Canonical Tensor Model throuth data analysis
-Dimensions, topologies, and geometries-

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Discrete Approaches to the Dynamics of Fields and Space-Time

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   propose a procedure, some notes

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   fuzzy spaces with/without boundary,
   orientable/unorientable,
   product space,…

Procedure for P_{abc}

Setups

- $g_{\mu\nu}(x)$: a metric tensor on a Cauthy surface Σ with dimension d
- $f_a(x)$: a basis function on Σ

Normalization condition:

$$\int d^d x \sqrt{g} f_a f_b = \delta_{ab}$$

From the definition of P_{abc}: $f_a f_b = P_{ab}{}^c f_c$,

$$P_{abc} = \int d^d x \sqrt{g} f_a f_b f_c \ \text{(naive def.)}$$

Procedure for P_{abc}

Note about f_a

- Result from spectral theory
  If Σ is a compact manifold, the eigenfunctions of the Laplace-Beltrami operator ∇^2 on Σ form an orthonormal basis for $L^2(\Sigma)$.

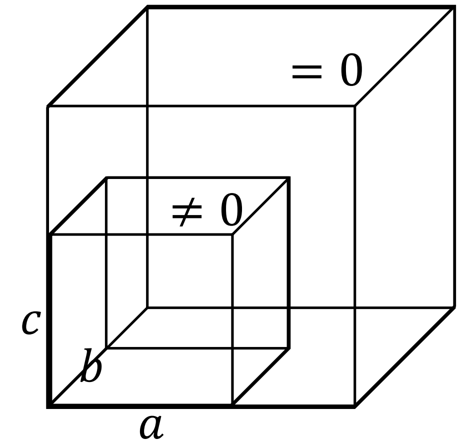
- So f_a can be obtained by the Helmholtz equation:

$$(\nabla^2 + m^2) f_a = 0,$$

$$\nabla^2 = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu).$$

- If Σ has boundary: $\partial\Sigma \neq \emptyset$, we need to impose a boundary condition (Dirichlet/Neumann) for f_a.
- $\{f_a | m^2 \leq J^2\}$ with some "sharp cutoff" J

Procedure for P_{abc}



Good definition of P_{abc}:

$$\widetilde{f}_a = f_a e^{-m^2/L^2} = e^{\nabla^2/L^2} f_a$$

$$P_{abc} = \int d^d x \sqrt{g}\, \widetilde{f}_a \widetilde{f}_b \widetilde{f}_c$$

$L \lesssim J$ would be the best choice.

Basis function has oscillational bahaviour

$$v_a^i v_a^j \sim \int_0^L da\, e^{iax} \quad \text{; oscillation}$$

but if there is the damping factor

$$v_a^i v_a^j \sim \int_0^L da\, e^{iax - a^2/L^2} \quad \text{;convergence}$$

Persistent Homology

Setup

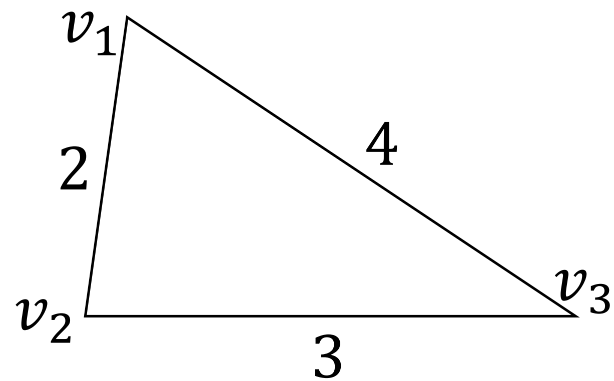
[Carlsson, 2009]

- Vertex set: V
- Distance: $d(i,j)$

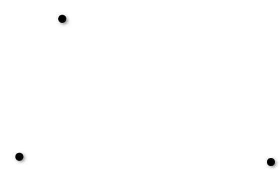
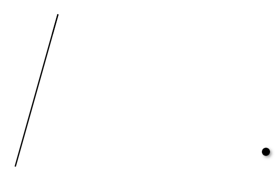
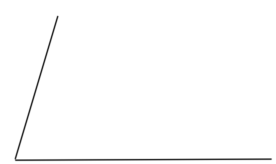
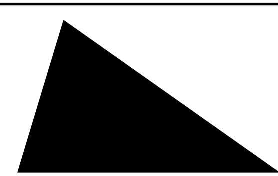
Def. Vietris-Rips stream $VR(u)$

$VR(u)$ is the mapping from a real parameter u to a simplicial complex that satisfy
(i)$[v] \in VR(V,u)$ for all vertices $v \in V$
(ii)n-simplex $[v_0 v_1 \dots v_n] \in VR(V,u)$
$\leftrightarrow d(v_i, v_j) \leq u$ for all edges $[v_i v_j] \in [v_0 v_1 \dots v_n]$

Persistent Homology



Simple example

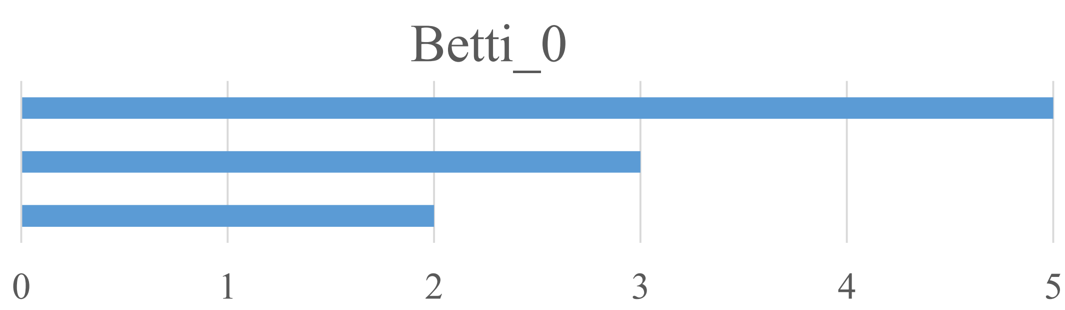
u	Vietris-Rips complex	Betti number
$0 \leq u < 2$		$B_0 = 3$
$2 \leq u < 3$		$B_0 = 2$
$3 \leq u < 4$		$B_0 = 1$
$4 \leq u$		$B_0 = 1$

