Mordell-Weil Torsion, Anomalies, Phase Transitions

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Motivation

- Elliptic fibrations
  - Geometrically engineered gauge theories
  - Captures global aspect of the gauge theory
- Base-free set-up

M-Theory

\[ CY_3 \]

5d \( \mathcal{N} = 1 \) Theory

\[ CY_3 \]

F-Theory

\[ CY_3 \]

6d \( \mathcal{N} = (1,0) \) Theory
**Elliptic Fibrations and Gauge Theories**

- (Semi-simple) Lie group \( G \), Lie algebra \( \mathfrak{g} \), Representation \( \mathbb{R} \)
- Dictionary between the elliptic fibration and the gauge theory

<table>
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<tr>
<th>Elliptic Fibration</th>
<th>Gauge Theory</th>
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<tr>
<td>Codimension 1 singularity</td>
<td>Gauge algebra ((\mathfrak{g}))</td>
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<tr>
<td>Codimension 2 singularity</td>
<td>Representation ((\mathbb{R}))</td>
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<td>Crepant resolution</td>
<td>Coulomb phase</td>
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<td>Flop</td>
<td>Phase transition</td>
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<td>Triple intersection polynomial</td>
<td>5d prepotential</td>
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<tr>
<td>Mordell-Weil group</td>
<td>The fundamental group of the gauge group ((\pi_1(G)))</td>
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Main Questions

➢ How the Mordell-Weil group of the elliptic fibration affect these supergravity theories?
  ▪ What is the effect on the Coulomb branch of a 5d gauge theory when a semi-simple group is quotiented by a subgroup of its center?
  ▪ What happens to the extended Mori cone of an elliptically-fibered Calabi-Yau threefold when the Mordell-Weil group is purely torsion?
  ▪ Moreover, what are the 6d uplift of such theories (if any)?
Algorithm to get geometric data

Step 1. Determine a singular Weierstrass model with Kodaira fibers associated to the desired Lie group $G$.

Step 2. Determine a crepant resolution of the singular Weierstrass model.

Step 3. Compute the pushforward formulas to push the total Chern class of the resolved elliptic fibration to its base.
   ➢ The generating function of Euler characteristics is computed.
   ➢ For a $d$-dimensional base, the Euler characteristic is given by the coefficient of $t^d$ in a power series expansion.
   ➢ Compute the Euler characteristics for Calabi-Yau threefolds.

Step 4. Compute the Hodge numbers using the fact that the base is a rational surface and Shioda-Tate-Wazir theorem.

Step 5. Determine the fiber structure of the resolved Weierstrass Model.

Step 6. Compute the geometric weights of the irreducible components of the singular fibers over codimension-two points.

Step 7. Compute the triple intersection polynomial.
Semi-simple Group with MW Torsion

Consider two non-trivial models of semi-simple Lie algebra with MW group $\mathbb{Z}_2$ that corresponds to the collision of the Kodaira fibers of type $I_2^{ns}+I_4$.

<table>
<thead>
<tr>
<th>$g = A_1 \oplus A_3$</th>
<th>$g = A_1 \oplus C_2$</th>
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</thead>
<tbody>
<tr>
<td>$G = \text{SU}(2) \times \text{SU}(4)$</td>
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<td>$\pi_1(G) = \mathbb{Z}_2 \times \mathbb{Z}_4$</td>
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</table>

Three possibilities for embedding $\mathbb{Z}_2$:
- $(\mathbb{Z}_2, 1)$, $(1, \mathbb{Z}_2)$ diagonal $\mathbb{Z}_2$

Possible quotient groups:
- $\text{SO}(3) \times \text{SU}(4), \text{SU}(2) \times \text{SO}(5), (\text{SU}(2) \times \text{SU}(4))/\mathbb{Z}_2$

Their centers:
- $\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2$

Bi-fundamental representation is only compatible with:
- $(\text{SU}(2) \times \text{SU}(4))/\mathbb{Z}_2$

Possible quotient groups:
- $\text{SO}(3) \times \text{Sp}(4), \text{SU}(2) \times \text{SO}(6), (\text{SU}(2) \times \text{Sp}(4))/\mathbb{Z}_2$

Their centers:
- $\mathbb{Z}_2$

Bi-fundamental representation is only compatible with:
- $(\text{SU}(2) \times \text{Sp}(4))/\mathbb{Z}_2$
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<th>Models</th>
<th>Algebraic data</th>
<th># Flops</th>
</tr>
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<tr>
<td>$P_{12}^{ss} + T_{14}^{ss}$</td>
<td>$G = (SU(2) \times Sp(4)) / \mathbb{Z}_2$&lt;br&gt;$\mathbb{Z}_2$&lt;br&gt;$R = (3, 1) \oplus (1, 10) \oplus (2, 4) \oplus (1, 5)$&lt;br&gt;$\chi = -4(9K^2 + 8K \cdot T + 3T^2)$</td>
<td>3</td>
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<td>3</td>
</tr>
<tr>
<td>$P_{12}^{ss} + \Gamma_1^s$</td>
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<td>12</td>
</tr>
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<td>20</td>
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\[ G = SU(2) \times Sp(4) \quad G = (SU(2) \times Sp(4)) / \mathbb{Z}_2 \]
Matter content for each model

