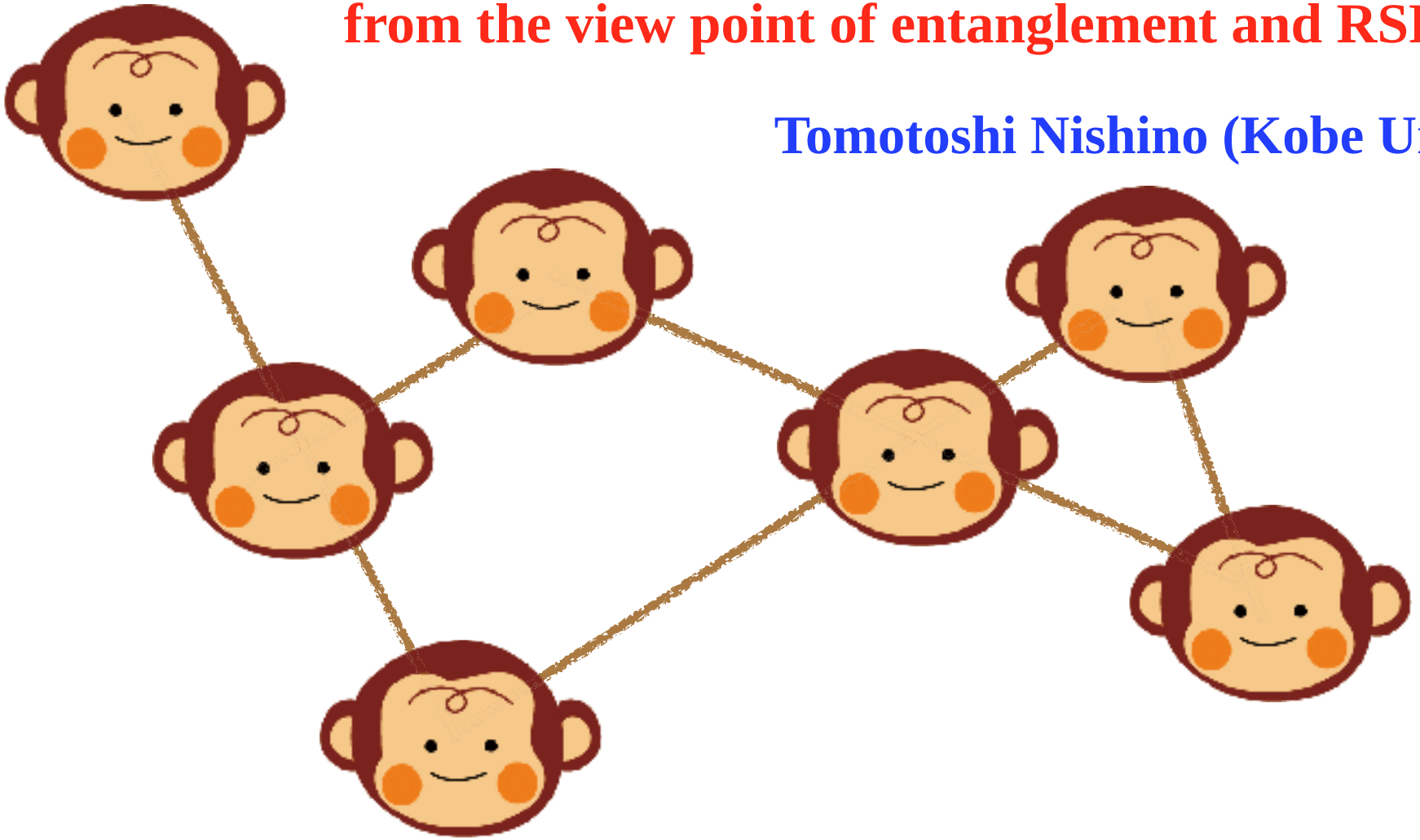
Tensor Network Formulations

from the view point of entanglement and RSRG

Tomotoshi Nishino (Kobe Univ.)



Realistic physical systems are normally entangled weakly.

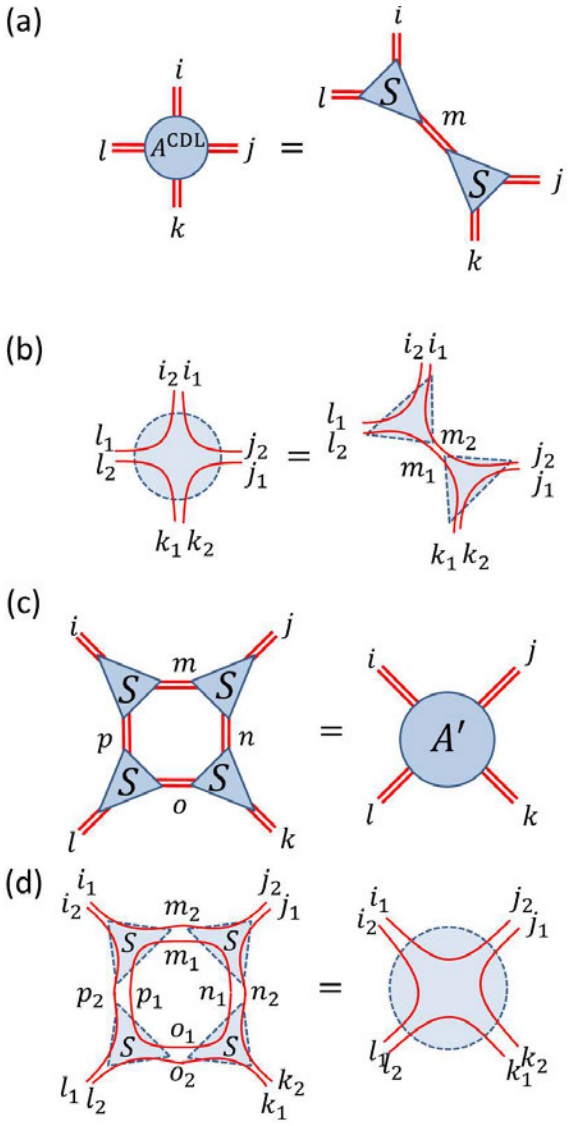
This property enables us to apply RG transformation.

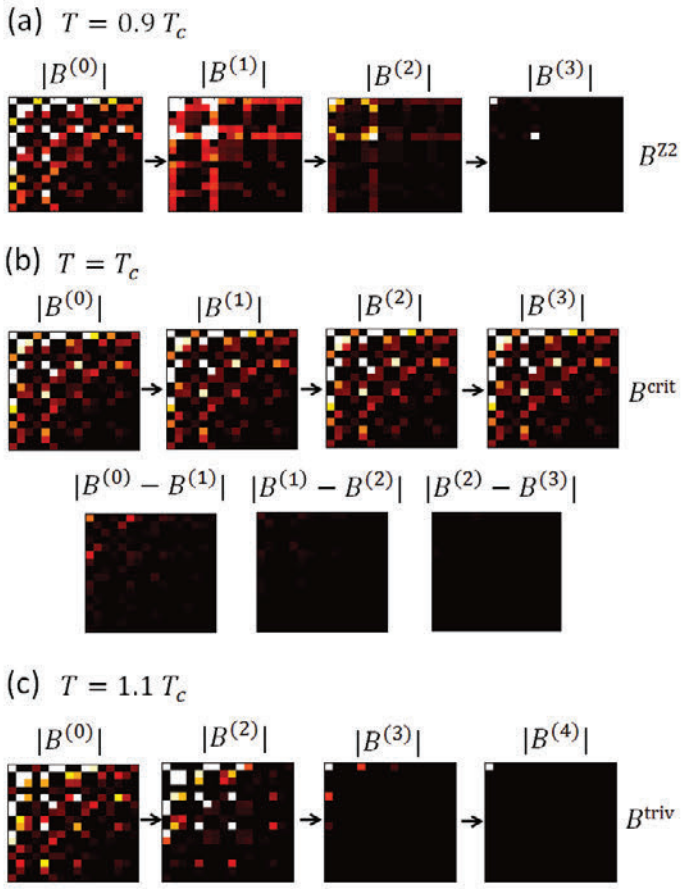
Recent? Progresses

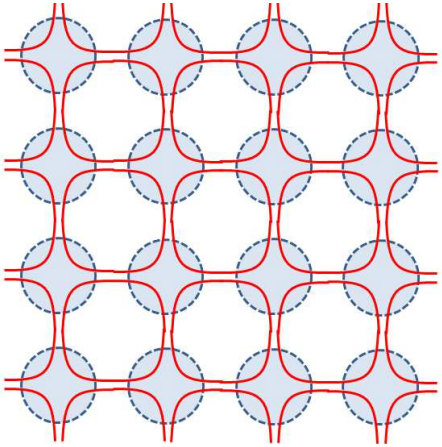
Evenly and Vidal

arXiv:1412.0732

Real Space Renormalization Group is of use!







RG flow is accurately traced at criticality.

Fixed point is numerically captured.

from recent numerical results by means of Tensor Network

2D Ising Model: Transition temperature can be obtained with 8 digits.
3D Ising Model: obtained Tc = 4.511546 (Monte Carlo: 4.511544)

(T. Xiang et al: arXiv:1201.1144)

(Evenbly and Vidal: arXiv:1412.0732)

estimation of conformal weights for 2D Ising Model

	exact	TRG(64)	TRG+env(64)	TEFR(64)	TNR(24)
c	0.5	0.49982	0.49988	0.49942	0.50001
σ	0.125	0.12498	0.12498	0.12504	0.1250004
ϵ	1	1.00055	1.00040	0.99996	1.00009
	1.125	1.12615	1.12659	1.12256	1.12492
	1.125	1.12635	1.12659	1.12403	1.12510
	2	2.00243	2.00549	-	1.99922
	2	2.00579	2.00557	-	1.99986
	2	2.00750	2.00566	-	2.00006
	2	2.01061	2.00567	-	2.00168

TABLE I. Exact values and numerical estimates of the central charge c and lowest scaling dimensions of the critical Ising model. TRG results are obtained using the original Levin and Nave's algorithm [4]. TRG+env results are obtained using an improved TRG method proposed in Ref. [5] under the name of "poor-man's SRG". TEFR results are taken from Ref. [7]. The first three numerical columns use bond dimension $\chi \equiv D_{cut} = 64$ and 1024 spins, while the TNR data uses $\chi = 24$ and 262,144 spins.

A Big Step in Real-Space RG

"Tensor" in tensor network formulation

$B_{ijk\ell}$

* **Elements are complex scalars.**
* **1-leg (vector), 2-leg (matrix), 3-leg, 4-leg, ….**
* **Not all of the legs are (purely) physical.**
* **Please forget about covariance and symmetries.**

(occasionally there are symmetries, though.)

* **A tensor may represent a certain area in the system.**

Contraction among tensors

$$\sum_{k\ell} B_{ijk\ell}\, A_{k\ell}$$

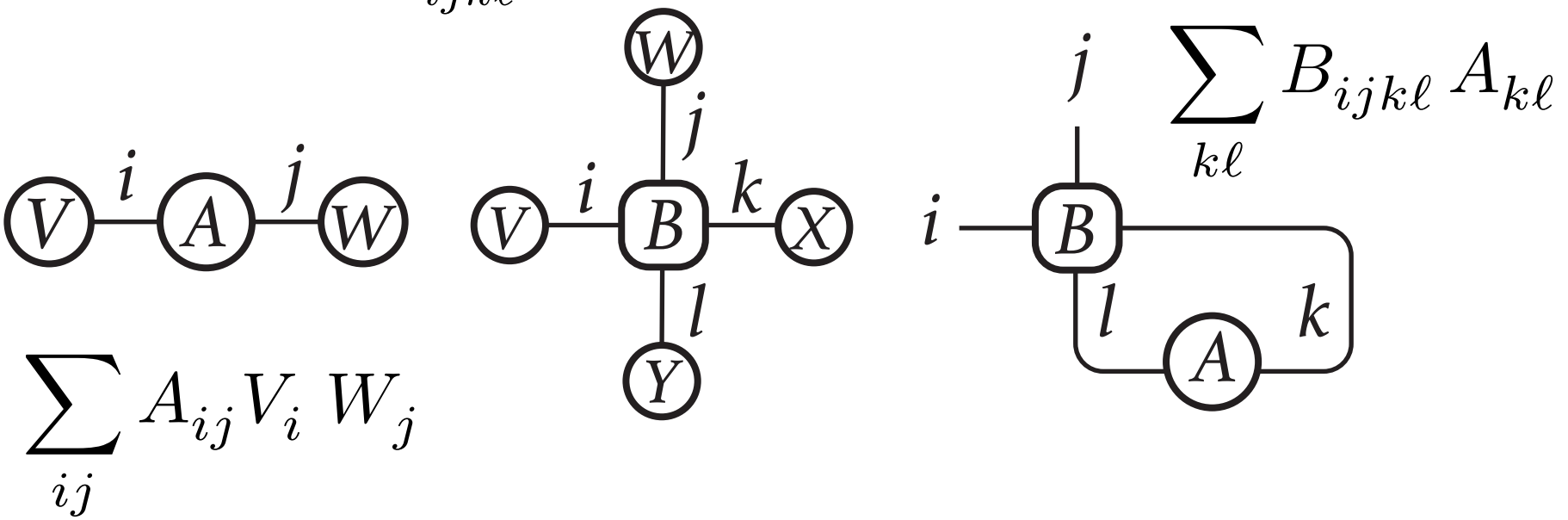
$$\sum_{ij} A_{ij} V_i\, W_j$$

$$\sum_{ijk\ell} B_{ijk\ell}\, V_i\, W_j\, X_k\, Y_\ell$$

* **We obtain a scalar or tensor, as a result.**

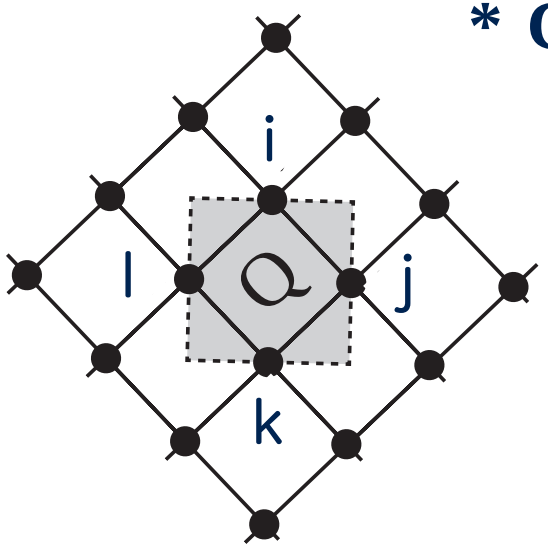
Diagrams

$$\sum_{ijk\ell} B_{ijk\ell}\, V_i\, W_j\, X_k\, Y_\ell$$



* **Tensor legs are represented by lines.**
* **Connected lines represent contraction.**
* **Open lines correspond to remaining indices.**
* **Diagrams are more often used than equations.**

Example: Tensors in Statistical Physics



* **Consider the square-lattice Ising Model.**

(We consider the diagonal lattice.)

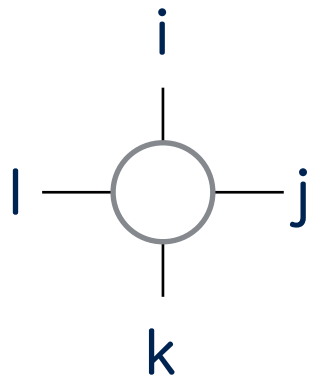
* **Black Dots: Ising spin variables.**

i, j, k, l are either 1 or -1.