Persistent Homology

Betti intervals (barcode)

- combinient to see the transition of the homology group of the fuzzy space
- long-life intervals characterize the global structure

u	Betti number
$0 \leq u < 2$	$B_0 = 3$
$2 \leq u < 3$	$B_0 = 2$
$3 \leq u < 4$	$B_0 = 1$
$4 \leq u$	$B_0 = 1$



Persistent Homology

Application for the fuzzy space

Some definitions

- Neighborhood $\mathcal{N}_c(i)$

$$j \in \mathcal{N}_c(i) \leftrightarrow v_a^i v_a^j > c$$

- Path $p(i,j)$

$$p(i,j) = (p_0, p_1, \ldots, p_n)$$
$$p_0 = i, p_n = j, \forall k (p_k \in \mathcal{N}_c(p_{k+1}))$$

- Distance $d(i,j)$

(i)If there exist paths $\{p(i,j)\}$

$$d(i,j) = \min\{n\}$$

(ii) i, j are not connected

$$d(i,j) = \infty$$

n-Sphere S^n

Setup [Higuchi,1987]

- Coordinates: $(\theta_1, \theta_2, \dots, \theta_n)$
- Metric: $d\Omega_n^2 = d\theta_n^2 + \sin^2\theta_n \, d\Omega_{n-1}^2, d\Omega_1^2 = d\theta_1^2$
- Helmholtz equation

$$(\nabla^2 + l_n(l_n + n - 1))f = 0$$

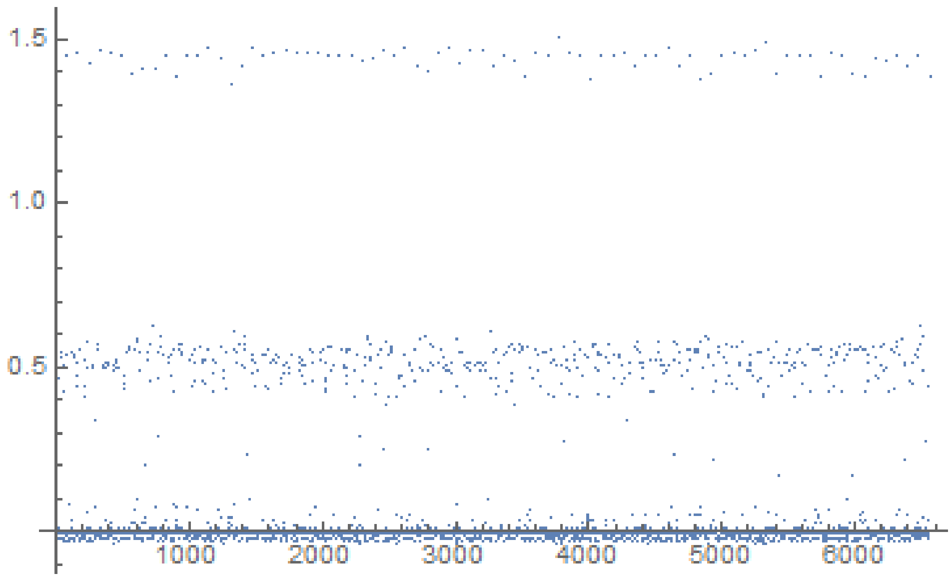
Basis function will be (extended) spherical harmonics

