On non-commutative crepant resolutions of some toric rings

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Motivation

R: Gorenstein normal domain admitting a **crepant resolution**:

$$\pi : Y \longrightarrow X := \operatorname{Spec} R \quad (\text{i.e., } K_Y = \pi^* K_X).$$

For some cases, there is a non-commutative ring Λ such that

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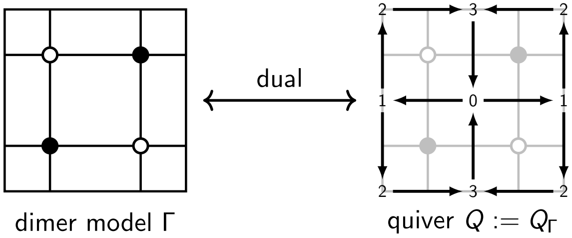
This ring Λ was formulated by M. Van den Bergh,
and is called a **non-commutative crepant resolution (= NCCR)**.

- We can investigate $\mathrm{D^b}(\operatorname{coh} Y)$ from the viewpoint of representation theory of Λ.
- NCCRs also play an important role in representation theory (e.g., Auslander-Reiten theory, Tilting theory).

Example of NCCRs

A **dimer model**, which is a bipartite graph on the torus $\mathbb{T} \cong \mathbb{R}^2/\mathbb{Z}^2$, gives an NCCR of a **three dimensional Gorenstein toric ring**.



- The path algebra of Q with certain relations $\Lambda := \mathbb{C}Q/\langle\text{relations}\rangle$.

If Γ satisfies the "consistency condition", then

- The center $R := \mathrm{Z}(\Lambda)$ of Λ is a 3-dimensional Gorenstein toric ring.
- Λ is an NCCR of R, especially $\mathrm{D^b}(\mathrm{coh}\, Y) \cong \mathrm{D^b}(\mathrm{mod}\, \Lambda)$.

Problems and Expectations

- The existence of NCCRs for higher dimensional toric rings is not known except a few cases.
- I hope if a toric ring R is obtained from a **"good combinatorial object"**, then it would help us to construct NCCRs of R.

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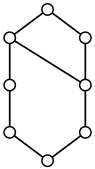
$\rightsquigarrow$ We consider a **Hibi ring** which is a toric ring arising from a **partially ordered set (= poset)**.

NCCRs of Hibi rings



the Hasse diagram of a poset P

the cone σ_P ass. to P

the Hibi ring $R = \mathbb{C}[\sigma_P^\vee \cap \mathbb{Z}^d]$

Theorem (N., Higashitani-N.)

Let R be a Hibi ring associated with a poset P.
Assume that R satisfies one of the following conditions:

- *the divisor class group* $\mathrm{Cl}(R)$ *is* $\mathbb{Z}$ *or* $\mathbb{Z}^2$,
- *R is isomorphic to the coordinate ring of the Segre embedding:* $\underbrace{\mathbb{P}^r \times \cdots \times \mathbb{P}^r}_{t} \hookrightarrow \mathbb{P}^{(r+1)^t-1}$.

Then, a Hibi ring R has an NCCR.