* **Boltzmann Weight for the shaded region:**

$$Q_{ijkl} = \exp[\, J(ij+jk+kl+li)/kT \,]$$

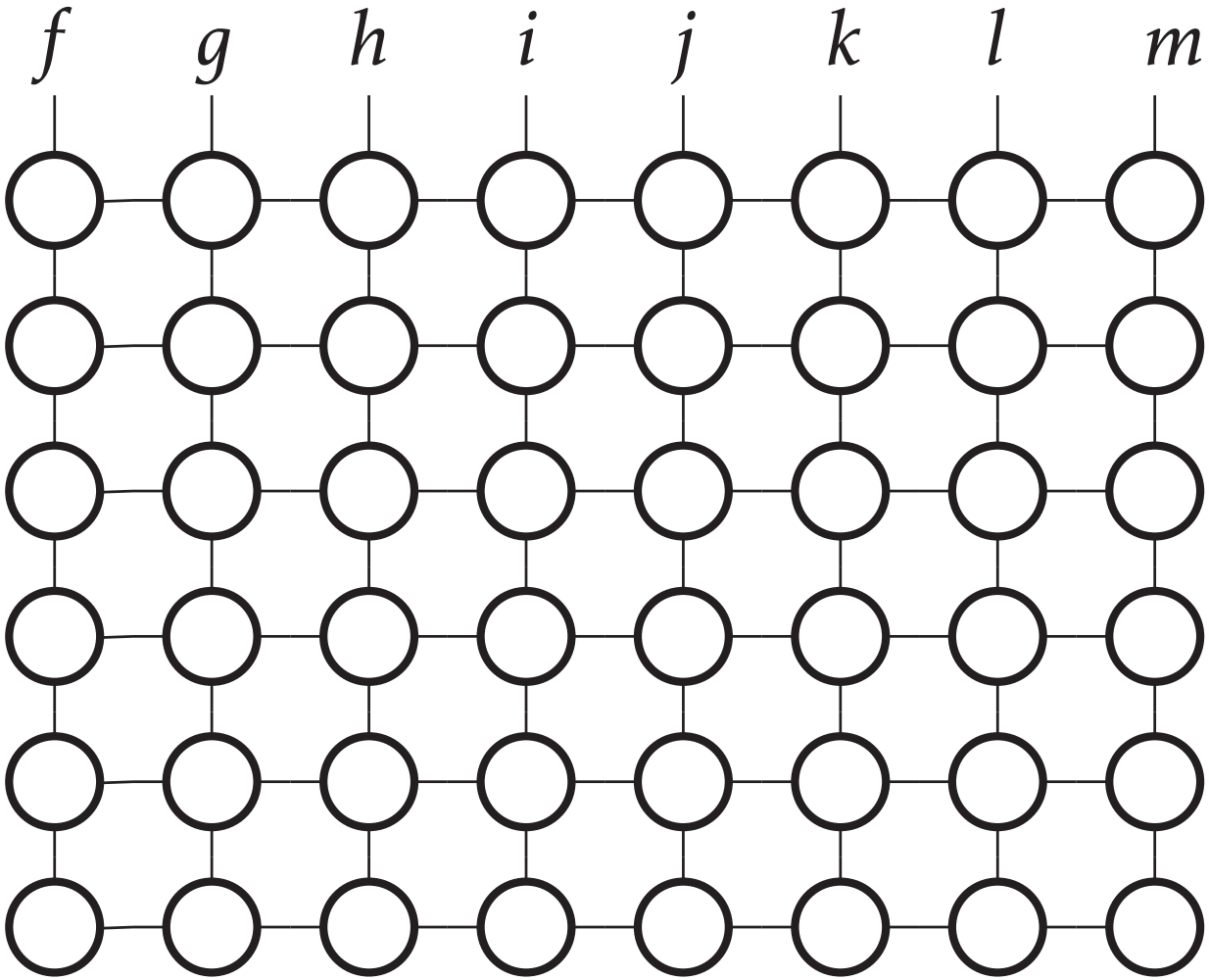
… this is already a 4-leg tensor.



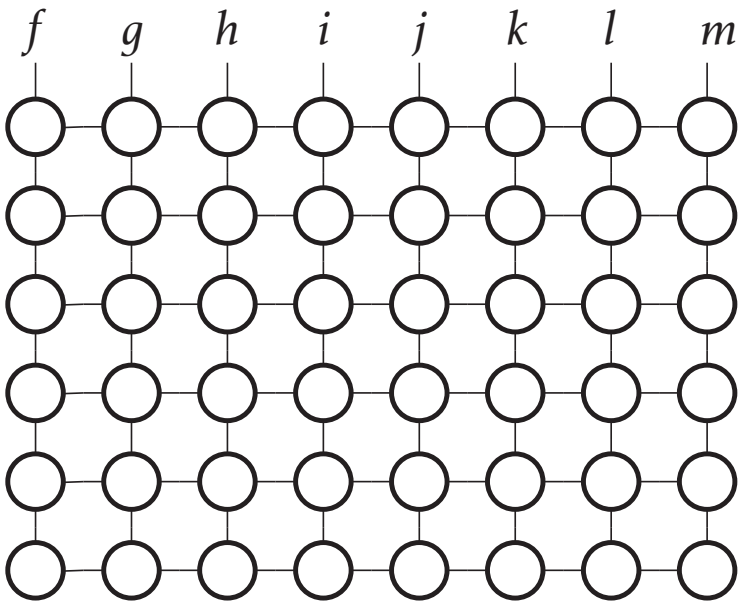
* **In case we consider models with continuous local degrees of freedom, tensor legs are also continuous.**

an example of the Tensor Network

* Legs at the top of the diagram are “alive”.



* There are various interpretations on the network diagram.



2D Ising Model

* Partition function of the square lattice Ising model with open boundary conditions for both sides and the bottom, and fixed ones at the top.

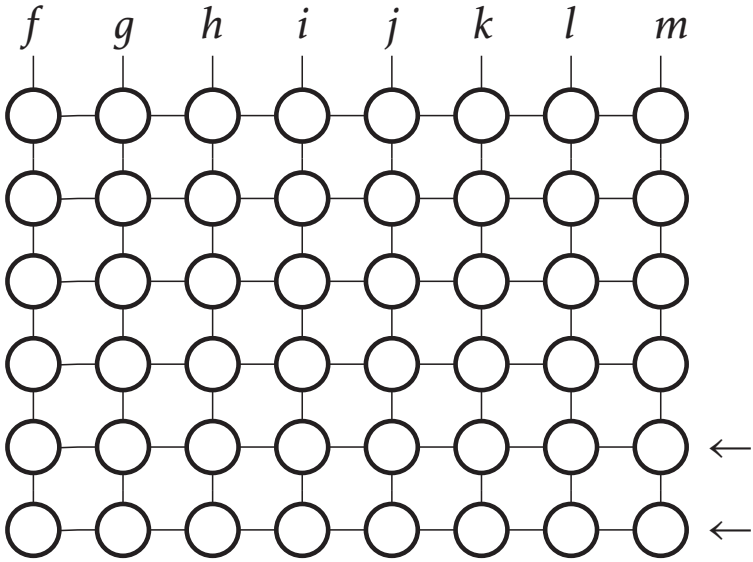
* **Tensor network is NOT new at all. It has been known for more than 70 years.**

1D Quantum Ising Model

* Through the quantum-classical correspondence (= discrete Path Integral), the above diagram can be interpreted as a quantum wave function.

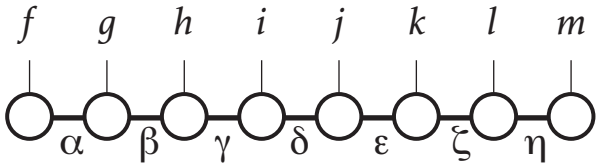
identified as a wave function

$$|\Psi\rangle = \sum_{f\sim m} \psi_{fghijk\ell m} |fghijk\ell m\rangle$$



* Look at the bottom of the diagram.

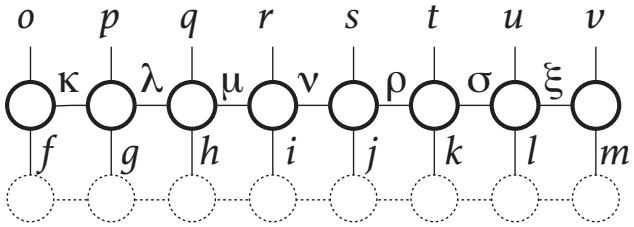
Matrix Product State (MPS)



$$|\Phi\rangle = \sum \phi_{fghijk\ell m} |fghijk\ell m\rangle$$
$$= \sum A^f_\alpha A^g_{\alpha\beta} A^h_{\beta\gamma} A^i_{\gamma\delta} A^j_{\delta\varepsilon} A^k_{\varepsilon\zeta} A^\ell_{\zeta\eta} A^m_\eta |fghijk\ell m\rangle$$

* Roman letters: physical indices
* Greek letters: auxiliary indices.

Matrix Product Operator (MPO) ~ Transfer Matrix



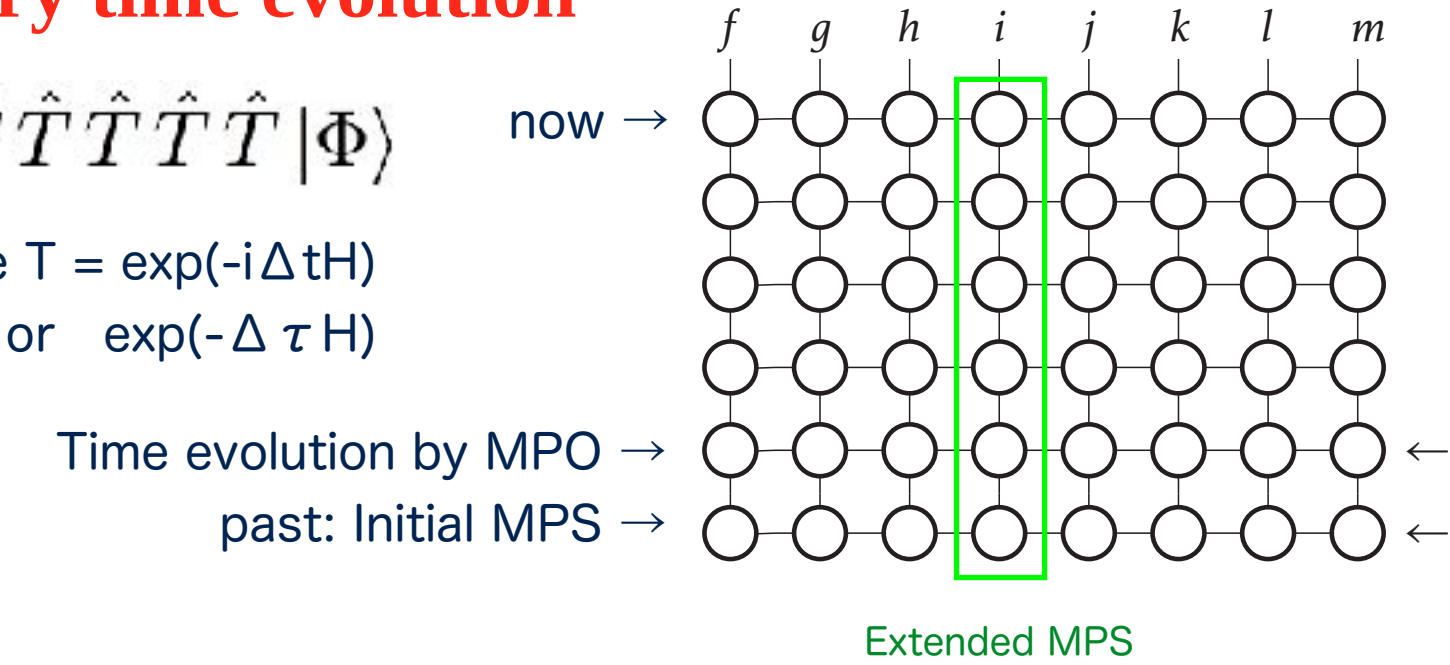
$$T^{opqrstuv}_{fghijk\ell m} = \sum_{\kappa\sim\xi} B^o_{f\kappa} B^p_{g\kappa\lambda} B^q_{h\lambda\mu} B^r_{i\mu\nu} B^s_{j\nu\rho} B^t_{k\rho\sigma} B^u_{\ell\sigma\xi} B^v_{m\xi}$$

real/imaginary time evolution

$$|\Psi\rangle = \hat{T}\hat{T}\hat{T}\hat{T}\hat{T}|\Phi\rangle$$

where T = exp(-iΔtH)

or exp(-Δτ H)



IT IS NOT EASY TO CONTRACT TENSORS!!

* Number of terms increases exponentially with respect to the number of contracted bonds.
* Dimension of wave function increases exponentially with respect to the size of the system.
* It is hopeless to handle Hilbert space directly.

From K-W approximation to TPS/PEPS

a Pioneering work: Kramers-Wannier Approx. Phys. Rev. 60, 263 (1941)

* Eigenvector of the transfer matrix of 2D Ising model is approximated by 2-state MPS.
* Tensor elements are adjusted so that variational partition function is maximized.
* One of the earliest example of "Numeircal Physics".

Statistics of the Two-Dimensional Ferromagnet. Part II

H. A. KRAMERS, *University of Leiden, Leiden, Holland*

AND

G. H. WANNIER, *University of Texas, Austin, Texas*

(Received June 12, 1941)

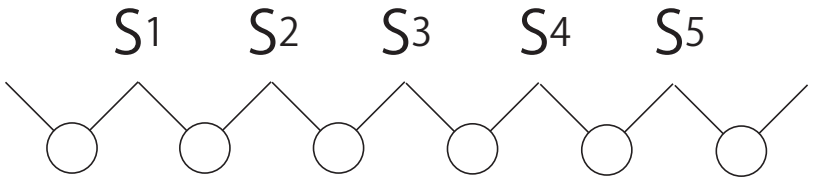
The study of the two-dimensional Ising model is continued. Its specific heat at the Curie point is investigated. The quantity in question is computed for six successive finite matrix problems and the conclusion is drawn that the specific heat is infinite at the Curie point. A new closed form approximation of the partition function λ is then developed by using the matrix method in its variational form. The two power series for λ at extreme temperatures are used as a test for this and various other approximations, and it is found that the new result is a considerable improvement over the existing solutions. Finally it is pointed out that these closed form solutions support our conclusion as to the place and nature of the Curie point transition.

Kramers-Wannier Approximation

Phys. Rev. 60, 263 (1941)

Estimate the partition function of 2D Ising model as Rayleigh ratio for the transfer matrix with respect to a variational state written in terms of matrix (= 2-leg tensor) product.

$$V(S_1, S_2, \ldots, S_N) = M(S_1, S_2)\, M(S_2, S_3) \ldots\ldots M(S_{N-1}, S_N)$$



Rayleigh Ratio

$$\lambda^n = \operatorname*{Max}_{c,\,d} \frac{\sum_{\mu_i \mu_i'} \mathcal{K}(\mu_i, \mu_i')\, c(\mu_i)\, d(\mu_i')}{\left\{\left[\sum_{\mu_i} c^2(\mu_i)\right]\left[\sum_{\mu_i'} d^2(\mu_i)\right]\right\}^{\frac{1}{2}}},$$

Maximize λ adjusting M

* V is equivalent to 1D Ising model under external field.

Numerical Physics (by hand!)

During the calculation, one has to treat 6-leg Ising ladder, and to obtain the maximal eigenvector of the transfer matrix. This was obtained by the Power method, performed by hand.

TABLE I. *Values of eigenvectors.*

VECTOR A_2	VECTOR A_5	VECTOR A_6	
1.000000	1.000000	1.000000	.094527
.249038	.145548	.137739	.017936
	.100846	.098916	.009995
VECTOR A_3	.129103	.112698	.017578
	.098804	.097652	.010148
1.000000	.019608	.016676	.002709
.186931	.075422	.071758	.008510
.105677	.148388	.114363	.022371
.231108	.096230	.096454	.065628
	.020594	.016343	.017673
VECTOR A_4	.011008	.010661	.008320
	.023617	.016817	.022060
1.000000	.070174	.069010	.056461
.159434	.023648	.016516	.022385
.103346	.066754	.062185	.060217
.160827	.218677	.141720	.216286
.099340			
.026148			
.079123			
.222893			

… who did it?