$$f_a = Y_{l_1, l_2, \dots, l_n}(\theta_1, \theta_2, \dots, \theta_n).$$

Damping factor

$$\exp(-l_n(l_n + n - 1)/L^2)$$

2-Sphere S^2

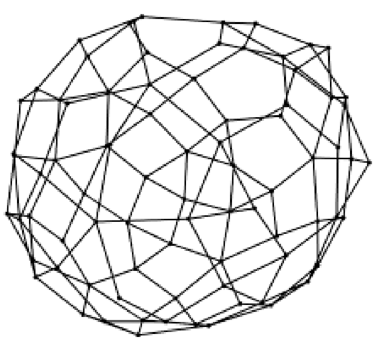


Condition

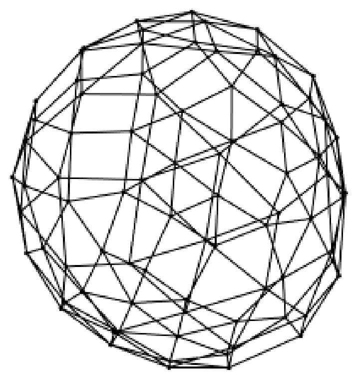
$N = 81, R = 81$

$J = 9, L = 8$

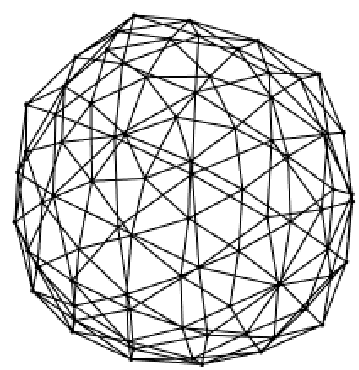
Distribution of $v_a^i v_a^j$



$c = 0.5$

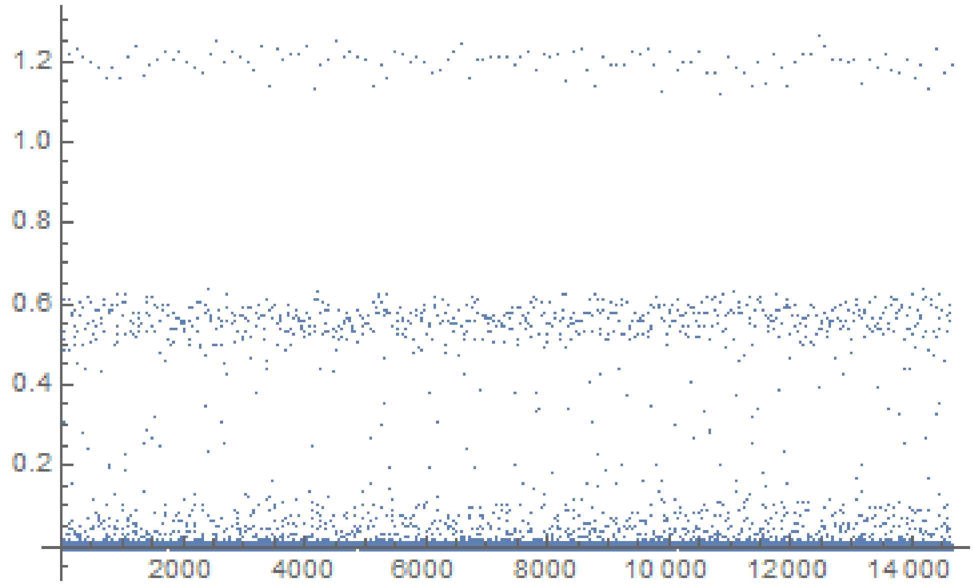


$c = 0.45$



$c = 0.1$

2-Sphere S^2

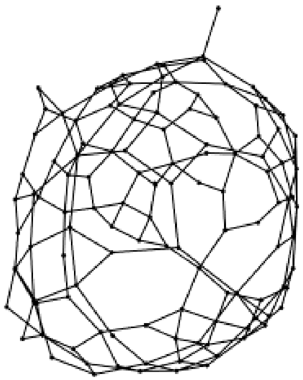


Condition

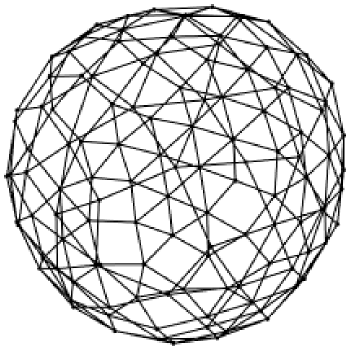
$N = 81, R = 121$

$J = 9, L = 8$

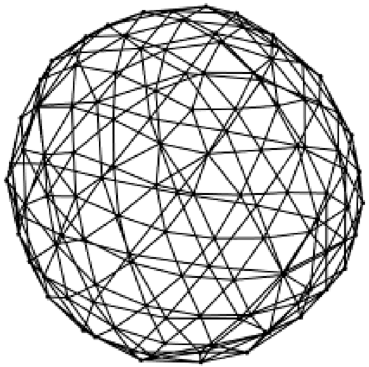
Distribution of $v_a^i v_a^j$



$c = 0.5$

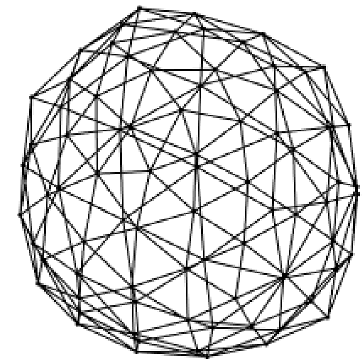


$c = 0.5$



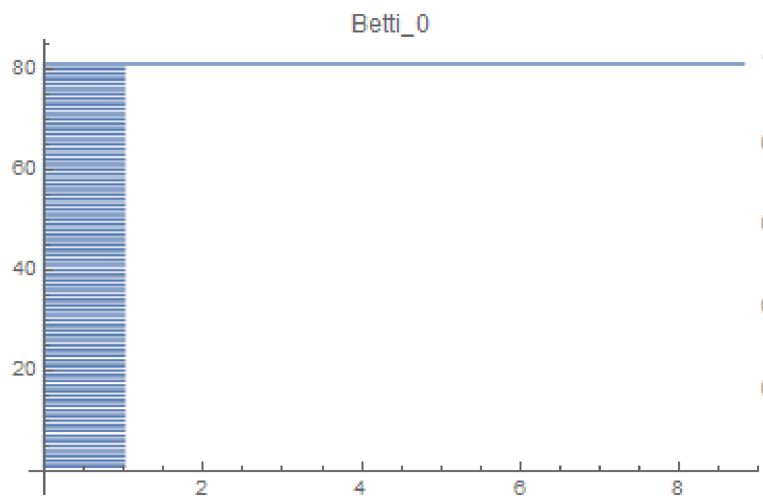
$c = 0.2$

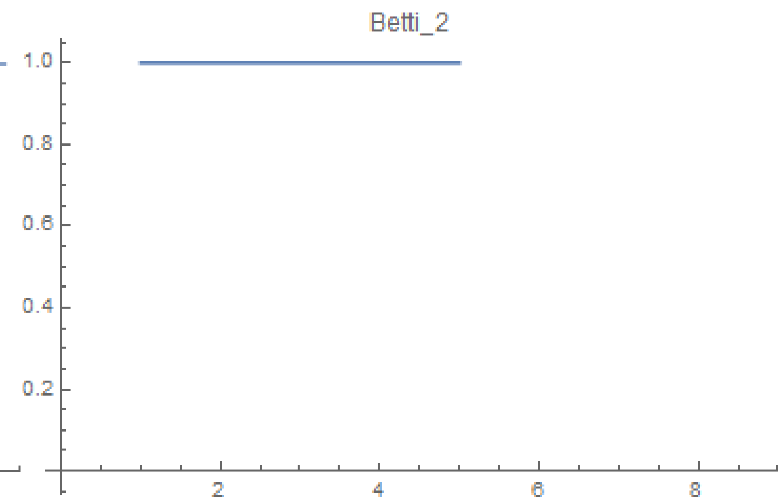
2-Sphere S^2



Betti intervals (for $\mathbb{Z}_2$, $R = 81$, $c = 0.1$)

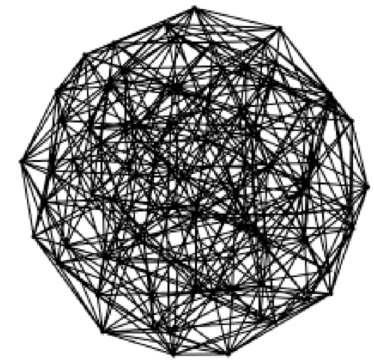
*There are no intervals for Betti_1.





