Frobenius manifolds and quantum groups

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The Dubrovin systems (or Frobenius manifolds) give a geometric formulation of Witten-Dijkgraaf-Verlinde-Verlinde equations governing deformations of 2D topological field theories.
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In the poster, we propose a quantization of the Dubrovin systems, and then explore its relation with quantum groups and Gromov-Witten type theory.
A linear system for a matrix valued function $F(z, u^1, ..., u^n)$

\[
\frac{\partial F}{\partial z} = \left( \frac{u}{z^2} + \frac{V(u)}{z} \right) F, \\
\frac{\partial F}{\partial u^i} = V_i(z, u) \cdot F.
\]

Here $u = \text{diag}(u^1, ..., u^n)$, $V(u)$ satisfies the Jimbo-Miwa-Ueno PDEs (compatibility of the system).
Dubrovin systems

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**Stokes matrix**: For any fixed $u$, the first equation has different canonical fundamental solutions in different sectors on $z$-plane. The Stokes matrix $S(u)$ measures the jump phenomenon of solutions.

**Isomonodromicity**: $S(u)$ don’t depend on $u$. 
We introduce a system of equations for a $Ug \otimes^2 [[\hbar]]$–valued function $F(z, u^1, ..., u^n)$:

$$\frac{\partial F}{\partial z} = \left( \frac{u \otimes 1}{z^2} + \hbar \frac{\Omega(u)}{z} \right) F,$$

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**Quantum Stokes matrix:** for any $u$, the element $S_{\hbar}(u) \in Ug \otimes^2 [[\hbar]]$ measuring the jump phenomenon of solutions.
Isomonodromic KZ systems

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Here $\Omega(u)$ satisfies a set of PDEs (compatibility of the system).

**Quantum Stokes matrix:** for any $u$, the element $S_{\hbar}(u) \in Ug \otimes^2 [[\hbar]]$ measuring the jump phenomenon of solutions.

**Theorem (Isomonodromicity)**

$S_{\hbar}(u)$ don’t depend on $u$. 
Theorem

The semiclassical limit of the IKZ system gives rise to Dubrovin systems, i.e.,

\[ \text{IKZ systems} \quad \hbar = 0 \quad \Downarrow \quad \text{Dubrovin systems} \]

In particular, any solution \( F \) of the Dubrovin system has a natural \( \hbar \)-deformation

\[ F_{\hbar} = F + F_1 \hbar + F_2 \hbar^2 + \cdots. \]
Semiclassical limit (a way of letting $\hbar$ equal 0)

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Quantum Stokes matrices and Yang-Baxter equations

Theorem

The q-Stokes matrices of IKZ systems satisfy Yang-Baxter equation.
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IKZ systems $\xrightarrow{\text{quantum Stokes matrices}}$ Quantum groups

$\hbar=0$ $\downarrow$

Dubrovin systems $\xrightarrow{\text{Stokes matrices}}$ Poisson Lie groups

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Corollary (Boalch)

The space of Stokes matrices of Dubrovin systems is identified with a Poisson Lie group.
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Question: find a field theoretic interpretation.
Following Givental, the solution $\mathbf{F}$ of a Dubrovin system is viewed as a symplectic transformation on certain loop space $H[[\hbar]]$.

We expect that the deformation $\mathbf{F}_\hbar = \mathbf{F} + \mathbf{F}_1 \hbar + O(\hbar^2)$ via IKZ system is a symplectic deformation of the transformation $\mathbf{F}$ on $H[[\hbar]]$. 
Symplectic actions on loop spaces

IKZ systems $\xrightarrow{\text{Solutions}}$ Symplectic actions on $H[[\hbar]]$

Dubrovin systems $\xrightarrow{\text{Solutions}}$ Symplectic actions on loop space $H$

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Refinement of Gromov-Witten type theory.

Solutions of Dubrovin systems have two deformation/quantization:

- $\hbar$-deformation via the IKZ system;
- $\epsilon$-deformation via Givental’s quantization.

The conjecture can combine these two into a quantization with two parameters. In terms of integrable hierarchies, the two parameters $\epsilon$ and $\hbar$ may correspond respectively to the dispersion and quantization parameters. It may be related to the prediction of Li from the topological string theory.
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Thank you very much!