimofte-Gaiotto-Gukov, Cecotti-Cordova-Vafa]

One may hope that the abelianization of complex classical Chern-Simons which we have found has a natural interpretation here.

This could give a source of geometric examples of abelianizations over 3-manifolds.

CONCLUSIONS



I described a relation between two versions of classical complex Chern-Simons theory:

- the $GL_N\mathbb{C}$ theory over M,
- the $GL_1\mathbb{C}$ theory with defects over the branched cover $\widetilde{M}$.

This relation gives a new method of computing in the $GL_N\mathbb{C}$ theory, and naturally accounts for / extends some known facts about that theory.

It still remains to find a good source of examples of a geometric nature, from class R or elsewhere.