Specific Heat

H. A. KRAMERS AND G. H. WANNIER Phys. Rev. **60**, 263 (1941)

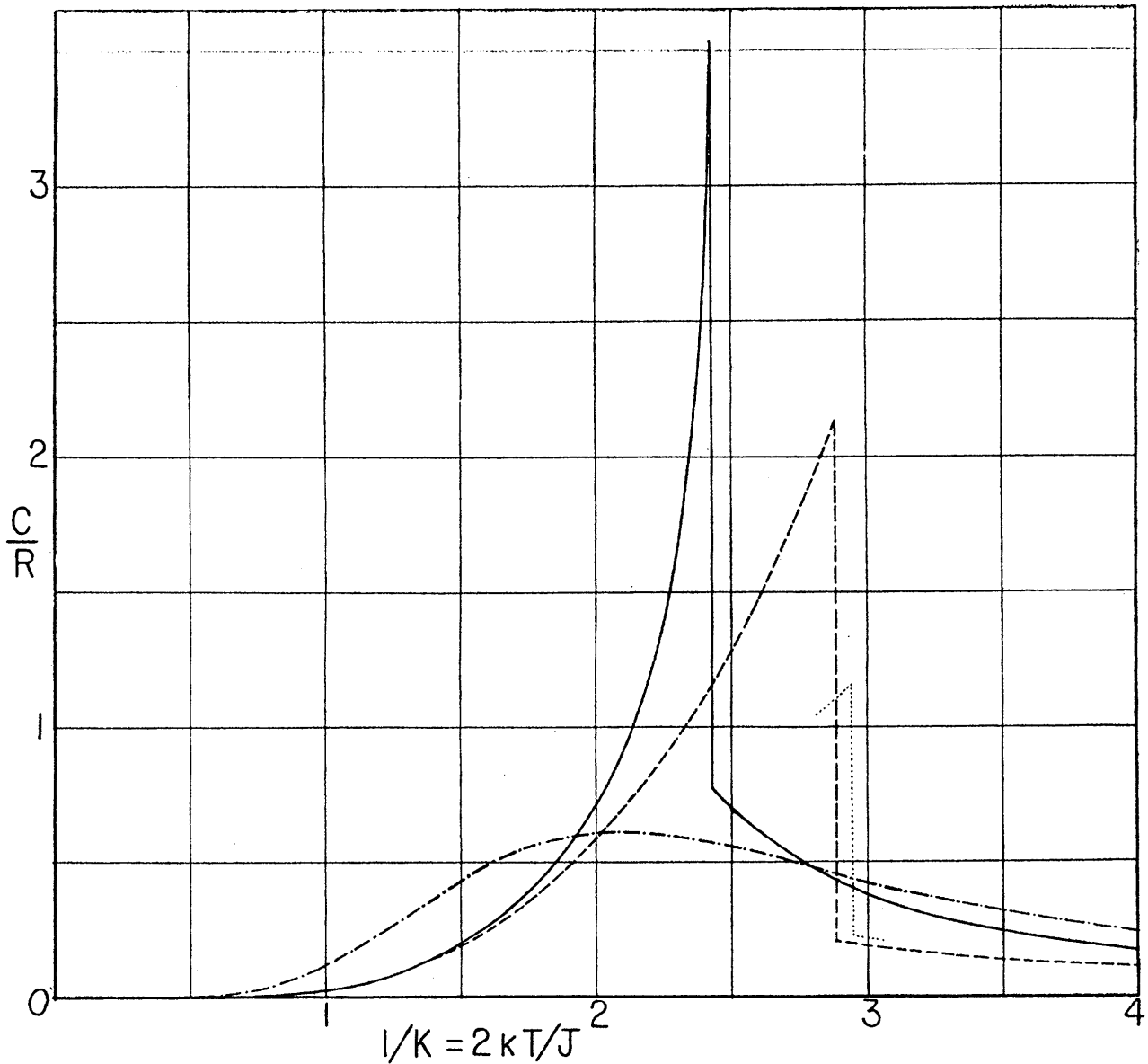


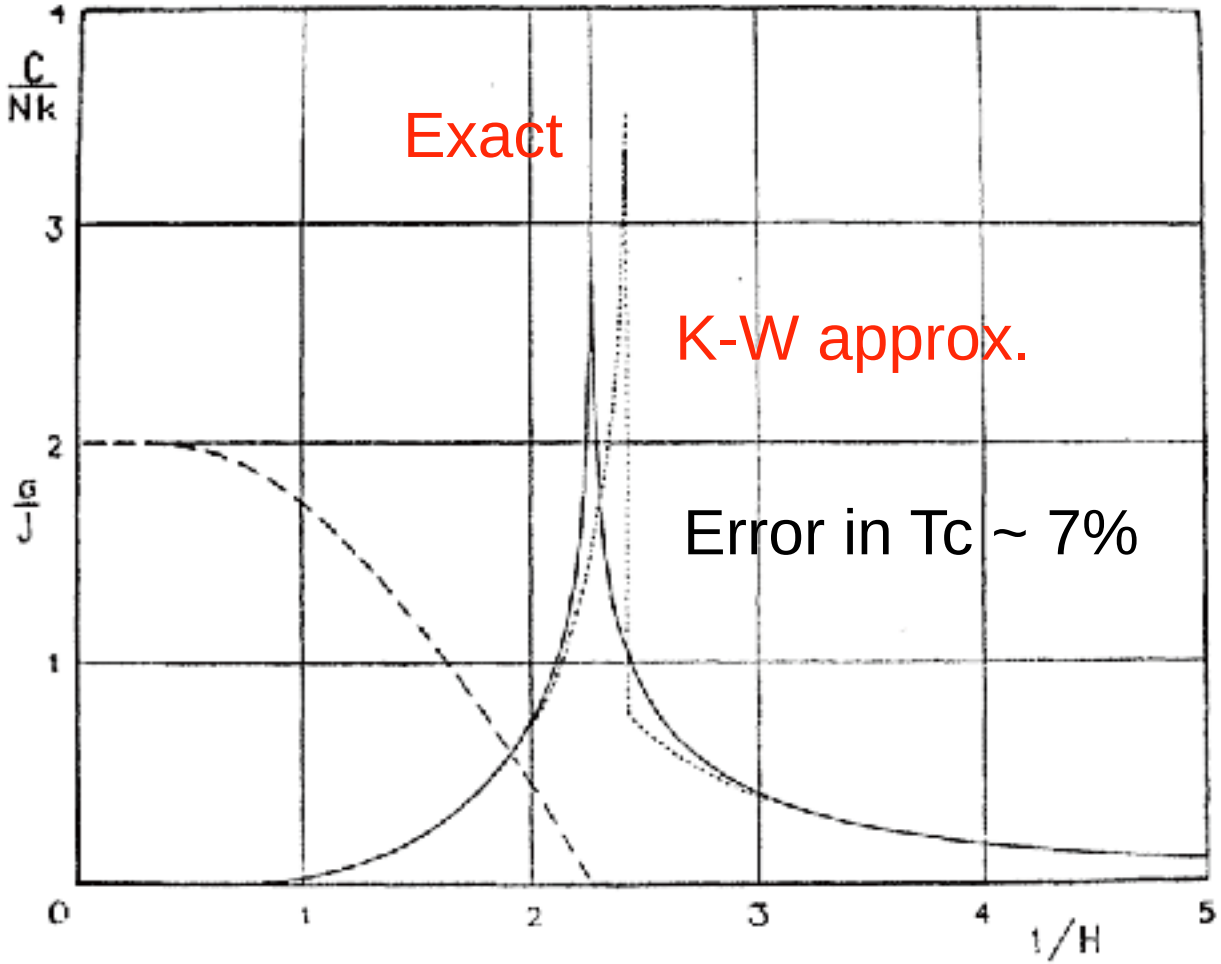
FIG. 8. Specific heat *versus* temperature curves. -·-·-· 2-screw approximation. ——— Variation method. – – – – Bethe method. ····· Kirkwood method.

K-W approximation and exact solution

L. Onsager, Phys. Rev. 65, 117 (1943)

Two ancestors of modern analyses

Note that difference in Free energy is very small.



These two view points are unified later by Baxter.

K-W approx. applied to 3D Ising Model

(cond-mat/9909097)

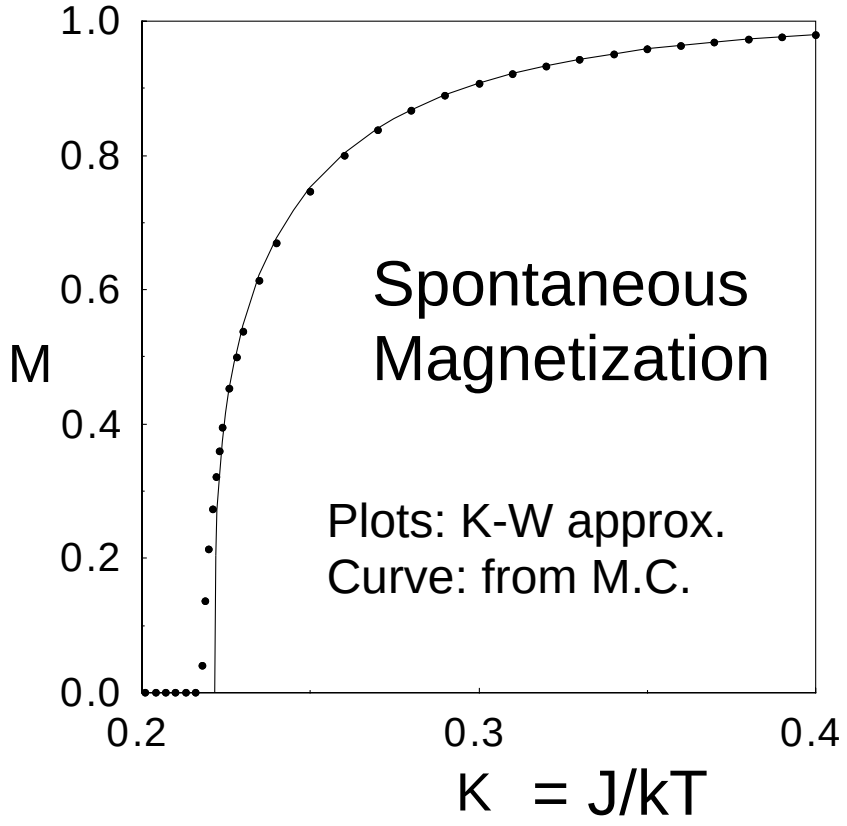


Fig. 2. The magnetization M as a function of K. The solid line denotes M^{MC} in Eq. (3·4) obtained from Monte Carlo simulations.[22),23)] In the vicinity of the calculated transition point, M is proportional to $\sqrt{K - K_{\mathrm{c}}}$.

Variational state is 2D Ising model under finite external field.

The K-W approximation works better in higher dimensions.

(So far, no application in 3+1 dimensional system.)

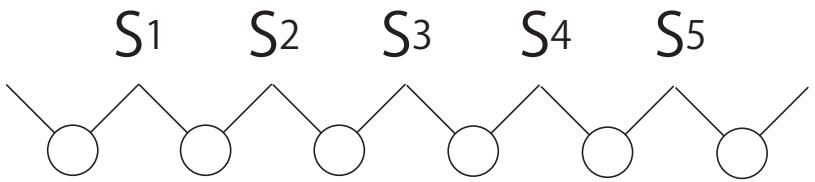
Quantum Systems

… a Similar variational state was considered for 1D quantum spin chains.

Nightingale and Bloete, Phys. Rev. B 33, 659 (1986)

As a variational form of the ground state for a given quantum number S of the z component of the total spin, we choose, again using the representation in which the s_i^z are diagonal,

$$\mathbf{v}(s_1, \ldots, s_n) = \prod_{i=1}^{h} A(s_i, s_{i+1}) \delta_{S, \sum_{i=1}^{n} s_i} , \qquad (9)$$



This article is known as the numerical proof (?) of the existence of the Haldane gap. (We return to this article again.)

Generalization by Baxter

J. Math. Phys. **9**, 650 (1968)

introduction of auxiliary variable

Dimers on a Rectangular Lattice

R. J. BAXTER
Research School of Physical Sciences, The Australian National University, Canberra, Australia

(Received 17 July 1967)

A set of matrix equations is derived which yields the statistical mechanical properties of a system of monomers and dimers on a rectangular lattice in the thermodynamic limit. As the matrices are strictly of infinite dimensionality, the equations cannot be solved directly, but if they are restricted to be of finite and quite small dimensionality, very good approximations to the thermodynamic properties are obtained.

Numerical!

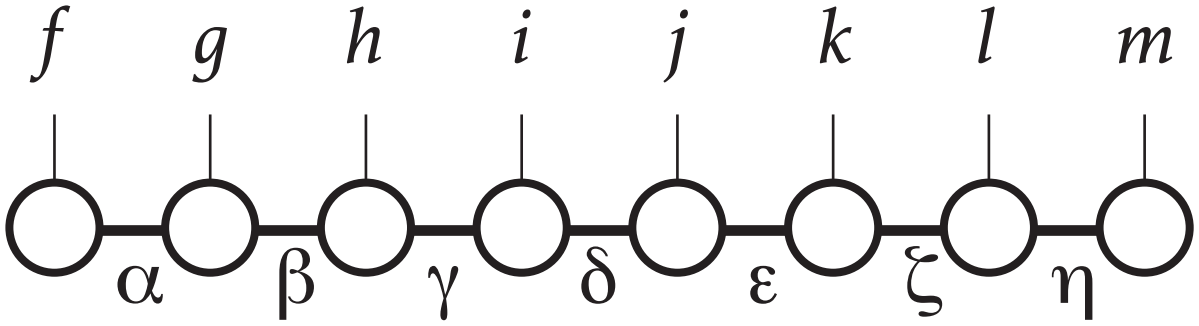
The calculations were performed on an IBM 360 computer using double-precision floating-point arithmetic accurate to 16 decimal places. A relative error for each of the $3r^2$ equations, represented by (3.19)–(3.21), was defined as the difference between the right- and left-hand sides divided by the absolute sum of all the additive terms in the equation (including the individual additive contributions to the matrix products). The iterations were assumed to have converged only when the relative error for each equation was less than 10^{-10}. At each iteration, the values of κ and ρ obtained from (3.12) and (3.24) were evaluated and no change was observed in their first ten significant figures during the last iteration. It is therefore believed that the numerical values obtained for these quantities are accurate to ten significant figures.

A set of matrix equations is derived which yields the statistical mechanical properties of a system of monomers and dimers on a rectangular lattice in the thermodynamic limit. As the matrices are strictly of infinite dimensionality, the equations cannot be solved directly, but if they are restricted to be of finite and quite small dimensionality, very good approximations to the thermodynamic properties are obtained.

J. Math. Phys. **9**, 650 (1968)

Matrix Product (Variational) State

$$\phi_{fghijklm} = \sum A^{f}_{\alpha} A^{g}_{\alpha\beta} A^{h}_{\beta\gamma} A^{i}_{\gamma\delta} A^{j}_{\delta\varepsilon} A^{k}_{\varepsilon\zeta} A^{\ell}_{\zeta\eta} A^{m}_{\eta}$$



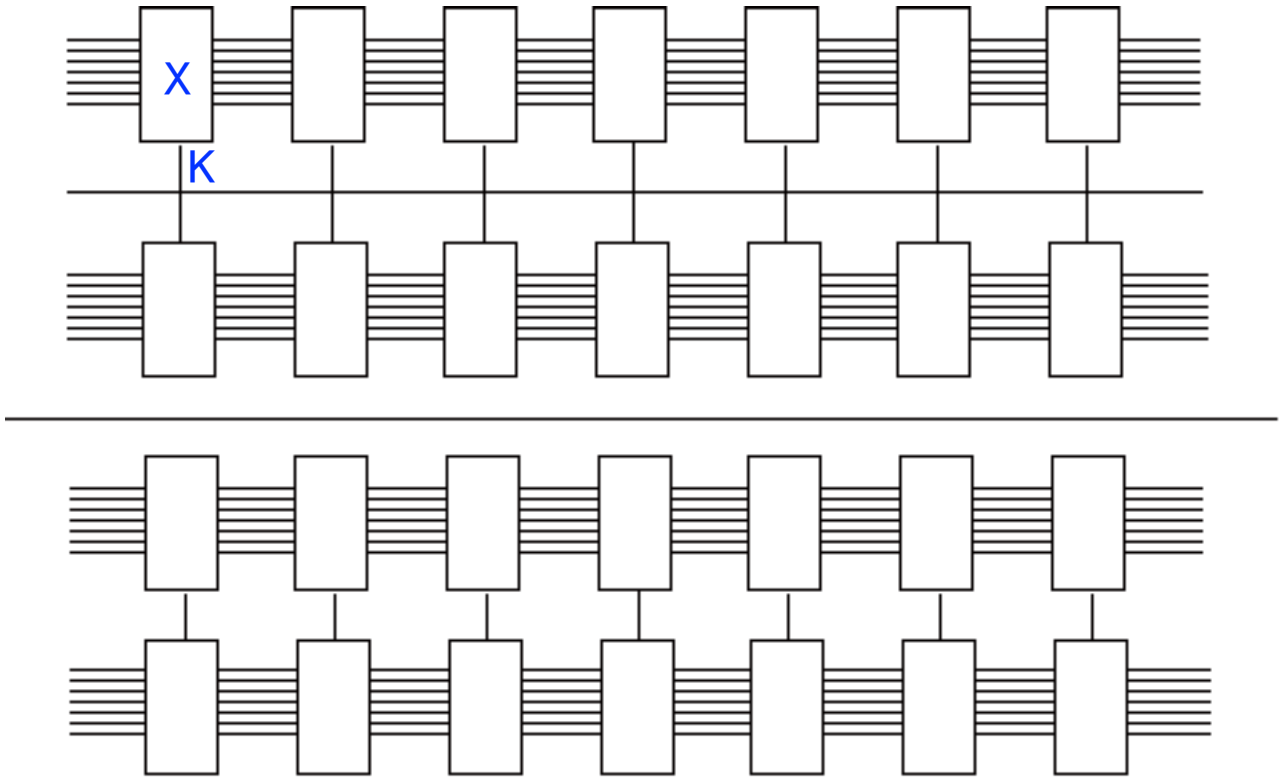
Greek letters represent D-state auxiliary variables.