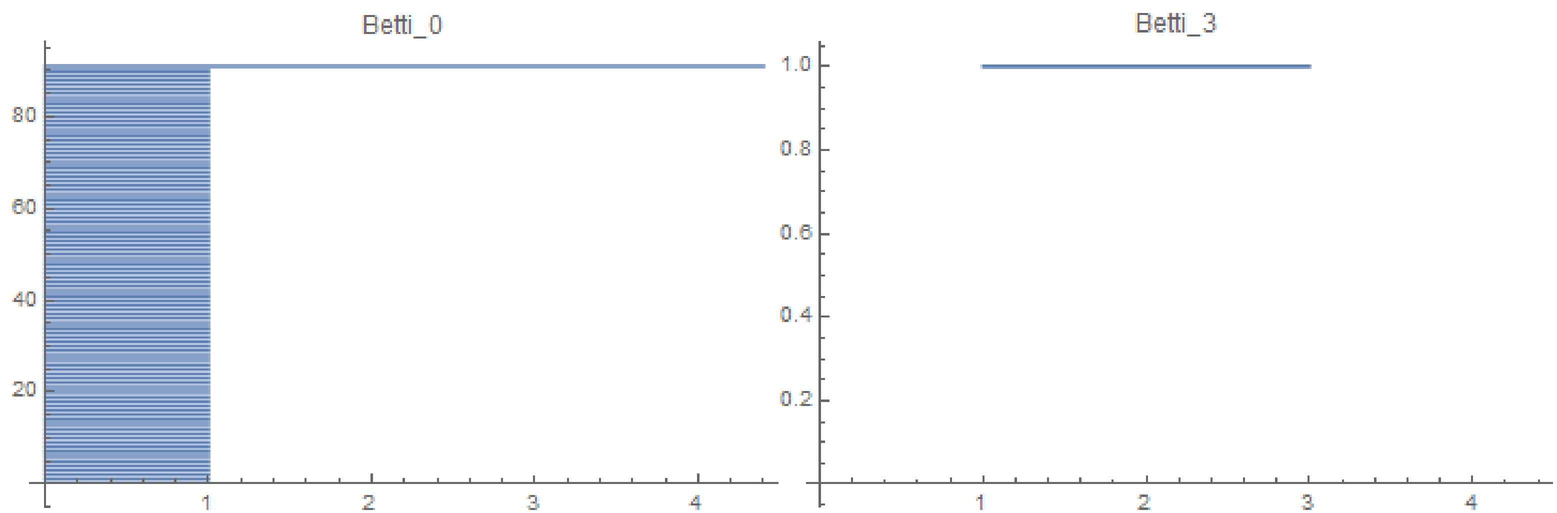
=> $(B_0, B_1, B_2) = (1,0,1)$

3-Sphere S^3



Betti intervals (for $\mathbb{Z}_2$)

*There are no intervals for Betti_1 and Betti_2.



