Frobenius manifolds and quantum groups

Xiaomeng Xu
MIT

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Tohoku University, Sendai, Japan

Goal

The Dubrovin systems (or Frobenius manifolds) give a geometric formulation of Witten-Dijkgraaf-Verlinde-Verlinde equations governing deformations of 2D topological field theories.

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In the poster, we propose a quantization of the Dubrovin systems, and then explore its relation with quantum groups and Gromov-Witten type theory.

Dubrovin systems

A linear system for a matrix valued function $F(z, u^1, ..., u^n)$

$$\frac{\partial F}{\partial z} = \left(\frac{u}{z^2} + \frac{V(u)}{z} \right) F,$$
$$\frac{\partial F}{\partial u^i} = V_i(z, u) \cdot F.$$

Here $u = \mathrm{diag}(u^1, ..., u^n)$, $V(u)$ satisfies the Jimbo-Miwa-Ueno PDEs (compatibility of the system).

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Isomonodromicity: $S(u)$ don't depend on u.

Isomonodromic KZ systems

We introduce a system of equations for a $U\mathfrak{g}^{\otimes 2}[[\hbar]]$–valued function $F(z, u^1, ..., u^n)$:

$$\frac{\partial F}{\partial z} = \left(\frac{u \otimes 1}{z^2} + \hbar \frac{\Omega(u)}{z} \right) F,$$
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Theorem (Isomonodromicity)

$S_\hbar(u)$ don't depend on u.

Semiclassical limit (a way of letting $\hbar$ equal 0)

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Theorem

The semiclassical limit of the IKZ system gives rise to Dubrovin systems, i.e.,

$$\boxed{\textit{IKZ systems}} \xrightarrow{\hbar=0} \boxed{\textit{Dubrovin systems}}$$

In particular, any solution F of the Dubrovin system has a natural $\hbar$-deformation $F_\hbar = F + F_1\hbar + F_2\hbar^2 + \cdots$.

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Corollary (Boalch)

The space of Stokes matrices of Dubrovin systems is identified with a Poisson Lie group.

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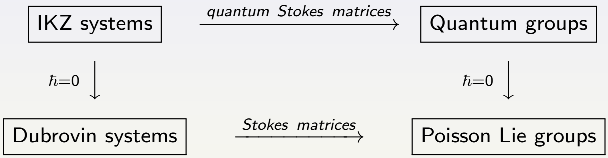

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Question: find a field theorietic interpretation.

Symplectic actions on loop spaces

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• Following Givental, the solution F of a Dubrovin system is viewed as a symplectic transformation on certain loop space H.

• We expect that the deformation $F_\hbar = F + F_1\hbar + O(\hbar^2)$ via IKZ system is a symplectic deformation of the transformation F on H.

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- $\hbar$-deformation via the IKZ system;
- ε-deformation via Givental's quantization.

The conjecture can combine these two into a quantization with two parameters. In terms of integrable hierarchies, the two parameters ε and $\hbar$ may correspond respectively to the dispersion and quantization parameters. It may be related to the prediction of Li from the topological string theory.

Thank you very much!