The state is applied to the dimer model.

Column-to-Column Transfer Matrix $R_{\lambda\mu|\lambda'\mu'} = \sum_{\beta} G^{\beta}_{\lambda\lambda'} G^{\beta}_{\mu\mu'}$

Extended C-T-C Transfer Matrix $S_{\alpha\lambda\mu|\alpha'\lambda'\mu'} = \sum_{\beta,\beta'} K(\alpha, \alpha' \mid \beta, \beta') G^{\beta}_{\lambda\lambda'} G^{\beta'}_{\mu\mu'}$

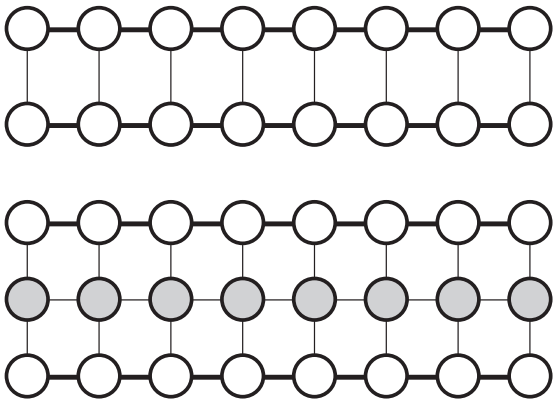
Variational Ratio $\Gamma = \mathbf{x}'\mathbf{V}\mathbf{x}/\mathbf{x}' \cdot \mathbf{x} = \mathrm{Tr}\, \mathbf{S}^n / \mathrm{Tr}\, \mathbf{R}^n$



.... this is an origin of MPS in statistical mechanics .... (another one is by Derrida.)

Expectation values can be calculated easily

※ Variational ratio can be calculated from left to right.

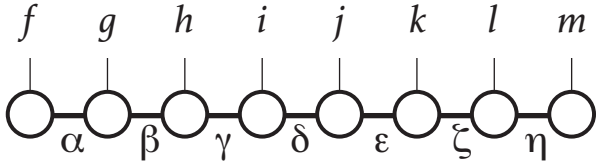


$$E = \frac{\langle \Phi | \hat{H} | \Phi \rangle}{\langle \Phi | \Phi \rangle} \geq E_0$$

Density Matrix Renormalization Group (DMRG) implicitly uses this superiority.

Structure of MPS

f…m run from 1 to d,
$\alpha \cdots \eta$ run from 1 to D.



$$|\Phi\rangle = \sum \phi_{fghijklm} |fghijklm\rangle$$
$$= \sum A^f_\alpha A^g_{\alpha\beta} A^h_{\beta\gamma} A^i_{\gamma\delta} A^j_{\delta\varepsilon} A^k_{\varepsilon\zeta} A^\ell_{\zeta\eta} A^m_\eta |fghijklm\rangle$$

It contains large number of variational parameters that can be tuned.

Baxter's numerical result on the dimer problem

r	κ/s			
	$s = 1.0$	$s = 4.0$	$s = 10.0$	$s = \infty$
1	1.937416664	1.444670083	1.356095932	1.299038106
2	1.940215341	1.460590906	1.381143005	1.335033348
3	1.940215344	1.460623453	1.381458447	1.337338271
4	1.940215351	1.460629381	1.381506501	1.337984697
5	1.940215351	1.460629397	1.381508315	1.338250017
6	1.940215351	1.460629398	1.381508512	1.338380390
Extrapolated	1.940215351	1.460629398	1.381508536	1.338506344

One finds a careful statement on his numerical result.

> It is clearly not possible to deduce rigorously from the above working whether or not the dimer system undergoes a phase transition, but the fact that the successive approximations vary smoothly with s and tend towards the known results at both the high- and low-density limits suggests very strongly that no transition occurs in this system.

... can anyone here can be as careful as Baxter? Apparently, I cannot ...

Baxter's Corner Transfer Matrix (CTM) method has been used since then.

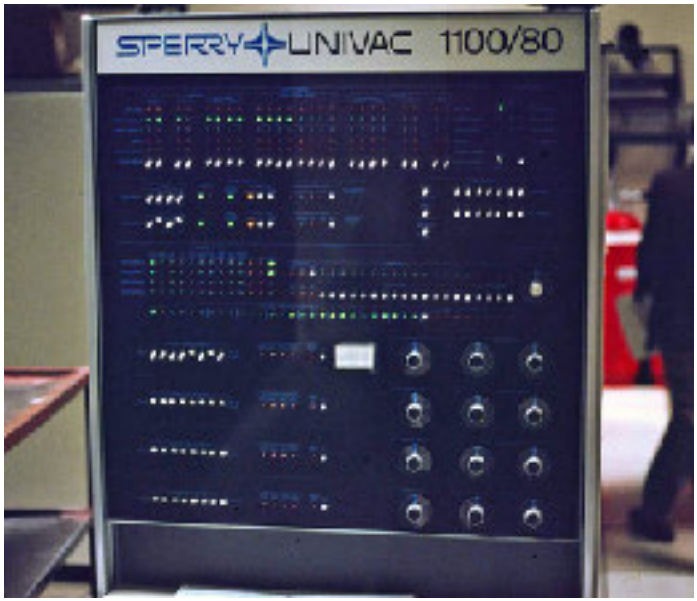
S.K. Tsang, J. Stat. Pays. 20 (1979) 95

Square Lattice Variational Approximations Applied to the Ising Model

S. K. Tsang[1]

Received September 5, 1978

The variational method developed by Baxter is applied to the zero-field Ising model on the square lattice. The problem is simplified to that of solving a relatively small system of nonlinear equations. The estimates to the spontaneous magnetization and the critical temperature from the sequence of variational approximations are obtained. The results converge rapidly to the exact ones. They exhibit a crossover phenomenon and satisfy a scaling relation.



Equation (76) has been solved numerically for a range of values of z below the critical point, using the Newton–Raphson method. The computation was done on a UNIVAC 1100/42 computer, using double-precision floating point arithmetic with 18 significant digits. A solution was assumed to have converged only when the relative change of each variable, through one iteration, was less than 10^{-14}. As a measure of the degree to which an equation was satisfied, we also calculated the difference between the right- and left-hand sides divided by the absolute sum of all the additive terms in the equation. This was never greater than 10^{-16}.

Cross-over in Critical behavior of expectation values / observables

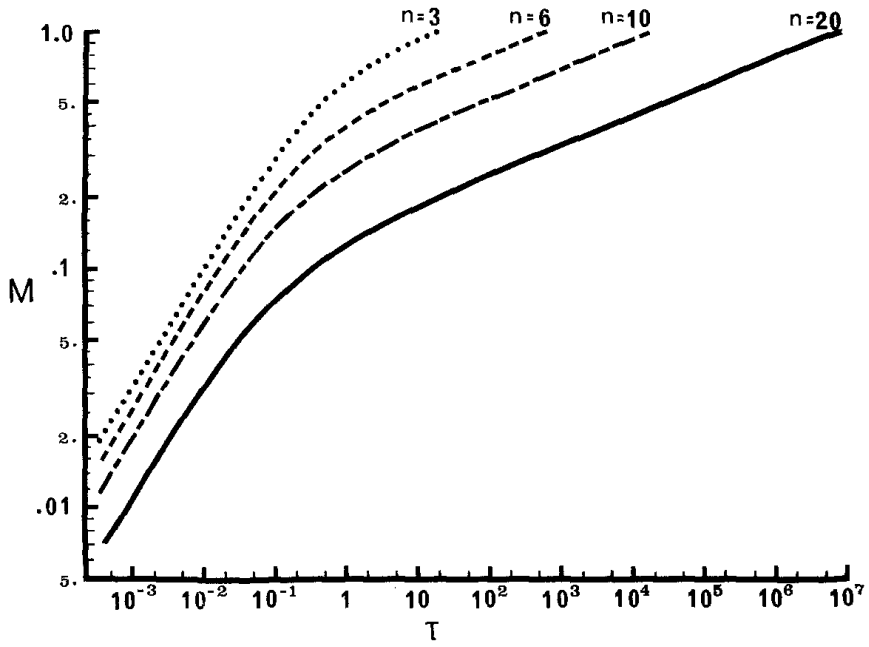


Fig. 2. Log–log plot of spontaneous magnetization vs. the temperature parameter τ for $n = 3, 6, 10, 20$. The crossover can be seen easily even for quite small values of n.

S.K. Tsang, J. Stat. Pays. 20 (1979) 95

Something important is found several times.

Liu et. al, Phys. Rev. B **82**, 060410 R 2010

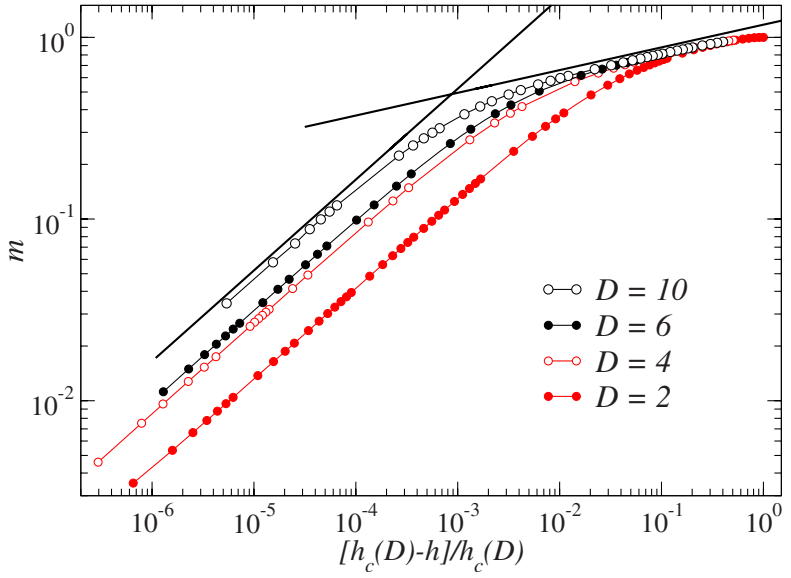


FIG. 4. (Color online) Demonstration of asymptotic MPS mean-field behavior and scaling crossover in one dimension. The D-dependent critical fields are: $h_c/J = 1.0717967$ ($D=2$), 1.0143343 ($D=4$), 1.0063523 ($D=6$), and 1.0021646 ($D=10$). The lines have slopes $\beta = 1/8$ and $1/2$.

History

... they put ancilla, and formed MPS ...

Exact MPS (of very large dimension)

Bethe Ansatz (1931), Onsager (1943)
----> Gaudin, Lieb, Mattis, Yang, Baxter, etc.....

MPS of small dimension (exact/approx.)

AKLT, Fannes, Zittartz, Derrida, etc.... (as we know quite well)

B. Derrida, M.R. Evans, V. Hakim, and V. Pasquier: J. Phys. A Math. Gen. **26** (1993) 1493.

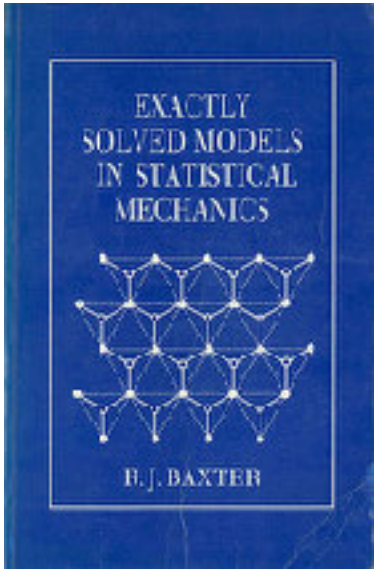
I. Affleck, T. Kennedy, E.H. Lieb, and H. Tasaki: Phys. Rev. Lett. **59** (1987) 799.

M. Fannes, B. Nachtergale, and R.F. Werner: Europhys. Lett. **10** (1989) 633.

Nightingale and Bloete, Phys. Rev. B 33, 659 (1986)
den Nijs and Rommelse, Phys. Rev. B 40, 4709 (1989)

CTM method is related to Density Matrix (and entanglement)

Later in 1982, Baxter states in his textbook, about the density matrix



Note from (13.1.11) and (13.1.17) that a significant variable is

$$\rho_1 = \text{Trace } A_d B_d C_d D_d \,. \tag{13.8.19}$$

The n eigenvalues of $A_d B_d C_d D_d$ are contained in the $2n$ eigenvalues of $\kappa^{-4} A_t B_t C_t D_t$, and the largest of each is unity. Let

$$\lambda = \text{largest eigenvalue of } \kappa^{-4} A_t B_t C_t D_t \text{ omitted from } A_d B_d C_d D_d \,. \tag{13.8.20}$$

Then in some sense this λ is a measure of the relative error in ρ_1 caused by truncating the equations to finite n. Since $\langle \sigma_1 \rangle$ is a derivative of $\ln \kappa$ and from (13.1.17) is proportional to ρ_1, this suggests that

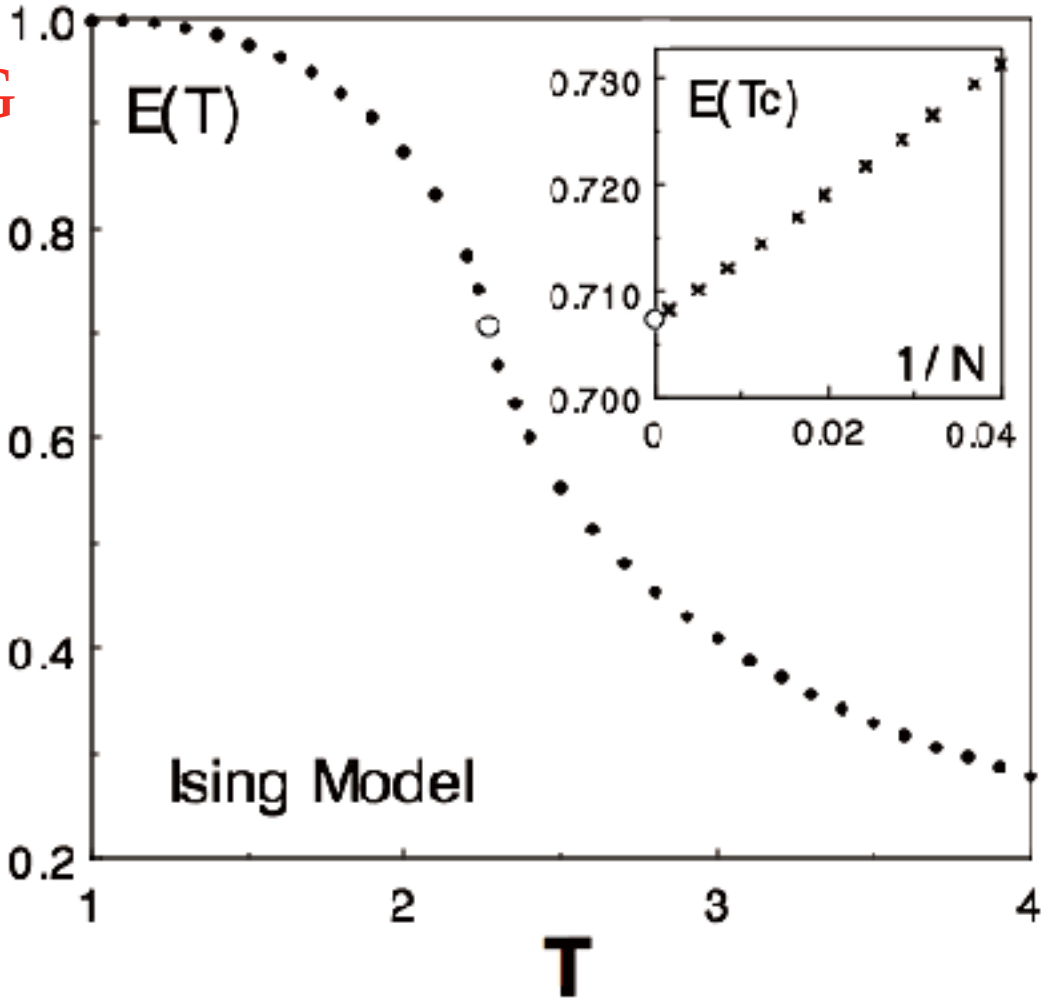
$$\text{relative error in } \kappa \simeq \lambda \,. \tag{13.8.21}$$

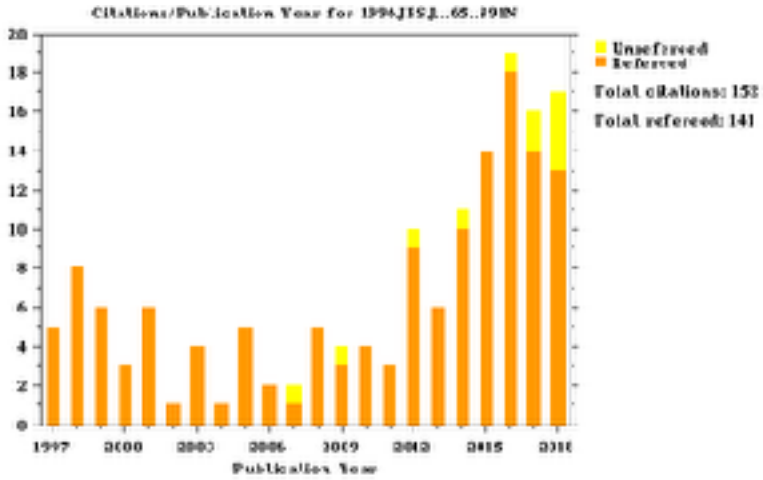
In 1986, Nightingale and Bloete stated in Phys. Rev. B **33**, 659 (1986),

> This method (*) was formulated by Baxter for classical models in statistical mechanics. The generalization to quantum mechanical systems is straightforward.