=> $(B_0, B_1, B_2, B_3) = (1,0,0,1)$

Line segment

Laplacian: ∂_x^2

Boundary: $x = \pm\pi$

Dirichlet boundary condition

$$f_a = \sin(nx), \cos\big((n - 1/2)x\big)$$
$$m^2 = n^2, (n - 1/2)^2$$

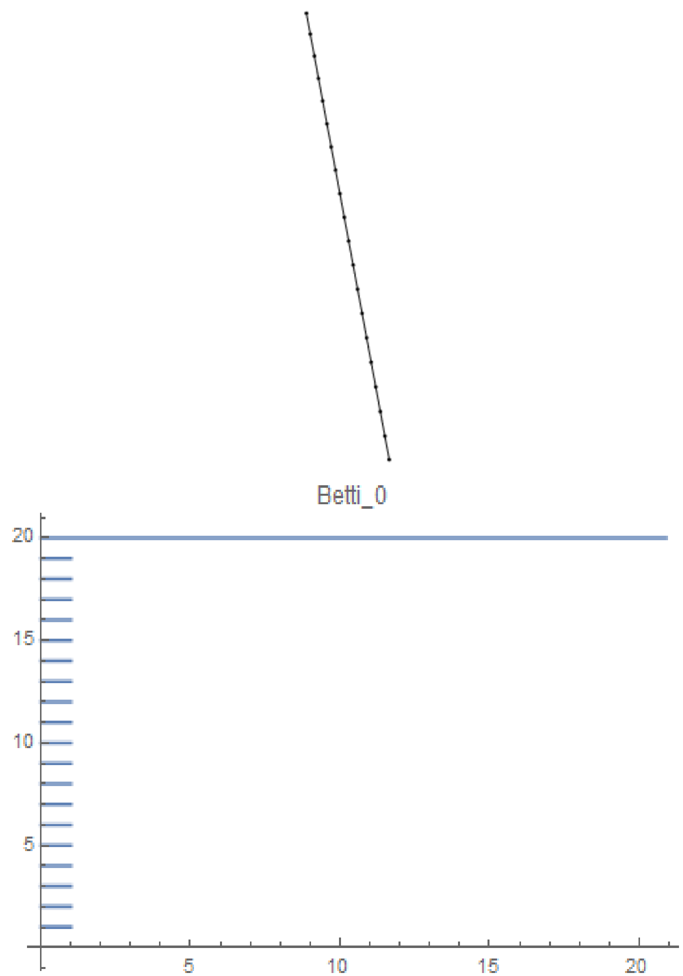
Neumann boundary condition

$$f_a = 1, \cos(nx), \sin((n - 1/2)x)$$
$$m^2 = 0, n^2, (n - 1/2)^2$$

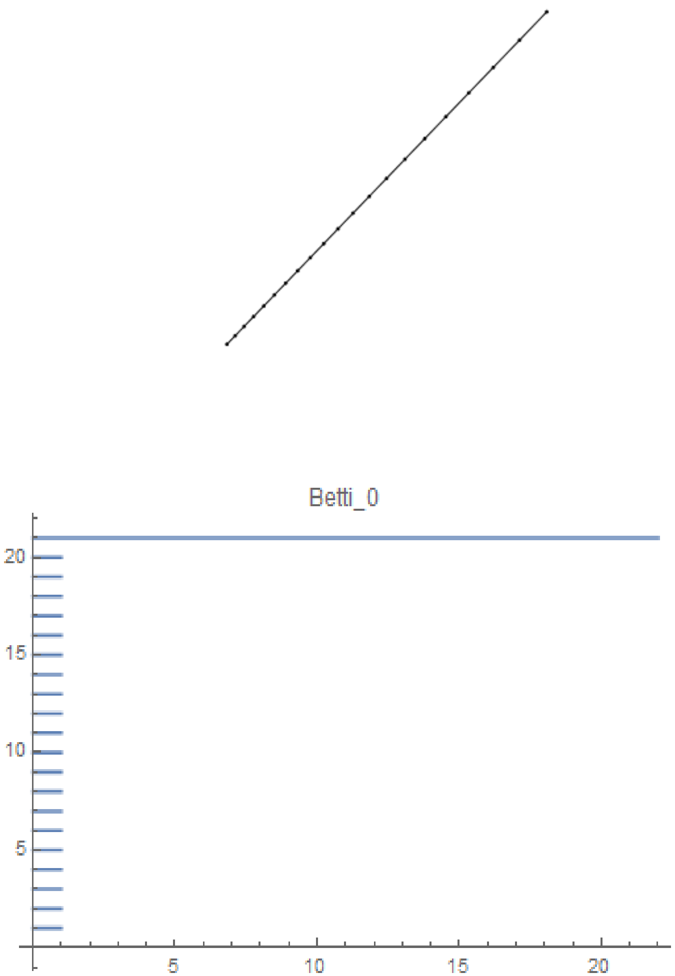
Damping factor

$$\exp(-m^2/L^2)$$

Line segment



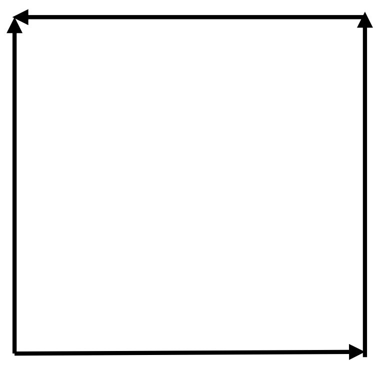
Dirichlet



Neumann

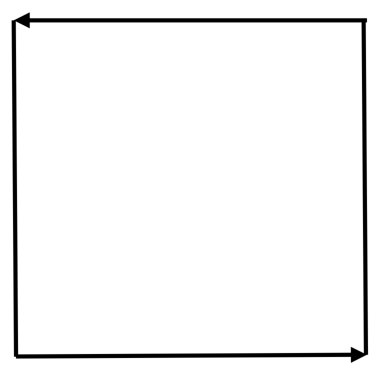