<table>
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<tr>
<th>$G$</th>
<th>Adjoint</th>
<th>Bifundamental</th>
<th>(Traceless) Antisymmetric, Fundamental</th>
</tr>
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<td>$(SU(2) \times Sp(4))/Z_2$</td>
<td>$n_{3,1} = g_S$</td>
<td>$n_{2,4} = S \cdot T$</td>
<td>$n_{1,5} = g_T - 1 + \frac{1}{2} T \cdot V(a_2)$</td>
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Charged hypers from triple intersection polynomials

\[ G = (\text{SU}(2) \times \text{SU}(4))/\mathbb{Z}_2 \]
\[ n_{3,1} = (2K + T)(3K + 2T) + 1, \quad n_{2,4} = n_{2,4} = -T(2K + T), \]
\[ n_{1,6} = -KT, \quad n_{1,15} = \frac{1}{2}(KT + T^2 + 2). \]

Comparing the triple intersection polynomial with the 5d prepotential completely fixed the number of hypers charged in each irreducible representations when there is a Mordell-Weil group \( \mathbb{Z}_2 \).

\[ G = (\text{SU}(2) \times \text{Sp}(4))/\mathbb{Z}_2 \]
\[ n_{3,1} = 6L^2 - 7LT + 2T^2 + 1 = g_s, \quad n_{2,4} = -2T(T - 2L) = 2(-4g_T + T^2 + 4), \]
\[ n_{1,5} = \frac{1}{2}(LT + T^2) = -g_T + T^2 + 1, \quad n_{1,10} = \frac{1}{2}(-LT + T^2 + 2) = g_T. \]

While for the cases with a trivial Mordell-Weil group, we are left with some linear relations.

\[ G = \text{SU}(2) \times \text{SU}(4) \]
\[ n_{1,4} + n_{1,4} = -2T(4K + S + 2T), \quad n_{2,4} + n_{2,4} = ST, \]
\[ n_{2,1} + 8n_{3,1} = -2S(2K - S + 2T). \]

\[ G = \text{SU}(2) \times \text{Sp}(4) \]
\[ n_{1,4} + n_{1,10} = -2(2KT + ST - 4), \quad n_{1,5} + n_{1,10} = T^2 + 1, \]
\[ n_{2,1} + 8n_{3,1} = -2S(2K - S + 2T) + 8. \]
Anomaly Cancellation

- Number of multiplets are given by: \( n_V^{(6)} = \dim G, \quad n_T = h^{1,1}(B) - 1 = 9 - K^2; \)
  \( n_H = n_H^0 + n_H^{ch} = h^{2,1}(Y) + 1 + \sum_i (\dim R_i - \dim R_i^{(0)}) \)

- Gravitational Anomalies are canceled when \( n_H - n_V^{(6)} + 29n_T - 273 = 0. \)

- For a semi-simple group with two simple components, \( G = G_1 + G_2, \) the remainder of the anomaly polynomial is given by
  \[
  I_8 = \frac{K^2}{8} (\text{tr} R^2)^2 + \frac{1}{6} (X_1^{(2)} + X_2^{(2)}) \text{tr} R^2 - \frac{2}{3} (X_1^{(4)} + X_2^{(4)}) + 4Y_{12}
  \]
  where
  \[
  \begin{align*}
  X_a^{(2)} &= \left( A_{a,\text{adj}} - \sum_i n_{R_{i,a}} A_{R_{i,a}} \right) \text{tr}_{F_a} F_a^2, \\
  X_a^{(4)} &= \left( B_{a,\text{adj}} - \sum_i n_{R_{i,a}} B_{R_{i,a}} \right) \text{tr}_{F_a} F_a^4 + \left( C_{a,\text{adj}} - \sum_i n_{R_{i,a}} C_{R_{i,a}} \right) \left( \text{tr}_{F_a} F_a^2 \right)^2, \\
  Y_{ab} &= \sum n_{R_{i,a},R_{j,b}} A_{R_{i,a}} A_{R_{j,b}} \text{tr}_{F_a} F_a^2 \text{tr}_{F_b} F_b^2.
  \end{align*}
  \]

- If the \( l_8 \) factors, then the anomalies are all canceled by Green-Schwartz mechanism.

- We check that all the anomalies are canceled once all the number of hypers in each representation are identified.
Thank you for listening! 😊