Density matrix renormalization group (DMRG) was established just 6 years later.

Corner Transfer Matrix RG
(Baxter CTM + DMRG)





arXiv:cond-mat/9507087

Okunishi and Nishino (1996)

… when i was living in Sendai …

Ising model on (5, 4) lattice can be studied by CTMRG

$$H = - J \sum \sigma_i \sigma_j$$

(i and j are NN sites)

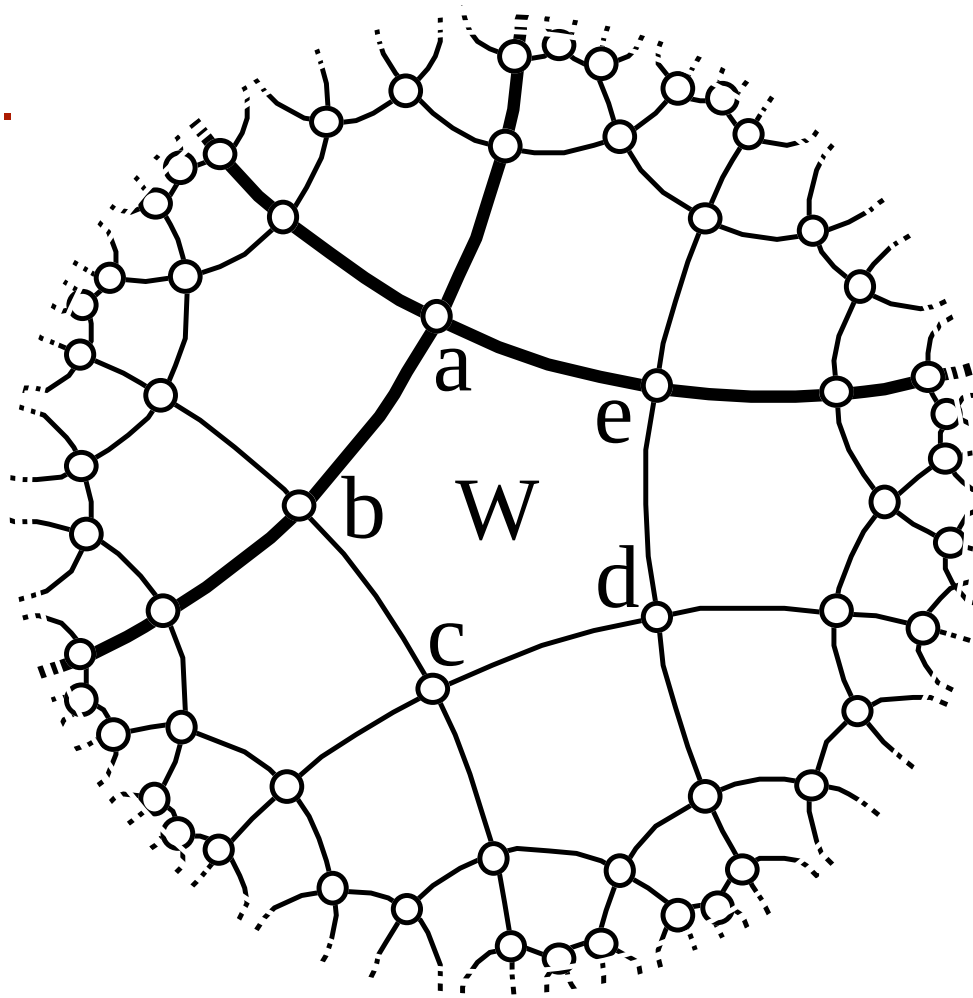
The space is negatively curved.

All the sites are equivalend

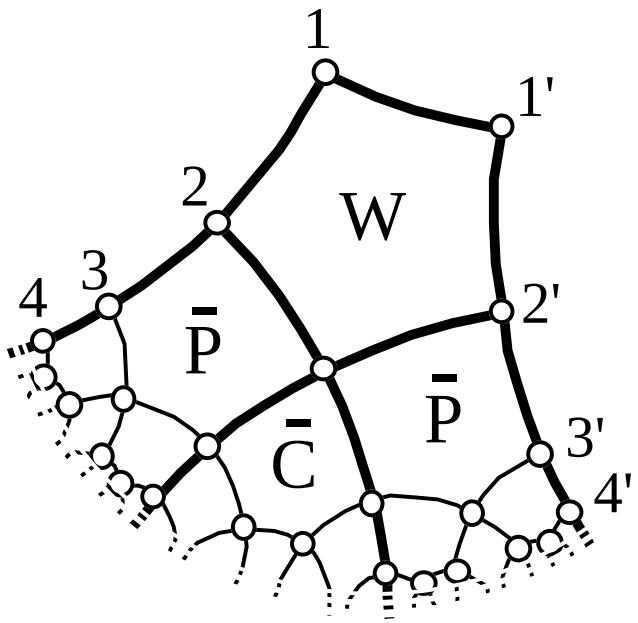
This is also an example
of the Tensor Network

The lattice has recursive nature, which enables us to apply CTMRG.

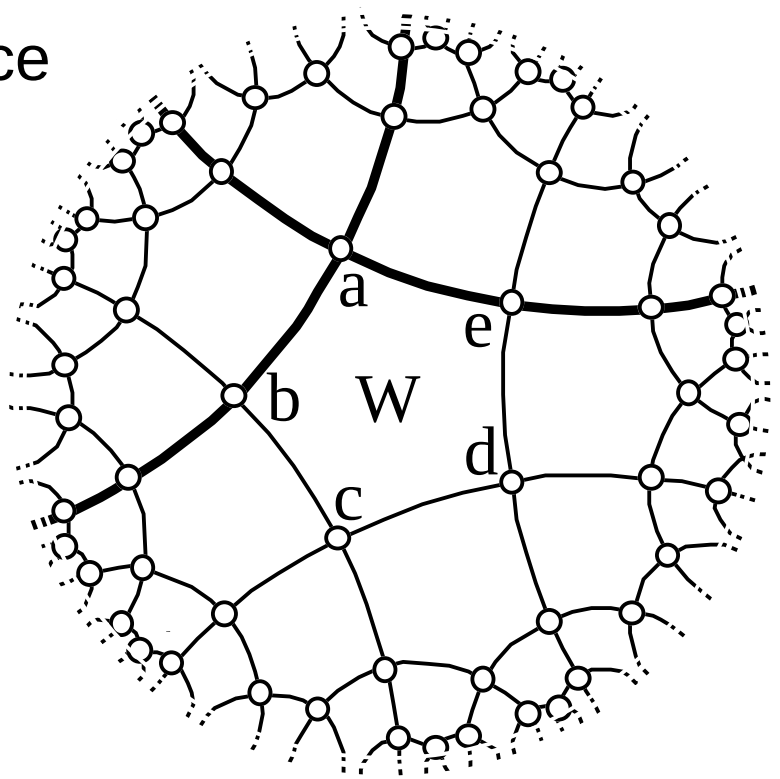
It turned out that the phase transition is mean-field like.



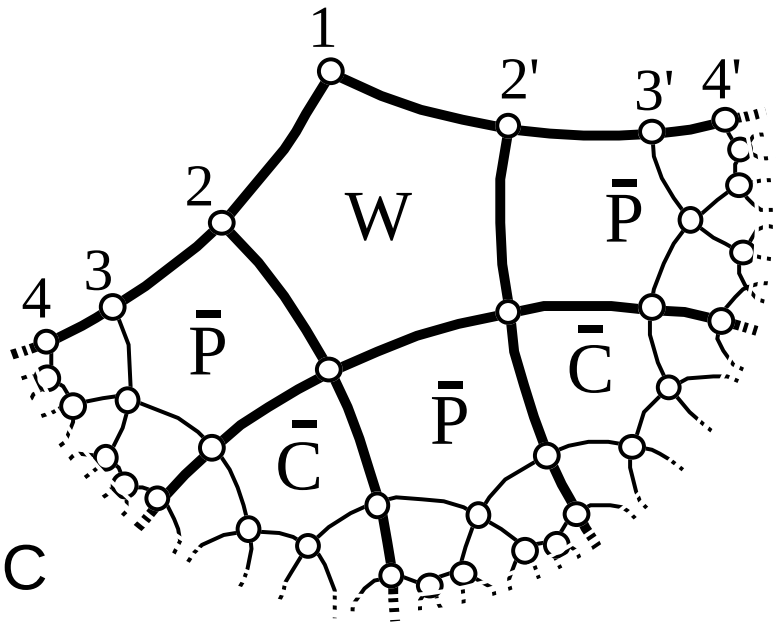
◎ Recursive Structure of the (5,4) lattice



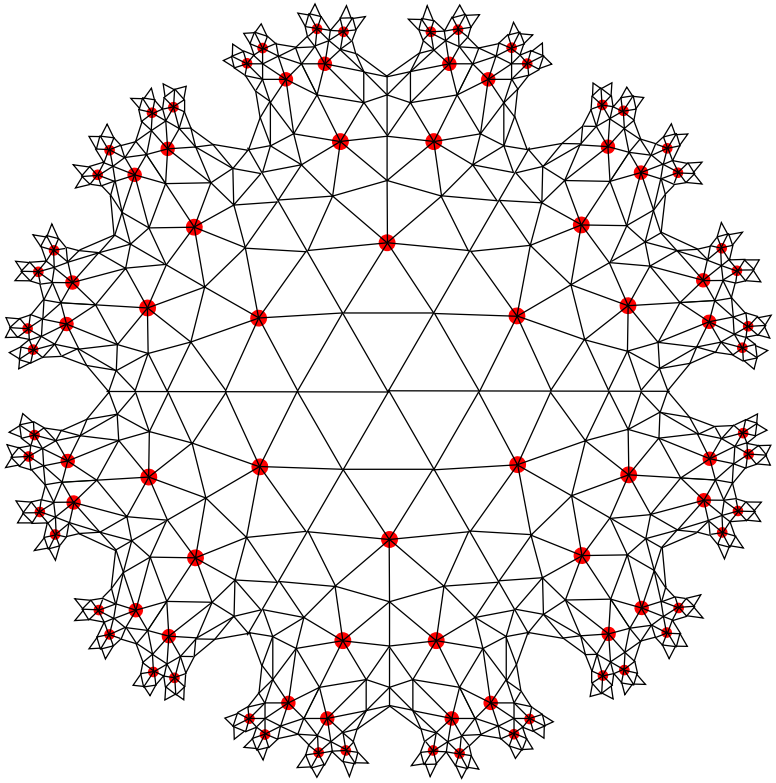
Structure of the Lattice

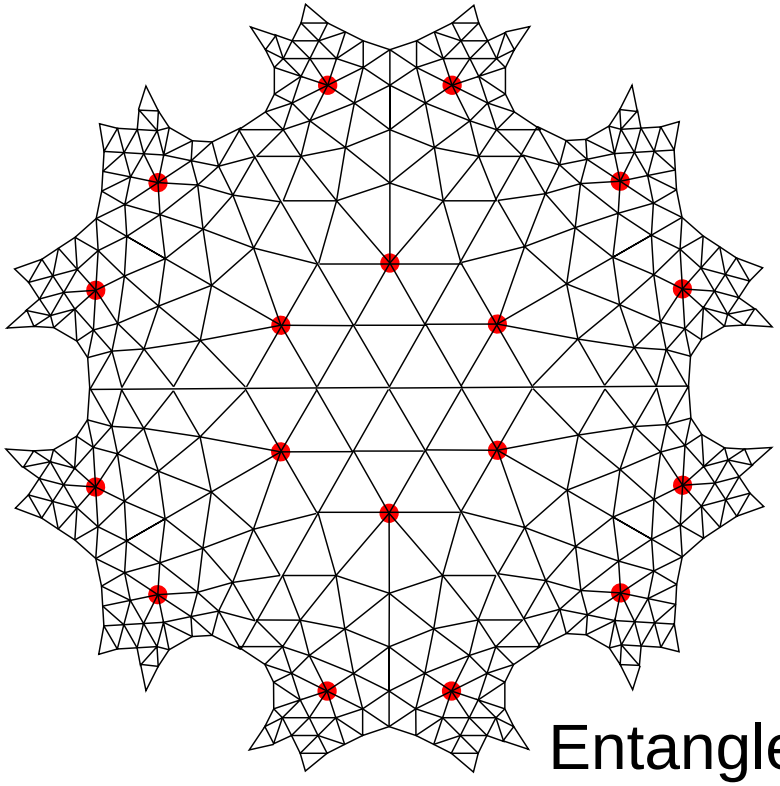


Half `row' transfer matrix P



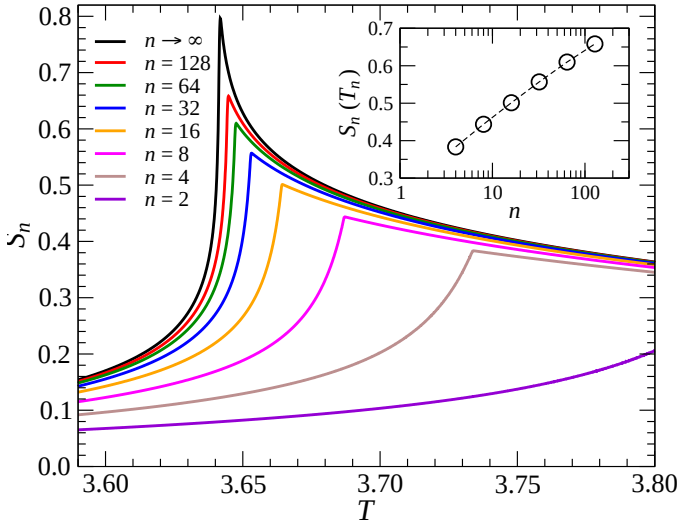
Corner transfer matrix C





Entanglement

◎ Weakly curved 2D lattice

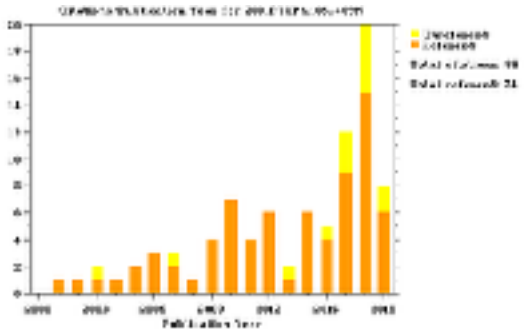


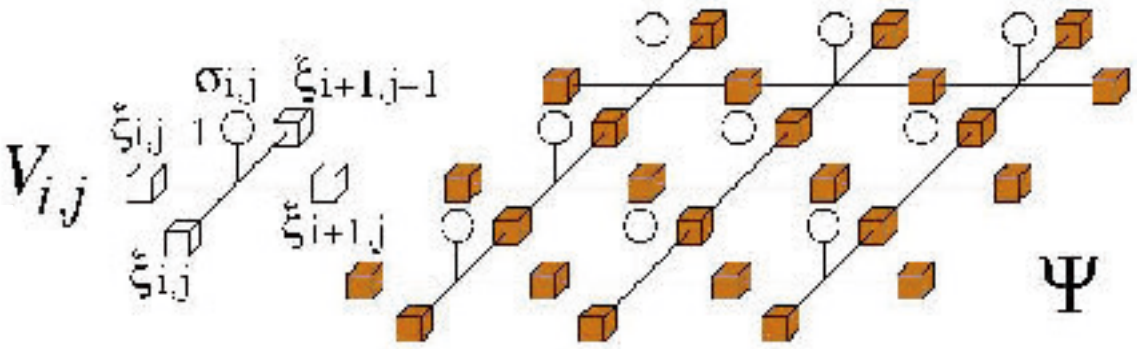
Baxter considered **higher dimensional generalization**. (in his textbook 1982)

They can also be extended to three dimensions: one obvious way being to write down the generalization of Fig. 13.9, which will involve a cube sliced into 27 pieces by 6 cuts! Unfortunately the resulting equations are quite complicated and involve 'corner tensors' with three indices. There is no analogue of matrix diagonalization for these tensors, and as yet the equations have not been investigated.