Unorientable case

- Klein bottle



$$f(x,y) = f(x+2\pi, y)$$
$$f(x,y) = f(-x, y+2\pi)$$

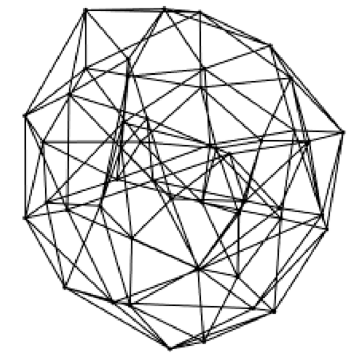
- Möbius strip



$$f(x,y) = f(-x, y+2\pi)$$

+ Dirichlet/Neumann

Unorientable case



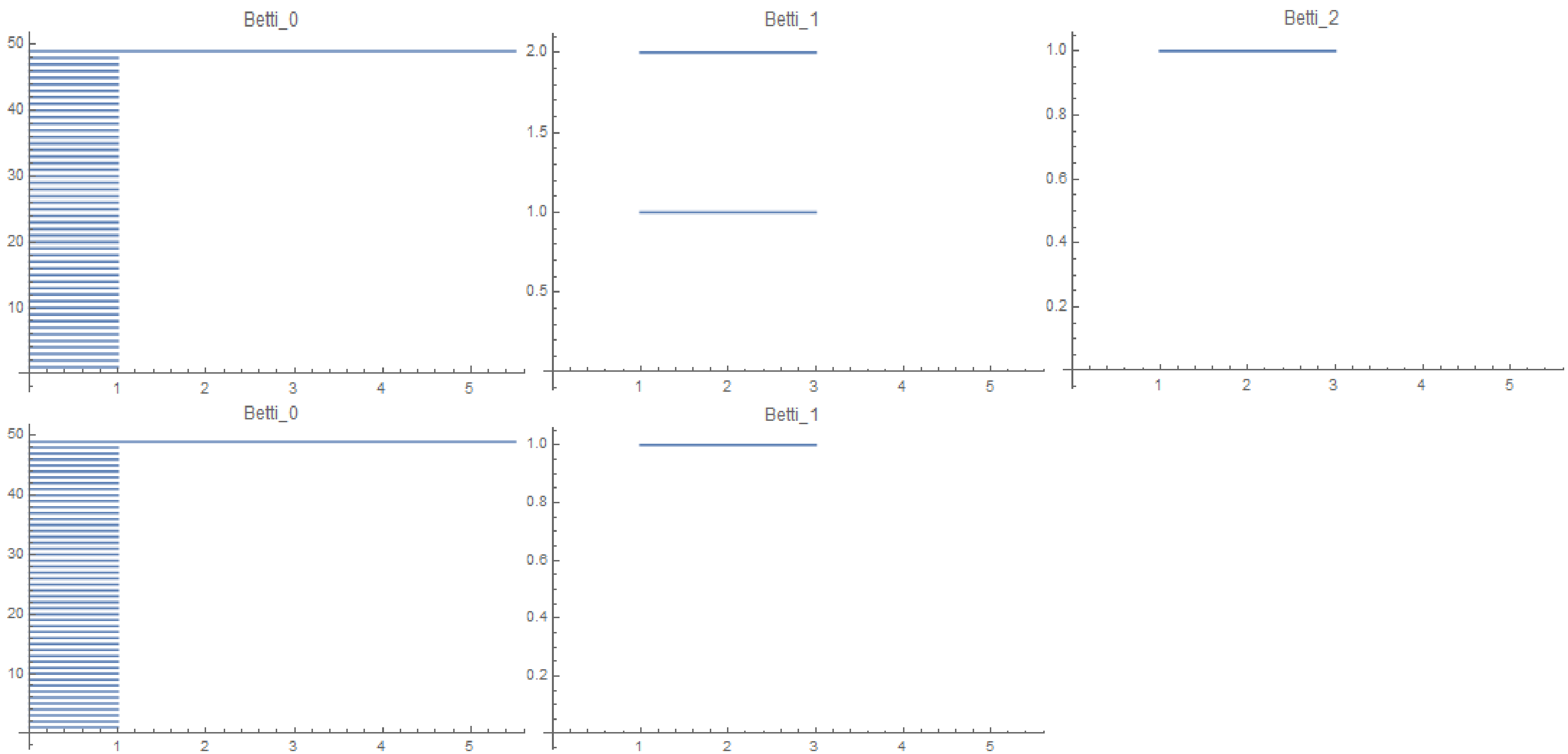
Klein bottle

$(B_0, B_1, B_2) = (1,2,1)$ for $\mathbb{Z}_2$

$(B_0, B_1, B_2) = (1,1,0)$ for $\mathbb{Z}_3$

$\mathbb{Z}_2$

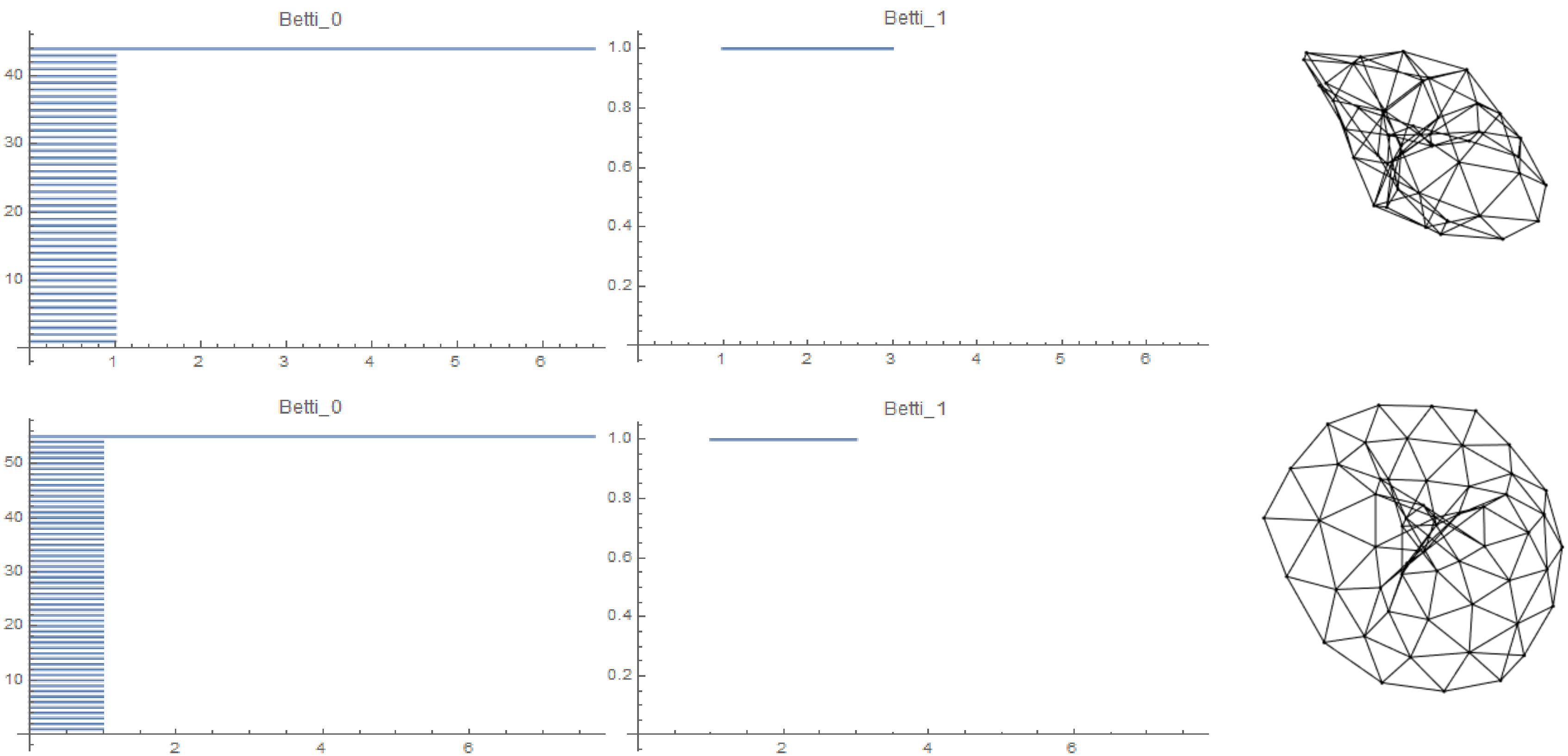
$\mathbb{Z}_3$



Unorientable case

Möbius strip (upper: Dirichlet, lower: Neumann)