This is nothing but the Tensor Product State, or PEPS

(cond-mat/0011103, 0101360, 0303376)

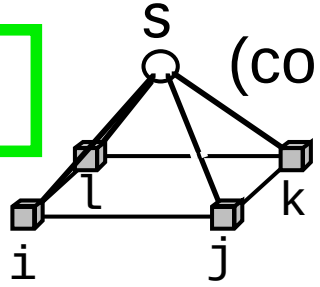




(Two-dimensional extension of MPS)

http://quattro.phys.sci.kobe-u.ac.jp/nishi/NiDMRG2000.html

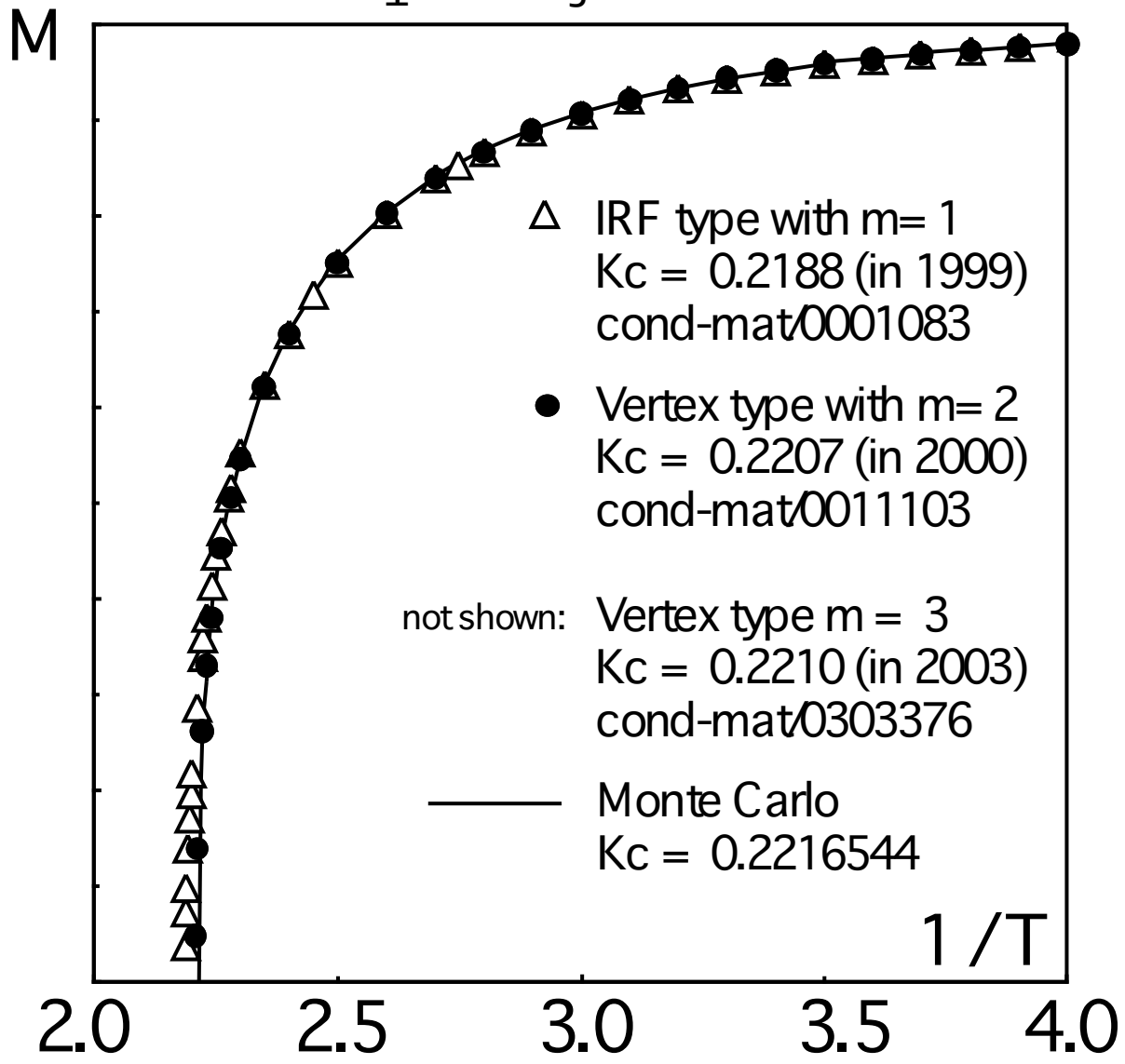
Application to 3D Ising Model



(cond-mat/0011103
0101360
0303376)

= Uniform Tensor Network State (or uniform PEPS)

Applied to 3D Ising:
Precision in Tc
m = 2: 0.6%
m = 3: 0.3%



in Quantum Physics

★Two-dimensional tensor network was proposed as an extension of the VBS state of the AKLT model.

I. Affleck, T. Kennedy, E.H. Lieb, and H. Tasaki: Commun.Math. Phys. 115 (1988) 477

★Hieida applied 2D tensor network to the deformed VBS state, and treat it numerically by means of DMRG.

Y. Hieida, K. Okunishi and Y. Akutsu: New. J. Phys. 1 (1999) 7.1.

MPS and Entanglement

tangle
entangle
disentangle

(distangle?!)

★Why MPS and TPS/PEPS are so accurate?

Singular Value Decomposition (SVD) is a key technique.

Singular Value Decomposition

* a linear decomposition for arbitrary matrix.

$$A_{ij} = \sum_{\xi} \lambda_{\xi} U_{i\xi} V^{*}_{j\xi}$$

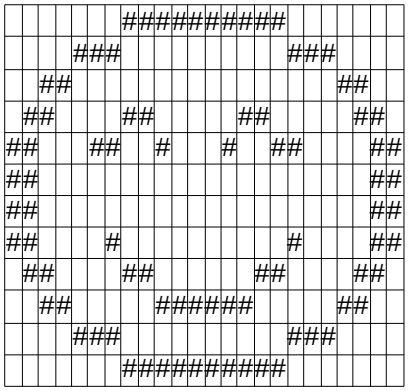
* ‘lambdas’ are the singular values, which are non-negative.

* U and V are orthogonal (or unitary) matrix, which satisfy the orthogonal relations.

$$\sum_{i=1}^{m} U_{i\xi} U^{*}_{i\eta} = \delta_{\xi\eta}, \quad \sum_{j=1}^{n} V_{j\xi} V^{*}_{j\eta} = \delta_{\xi\eta}$$

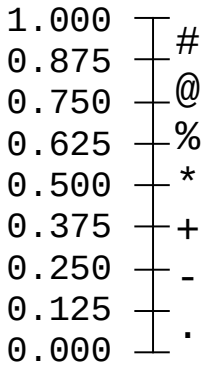
* Normally, singular values decays rapidly, and tiny ones can be omitted. This is one of the key point in the tensor network formulation.

compression of an image



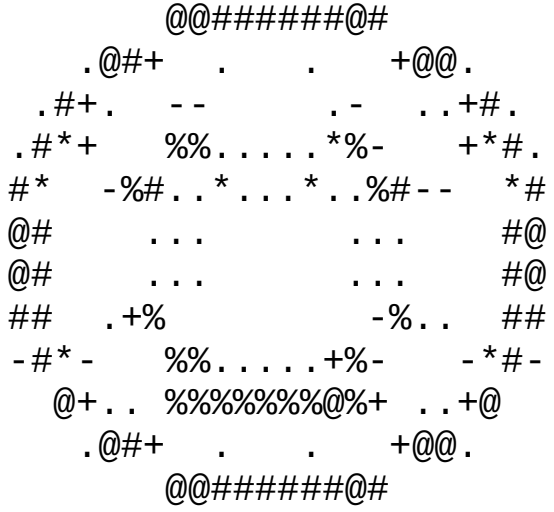
ORIGINAL

- Singular Values:

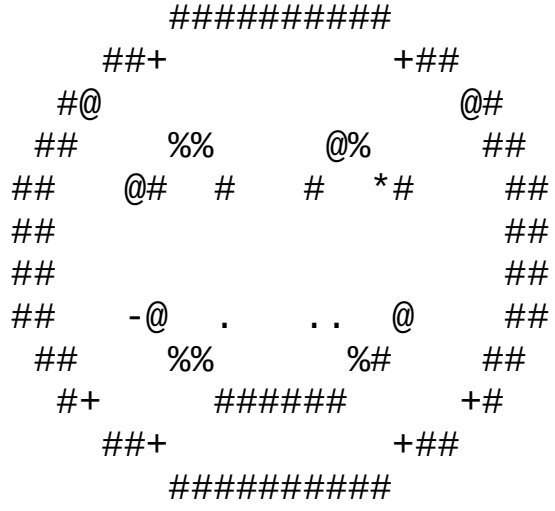


5.76078916114264
4.49306136234049
3.53367812591030
2.93100645840880
2.75155825714451
1.47701283908936
0.936697418652581
0.831935681254495
0.475256853905076
3.315897423129413E-016
1.400243040313679E-016
0.000000000000000E+000

- 4 dominant singular values are sufficient to recover rough image.

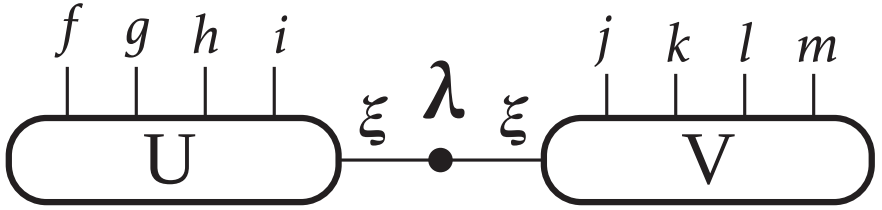


1+2+3+4



1+2+3+4+5+6+7

Schmidt decomposition of a quantum state



perform SVD at the center

$$\psi_{(fghi)(jk\ell m)} = \sum_{\xi} \lambda_{\xi}\, U_{(fghi)\xi}\, V^{*}_{(jk\ell m)\xi}$$

corresponding state

$$\sum_{\xi} \lambda_{\xi} \left[\sum_{fghi} U_{(fghi)\xi} \, |fghi\rangle \right] \left[\sum_{jk\ell m} V^{*}_{(jk\ell m)\xi} \, |jk\ell m\rangle \right]$$

Linear Combination

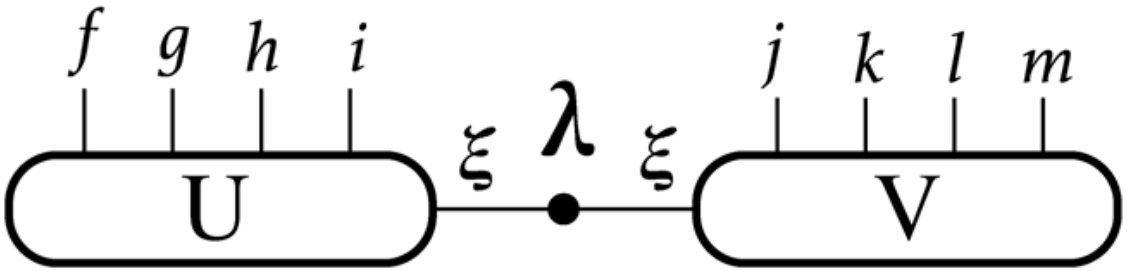
$$|\,\xi\,\rangle_{\text{left}} = \sum_{fghi} U_{(fghi)\xi} \, |fghi\rangle$$

$$|\,\xi\,\rangle_{\text{right}} = \sum_{jk\ell m} V^{*}_{(jk\ell m)\xi} \, |jk\ell m\rangle$$

Schmidt decomposition

$$|\Psi\rangle = \sum_{\xi} \lambda_{\xi} \, |\,\xi\,\rangle_{\text{left}} \, |\,\xi\,\rangle_{\text{right}}$$

Entanglement



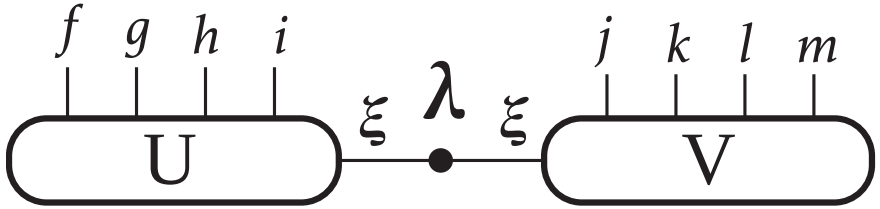
* When SVD is applied to the distribution P(fghijklm), dividing the indices into fghi and jklm, the singular values represent some sort of FREQUENCY of linear combinations of states.

* The von Neumann entropy

$$S = -\sum_{\xi} p_{\xi} \log p_{\xi} = -\sum_{\xi} (\lambda_{\xi})^2 \log (\lambda_{\xi})^2$$

is often called as the Entanglement Entropy, since it quantifies the quantum entanglement, when a quantum state is considered. The terminology is also used in the field of statistical mechanics.

Compression of information



Wave function and SVD $$\psi_{(fghi)(jk\ell m)} = \sum_{\xi} \lambda_\xi \, U_{(fghi)\xi} \, V^*_{(jk\ell m)\xi}$$

Approximation $$\tilde{\psi}_{fghijk\ell m} = \sum_{\xi=1}^{\chi} \lambda_\xi \, U_{(fghi)\xi} \, V^*_{(jk\ell m)\xi}$$

This process do not spoil the entanglement drastically.

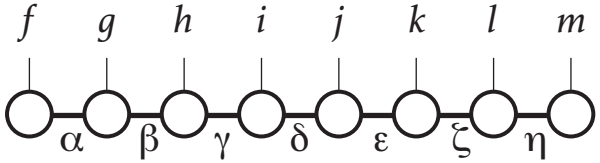
original Entanglement Entropy $$S = -\sum_{\xi} p_\xi \log p_\xi = -\sum_{\xi} (\lambda_\xi)^2 \log (\lambda_\xi)^2$$

after the compression $$\tilde{S} = -\sum_{\xi=1}^{\chi} p_\xi \log p_\xi = -\sum_{\xi=1}^{\chi} (\lambda_\xi)^2 \log (\lambda_\xi)^2$$

How MPS is obtained?

perform SVD from the left $\psi_{fghijk\ell m} = \sum_{\alpha} U_{f\alpha} \left[V_{(ghijk\ell m)\alpha} \lambda_{\alpha} \right]$

do SVD again. $= \sum_{\alpha\beta} U_{f\alpha} U'_{(\alpha g)\beta} \left[V'_{(hijk\ell m)\beta} \lambda'_{\beta} \right]$



$= \sum_{\alpha\beta\gamma} U_{f\alpha} U'_{(\alpha g)\beta} U''_{(\beta h)\gamma} \left[V''_{(ijk\ell m)\gamma} \lambda''_{\gamma} \right]$

In this manner, one can transform an arbitrary state to MPS.

※ We state only the possibility of expression by use of MPS.

※ Greek indices are restricted to χ

※ If the system is critical, one has to use larger χ

Entanglement structure is a key point in TN formulation.

Renormalization Group picture

Diagram for a Tree Tensor Network

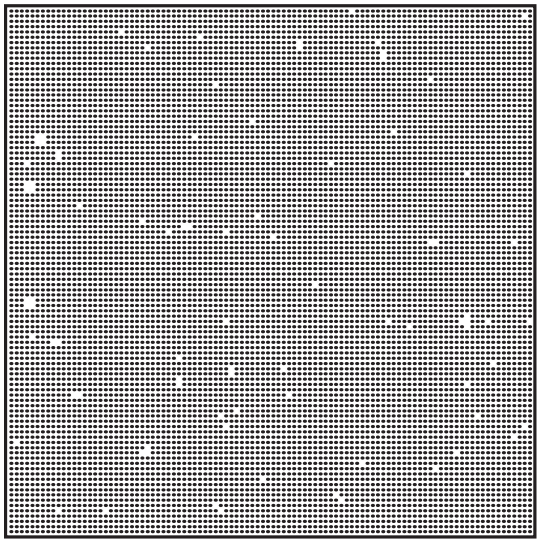
Scale Invariant?