$(B_0, B_1, B_2) = (1,1,0)$ for $\mathbb{Z}_2$ and $\mathbb{Z}_3$



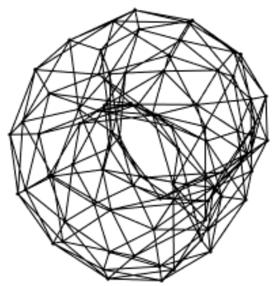
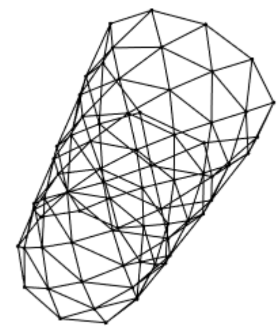
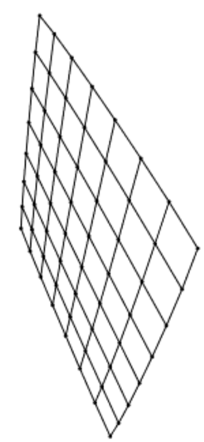
Product space

Let Σ be product space $\Sigma_1 \times \Sigma_2$.

	Σ_1	Σ_2	$\Sigma = \Sigma_1 \times \Sigma_2$
index	a_1	a_2	$a = (a_1, a_2)$
coordinates	$x_1^{\mu_1}$	$x_2^{\mu_2}$	$x^\mu = (x_1^{\mu_1}, x_2^{\mu_2})$
Laplacian	∇_1^2	∇_2^2	$\nabla^2 = \nabla_1^2 + \nabla_2^2$
basis	$f_{a_1}(x_1)$	$f_{a_2}(x_2)$	$f_a(x) = f_{a_1}(x_1) f_{a_2}(x_2)$
tensor	$P_{a_1 b_1 c_1}$	$P_{a_2 b_2 c_2}$	$P_{abc} = P_{a_1 b_1 c_1} P_{a_2 b_2 c_2}$

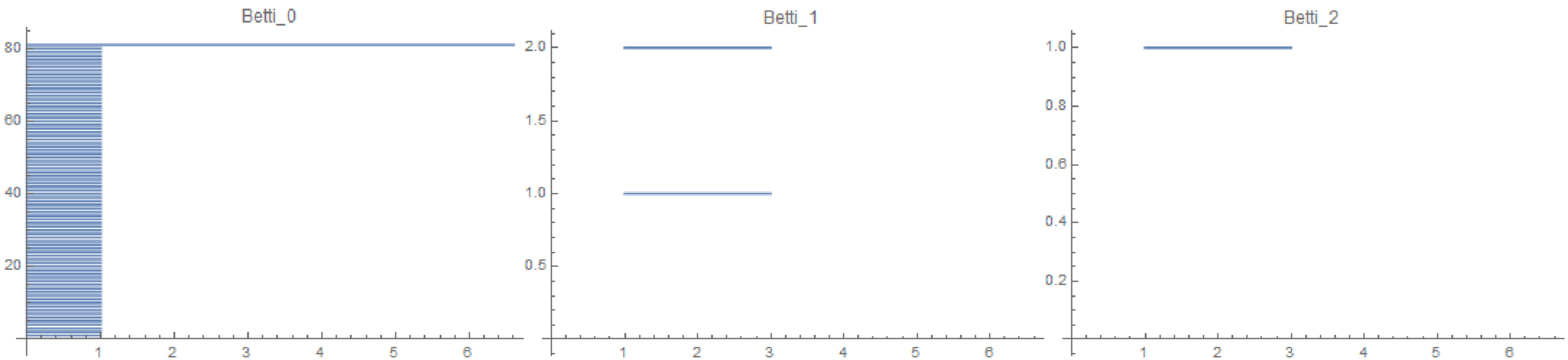
By using a damping factor like $\exp(-m^2/L^2)$,
the induced dumping factor $\exp(-(m_1^2 + m_2^2)/L^2)$ is also
a natural one in the product space.

Product spaces

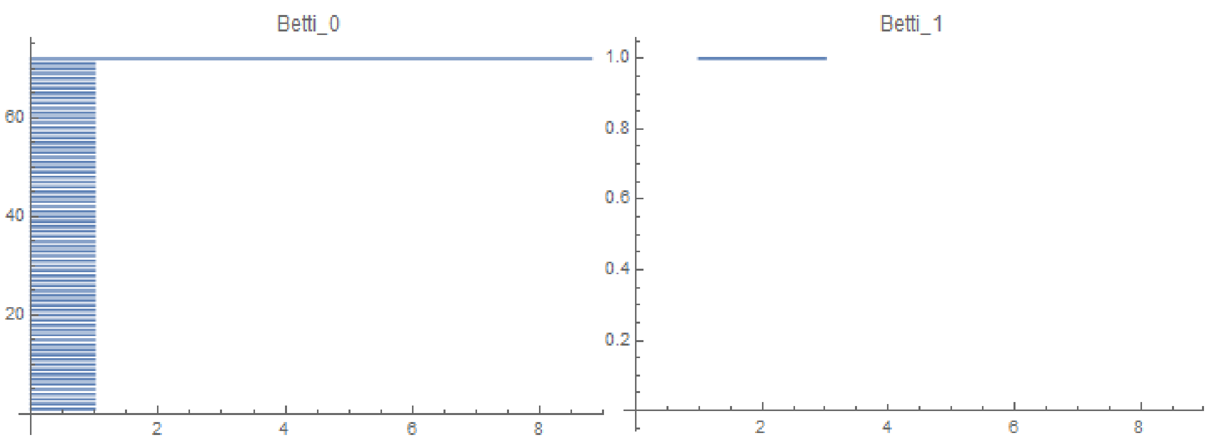
	Circle	Line Segment
Circle	Torus 	Cylinder 
Line Segment		Square 