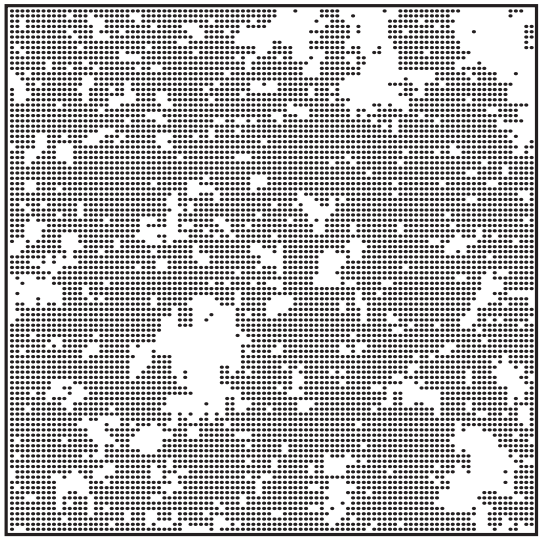
RG and Critical Phenomena

Low T

T = Tc

Snapshot of the 2D Ising model



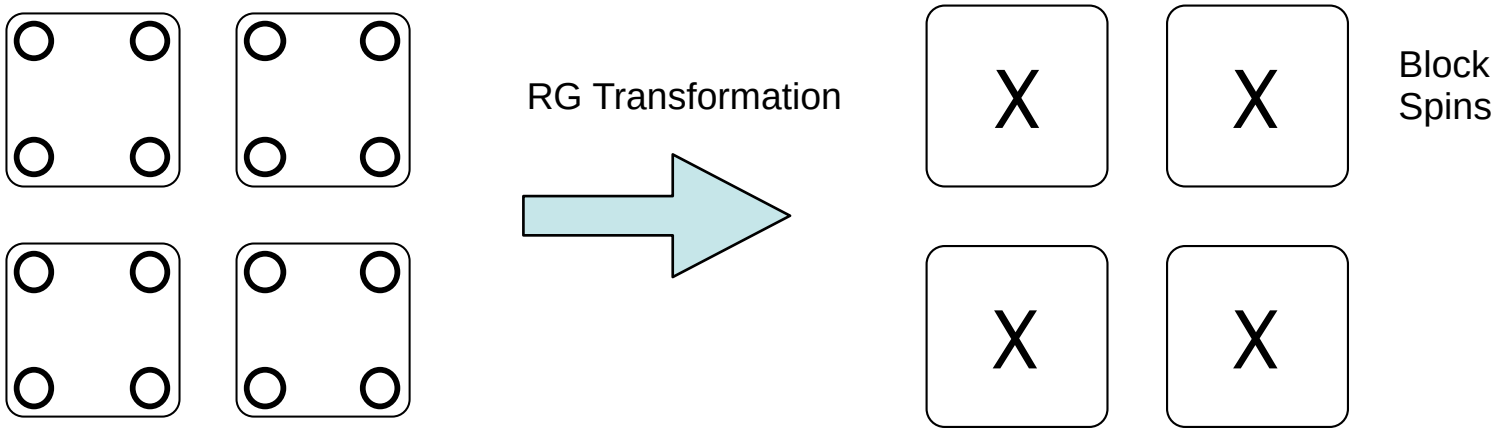


Scale invariance

Real Space Renormalization

Kadanoff,
Wilson, etc.

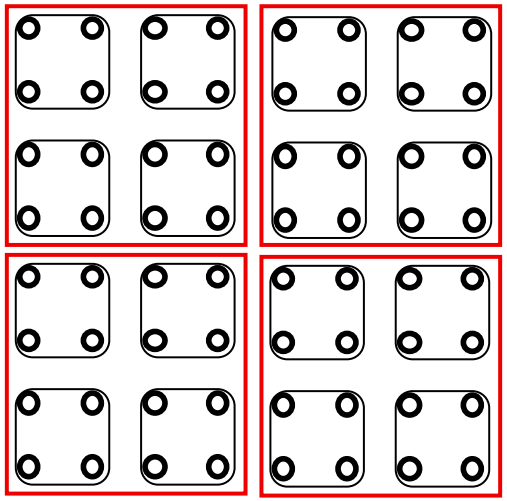
Circles are local degrees of freedom



(a) Treat several spins as an effective spin. (Block Spin Transformation)
(b) Keep only relevant freedom. (Renormalization)
(c) Scale change, which is 1/2 in the above case. (Rescaling)
(d) Consider interaction among blocks. (RG flow)

Conventional RSRG: Conceptually good, but not Quantitative

T > Tc Interaction decreases with RG steps.(disordered)
T = Tc Interaction remains constant. (critical)
T < Tc Interaction increases with RG steps. (ordered)



★ when interaction parameter K is slightly smaller than its critical value

$$K = Kc - \delta \rightarrow Kc - \lambda\delta \rightarrow Kc - \lambda^2\delta \rightarrow Kc - \lambda^3\delta \rightarrow ....$$

Suppose that when $\lambda^M \delta = C$, the correlation becomes local. Then correlation length ξ can be obtained from

$2^M = \xi$ and $\lambda^M \delta = C$

★ Result

$$\xi = \text{const.}\ (Kc - K)^{-\nu} \qquad \nu = \log 2 / \log \lambda$$

Reference: Kadanoff et al: Rev. Mod. Phys. 86 (2014) 647.

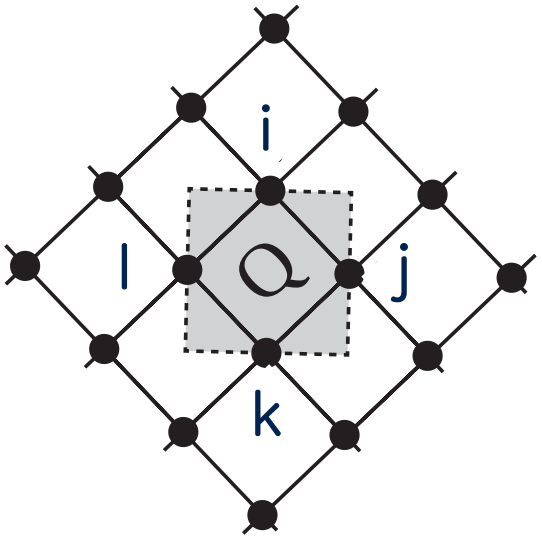
Block-spin transformations are chosen (partially) intuitively.

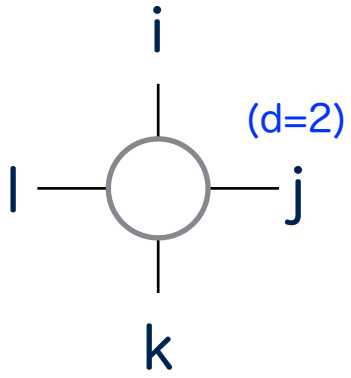
How to improve the situation?

※ Choice of block spin transformation is not automatic.

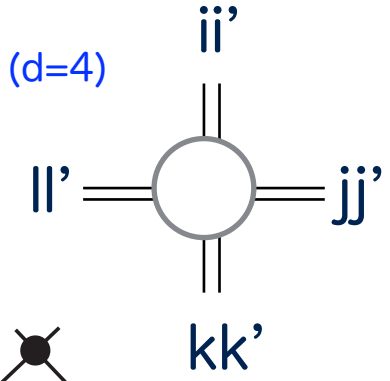
※ New site degrees of freedoms are identified as the block.

Let us reconsider from tensor view point. $Q_{k\ell ij} = \exp\left[\frac{J}{k_{\mathrm{B}}T}(ij + ik + j\ell + k\ell)\right]$



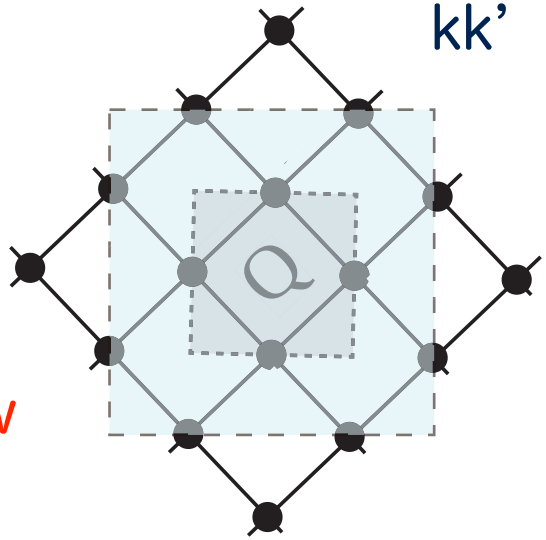


Extend the block size



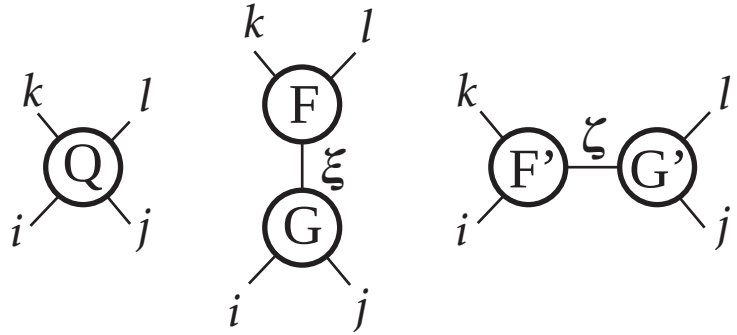
※Block touches only through their side

→ those tensor legs that are on the sides should be considered as the new degrees of freedom.



Tensor Renormalization Group

RG steps: first, perform SVD for the local tensor.



$$Q_{k\ell ij} = \sum_{\zeta} \lambda_{\xi} U_{(k\ell)\,\xi} V^{*}_{(ij)\,\xi}$$

$$= \sum_{\xi} \left[\sqrt{\lambda_{\xi}}\ U_{(k\ell)\,\xi} \right] \left[\sqrt{\lambda_{\xi}}\ V^{*}_{(ij)\,\xi} \right] = \sum_{\xi} F_{k\ell\xi}\ G_{ij\xi}$$

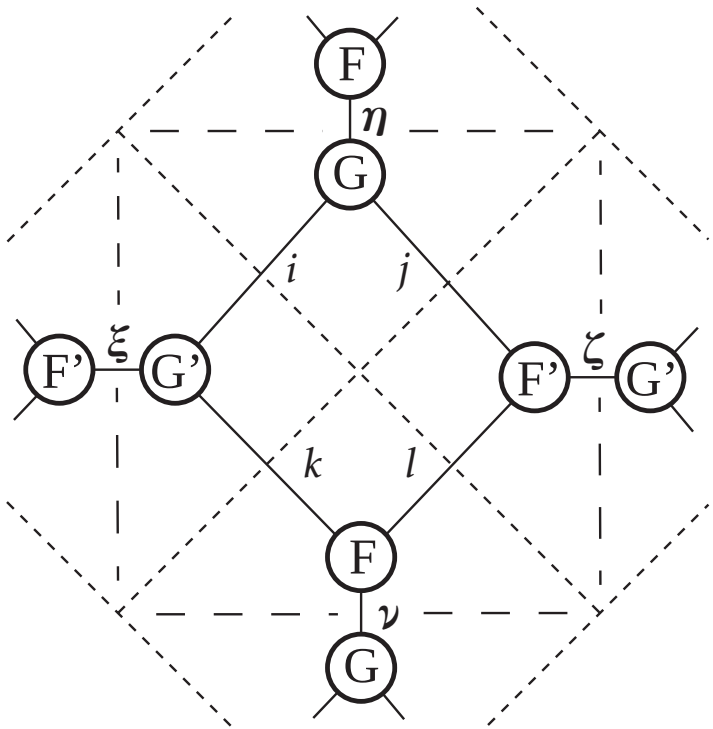
Restrict greek indices!

It can be decomposed vertically

$$Q_{k\ell ij} = \sum_{\zeta} \lambda'_{\zeta} U'_{(ki)\,\zeta} V'^{*}_{(\ell j)\,\zeta} = \sum_{\zeta} F'_{ki\zeta}\ G'_{\ell j\zeta}$$

Contract 4-tensors that are connected

$$Q^{(2)}_{\xi\eta\nu\zeta} = \sum_{ijk\ell} G'_{ik\xi}\ G_{ij\eta}\ F_{k\ell\nu}\ F'_{j\ell\zeta}$$

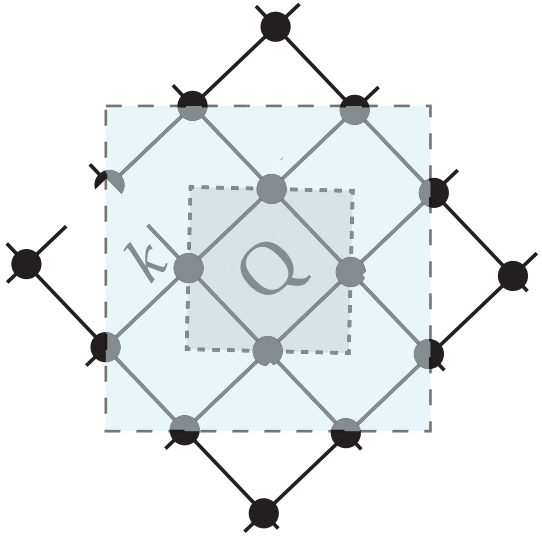


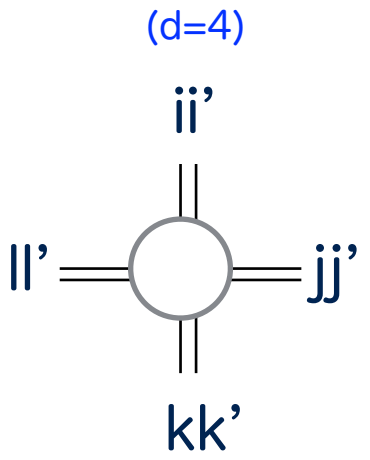
Repeat the above procedur

$$Q^{(3)}_{\alpha\beta\delta\gamma} = \sum_{\mu=1}^{256} F^{(3)}_{(\alpha\beta)\mu} G^{(3)}_{(\delta\gamma)\mu} \sim \sum_{\mu=1}^{\chi} F^{(3)}_{(\alpha\beta)\mu} G^{(3)}_{(\delta\gamma)\mu}$$

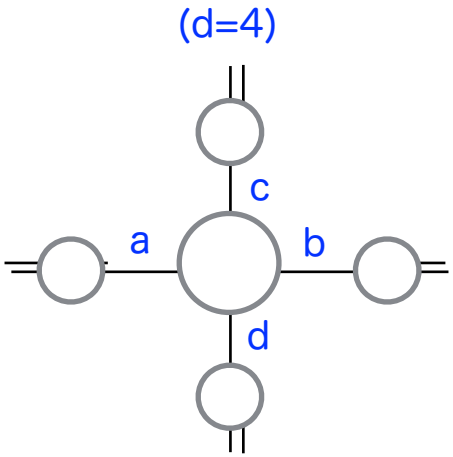
There are several computational methods.

(TERG, HOTRG, TNR, Loop-TNR, etc.)





>> SVD >>



What is common?

※Entanglement structure of the system is almost kept

※Focusing on bonds (links) other than sites.

※What is renormalized is not Hamiltonian but tensors.

※From the RG view point, it is curious to consider how the renormalized Hamiltonian looks like when it is written by equations, in particular when the system is critical.

Recent? numerical results by means of Tensor Network

2D Ising Model: Transition temperature can be obtained with 8 digits.

3D Ising Model: obtained Tc = 4.511546 (Monte Carlo: 4.511544)

(~ 24 states are kept) **(T. Xiang et al: arXiv:1201.1144)**

estimation of conformal weights for 2D Ising Model

(Evenbly and Vidal: arXiv:1412.0732)

	exact	TRG(64)	TRG+env(64)	TEFR(64)	TNR(24)
c	0.5	0.49982	0.49988	0.49942	0.50001
σ	0.125	0.12498	0.12498	0.12504	0.1250004
ϵ	1	1.00055	1.00040	0.99996	1.00009
	1.125	1.12615	1.12659	1.12256	1.12492
	1.125	1.12635	1.12659	1.12403	1.12510
	2	2.00243	2.00549	-	1.99922
	2	2.00579	2.00557	-	1.99986
	2	2.00750	2.00566	-	2.00006
	2	2.01061	2.00567	-	2.00168

TABLE I. Exact values and numerical estimates of the central charge c and lowest scaling dimensions of the critical Ising model. TRG results are obtained using the original Levin and Nave's algorithm [4]. TRG+env results are obtained using an improved TRG method proposed in Ref. [5] under the name of "poor-man's SRG". TEFR results are taken from Ref. [7]. The first three numerical columns use bond dimension $\chi \equiv D_{cut} = 64$ and 1024 spins, while the TNR data uses $\chi = 24$ and 262,144 spins.