Product spaces

Torus $(B_0, B_1, B_2) = (1,2,1)$

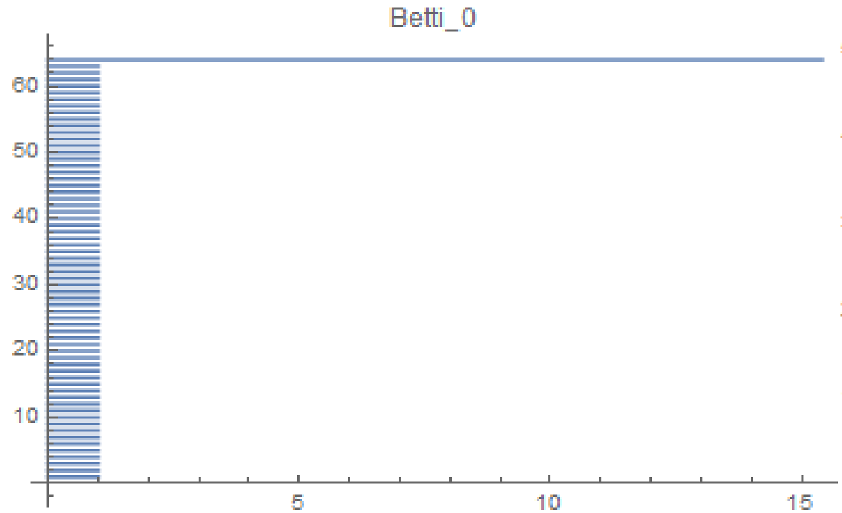


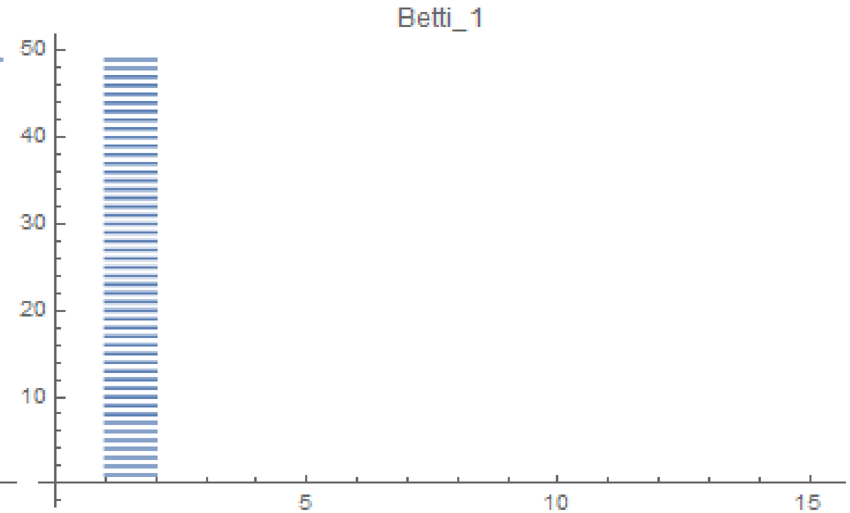
Cylinder $(B_0, B_1, B_2) = (1,1,0)$



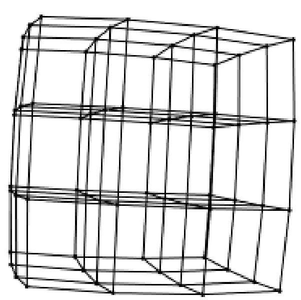
Product spaces

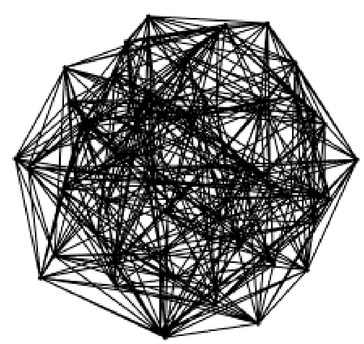
Square $(B_0, B_1, B_2) = (1,0,0)$





others



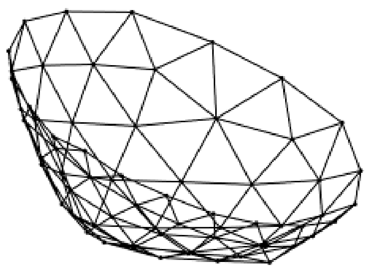


Klein bottle × Circle

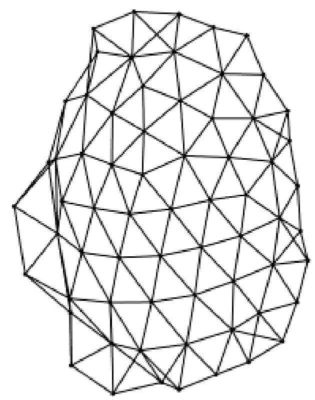
de Sitter space

- spacetime metric

$$ds^2 = -(1-r^2)dt^2 + \underbrace{\frac{dr^2}{1-r^2} + r^2 d\Omega_{D-2}^2}_{\equiv g_{\mu\nu}}$$



Dirichlet



Neumann

Summary

- A procedure to calculate P_{abc} associated with the fuzzy space was proposed.
- By using persistent homology method, the global structure of the fuzzy space was explored.

Future prospects

- non-compact manifold? (like black hole)
- thermodynamics?
- etc. …