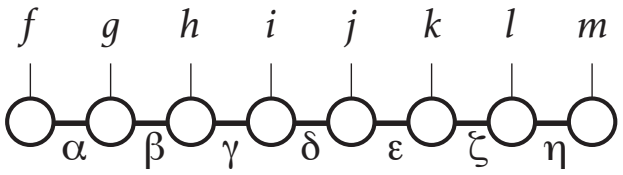
A Big Step in Real-Space RG

a milstone

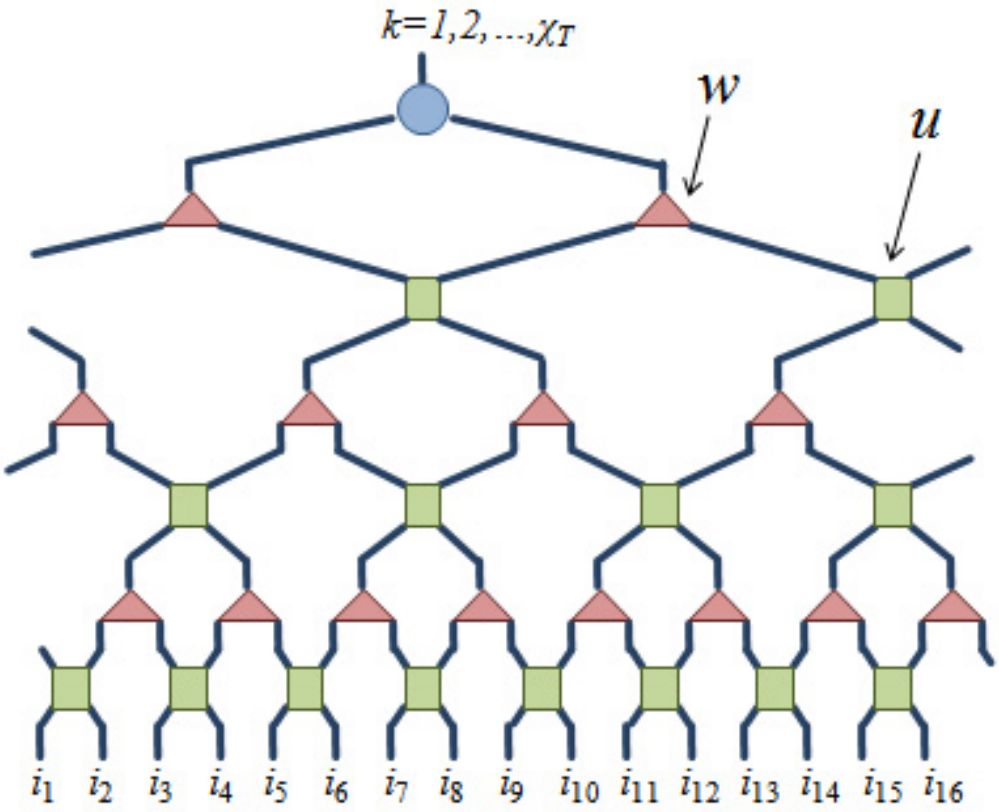
Evenbly and Vidal

arXiv:0707.1454

conventional MPS



Binary 1D MERA:



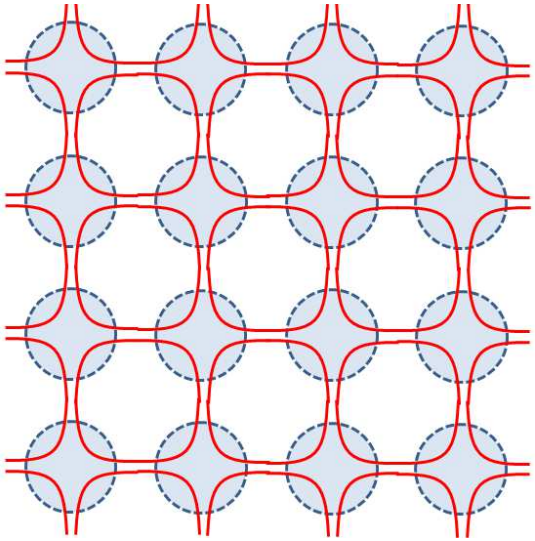
MERA, the multi-scale entanglement renormalization Ansatz, which has been applied to 1D quantum systems. This tensor network efficiently describes system at criticality.

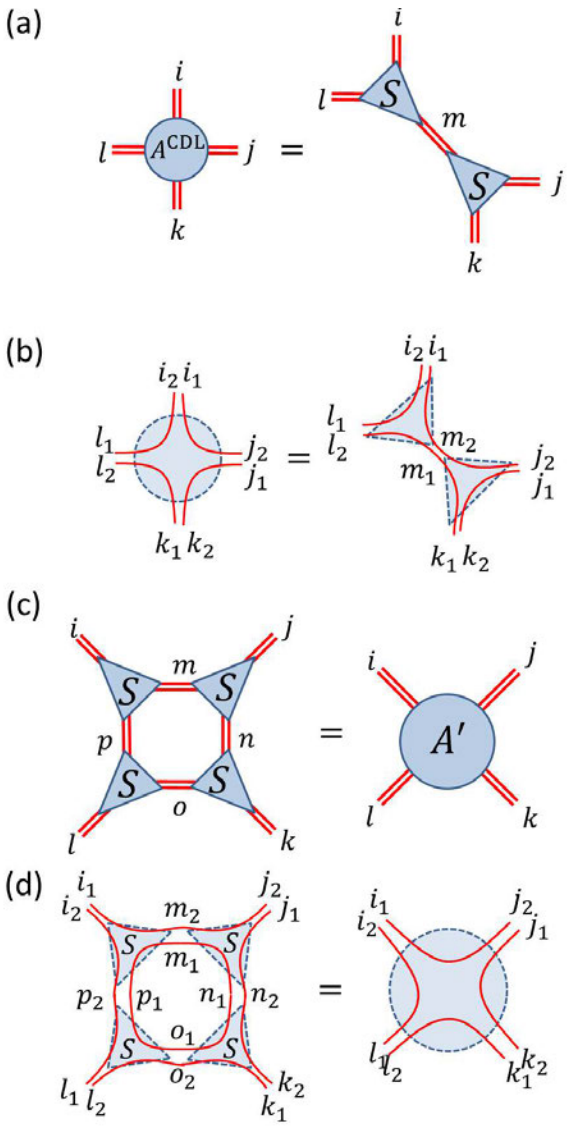
Problem of CDL

Evenly and Vidal

arXiv:1412.0732

Entanglement Structure in the Lattice models,
in particular at criticality.



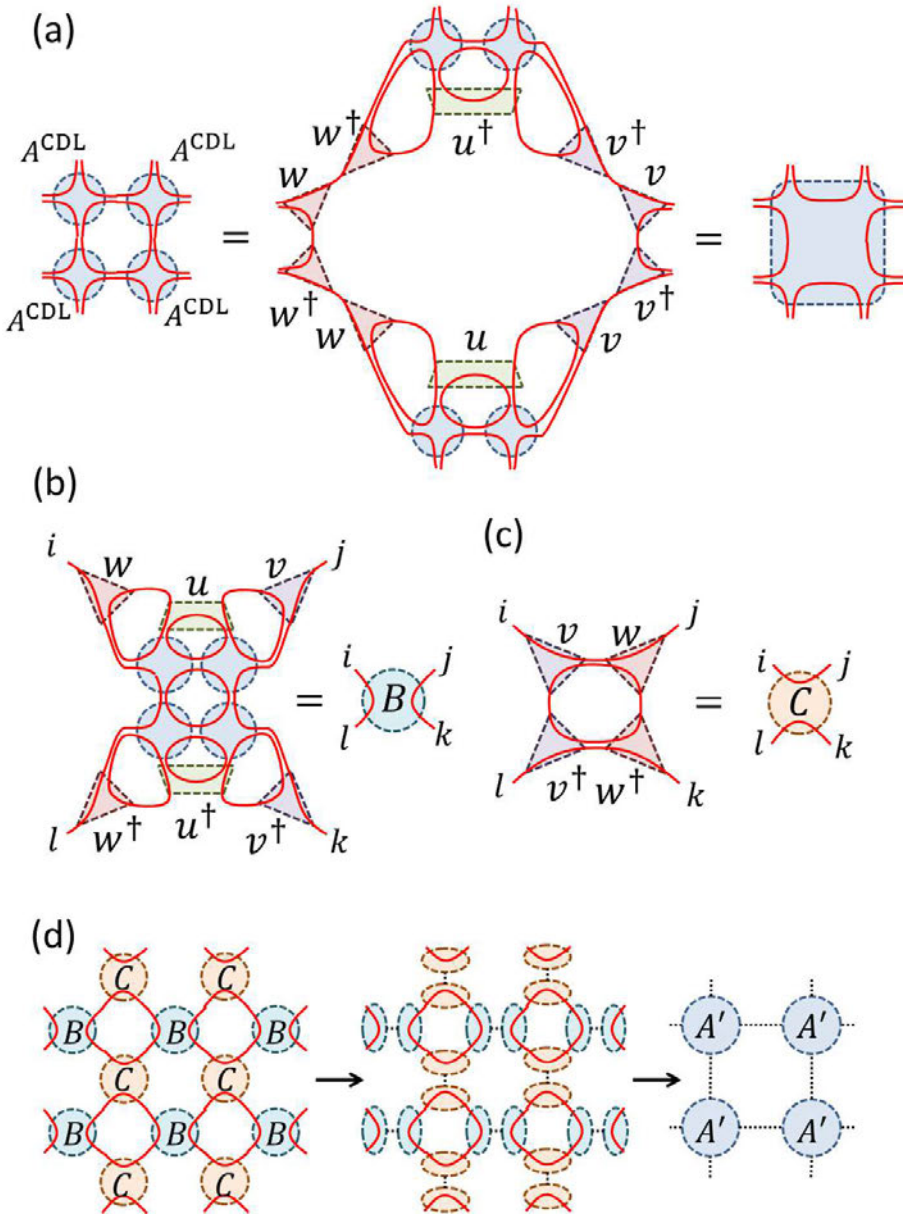


In TRG formulation, so called the Corner Double Line (CDL) remains even after the RG transformation. Since the original degrees of freedom partially (?) remains, the presence of CDL (drastically!) spoils the numerical precision in RSRG.

a solution for CDL

Evenly and Vidal

arXiv:1412.0732



Solution: apply the disentangler before performing RG transformation by isometry.

This scheme was first introduced in MERA, the multi-scale entanglement renormalization Ansatz, applied to 1D quantum systems.

Evenly and Vidal

arXiv:0707.1454

TRG v.s. TNR

Evenly and Vidal

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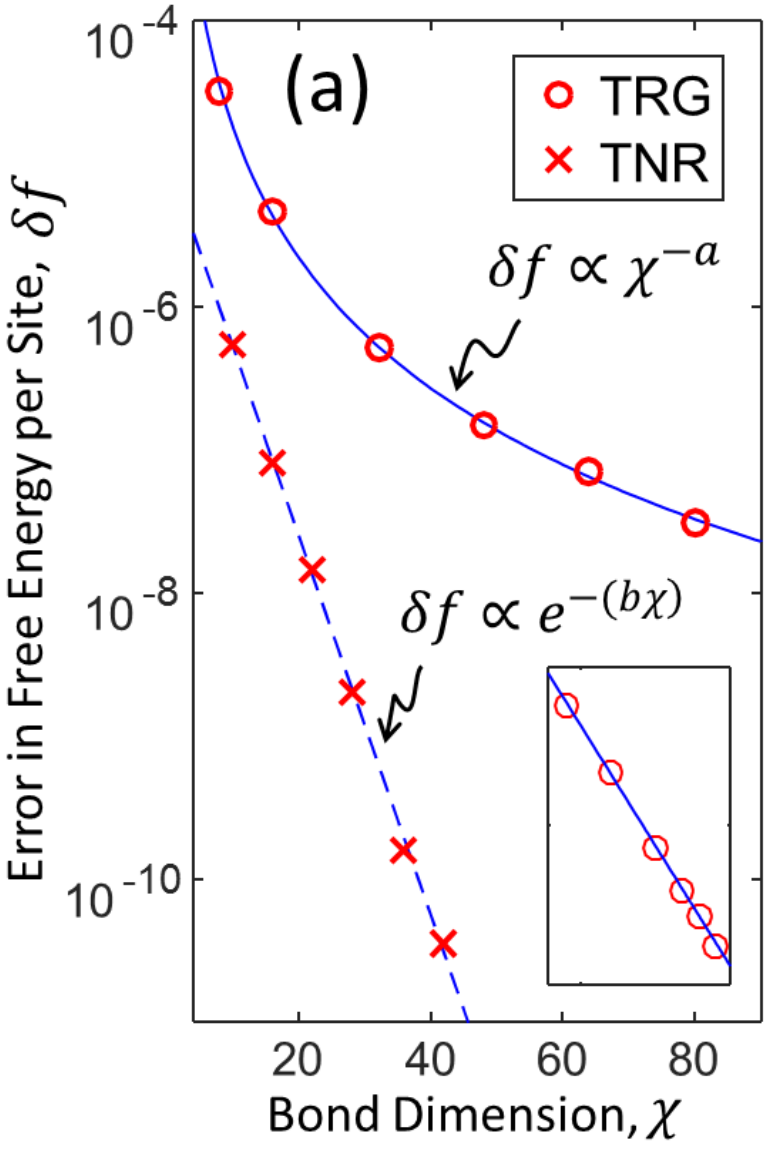

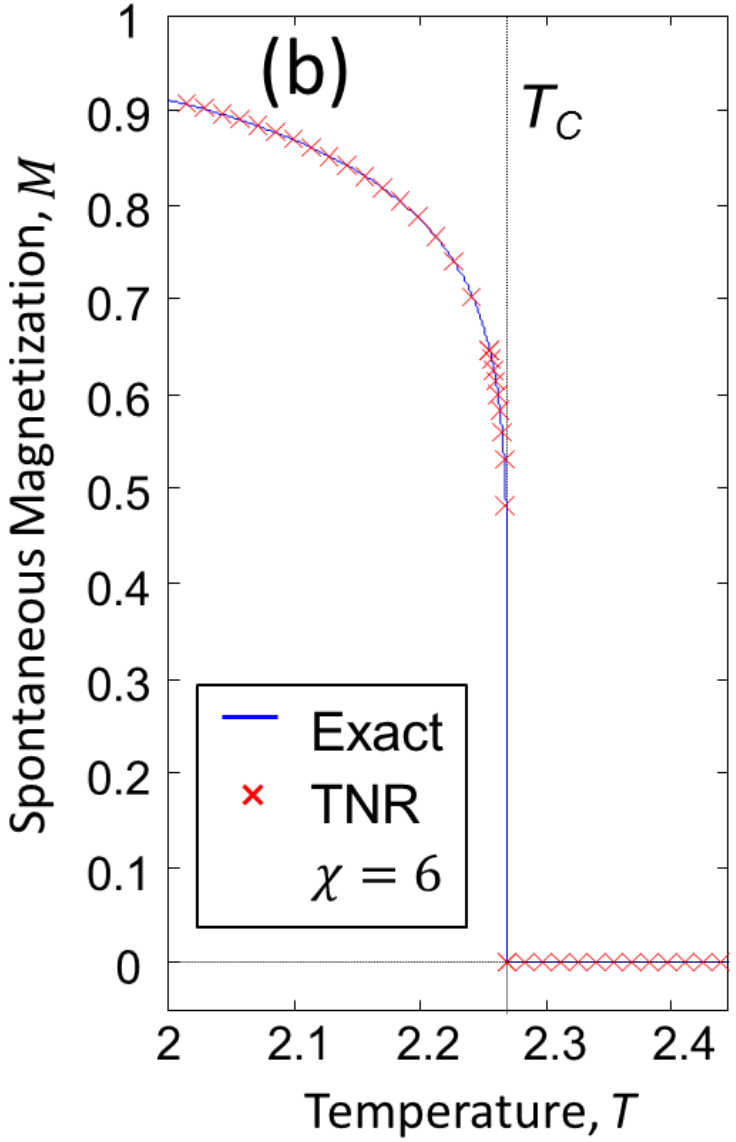

numerical results by Tensor Network Renormalization

2D Ising Model: Transition temperature can be obtained with 8 digits.
3D Ising Model: obtained Tc = 4.511546 (Monte Carlo: 4.511544)

(T. Xiang et al: arXiv:1201.1144)

estimation of conformal weights for 2D Ising Model

(Evenbly and Vidal: arXiv:1412.0732)

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not known …

From the theoretical view point, it is curious to consider how the renormalized Hamiltonian looks like when it is written by equations, in particular when the system is critical.

Tensor Network Formulations

from the view point of entanglement and RSRG

Tomotoshi Nishino (Kobe Univ.)