Understanding zero-temperature criticality of Ising model on 2d dynamical triangulations

Yuki Sato

(Nagoya U)

Sep. 10, 2018

YS and T. Tanaka, Phys. Rev. D98 (2018) no.2

Work in progress w/ J. Ambjorn

@ Discrete Approaches to the Dynamics of Fields and Space-Time, Tohoku U

Ising model on 2d dynamical triangulations (DT) [Kazakov, 1986]

(1) Continuous phase transition at non-zero temperature T_c.

(2) Physics around the critical point is described by

2d gravity coupled to Majorana fermion

Reconsider criticality of Ising model on 2d DT [YS, Tanaka, 2017]

(1) Introduce a "loop-counting" parameter θ.

(2) Tuning θ, one can reduce $T_c(\theta)$ to absolute zero.

(3) Continuum theories around absolute zero are NOT

2d gravity coupled to Majorana fermions.

Ising model on 2d dynamical triangulations (DT) [Kazakov, 1986]

(1) Continuous phase transition at non-zero temperature T_c.

(2) Physics around the critical point is described by

2d gravity coupled to Majorana fermion

Understand
the difference between the two!!
[YS, Ambjorn]

Reconsider criticality of Ising model on 2d DT [YS, Tanaka, 2017]

(1) Introduce a "loop-counting" parameter θ.

(2) Tuning θ, one can reduce $T_c(\theta)$ to absolute zero.

(3) Continuum theories around absolute zero are NOT

2d gravity coupled to Majorana fermions.

Ising model on 2d dynamical triangulations (DT) [Kazakov, 1986]

(1) Continuous phase transition at non-zero temperature T_c.

(2) Physics around the critical point is described by

2d gravity coupled to Majorana fermion

New 1-parameter family of continuum theories! ← Understand the difference between the two!! [YS, Ambjorn]

Reconsider criticality of Ising model on 2d DT [YS, Tanaka, 2017]

(1) Introduce a “loop-counting” parameter θ .

(2) Tuning θ , one can reduce $T_c(\theta)$ to absolute zero.

(3) Continuum theories around absolute zero are NOT

2d gravity coupled to Majorana fermions.

Outline

(1) Ising model on 2d DT

(2) zero-temperature vs. finite temperature

(3) Discussion

Ising model on 2d DT

Ising model on honeycomb lattice (G):

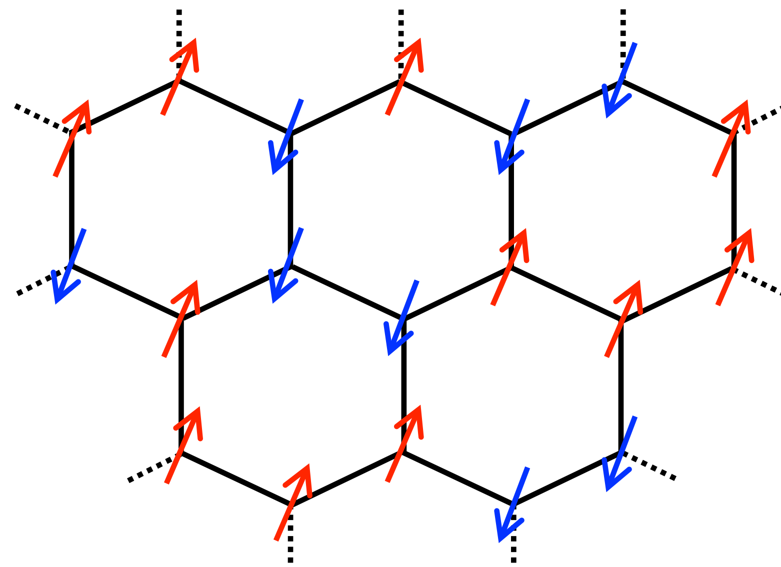
$$Z_G(\beta) = \sum_{\sigma} \prod_{<i,j>} e^{\beta \sigma_i \sigma_j}$$

Ising spin:

$$\sigma_i = \pm 1$$

Exactly solved in the thermodynamic limit:

[Weiner, 1950, Houtappel, 1950]



(1) 2nd order phase transition at

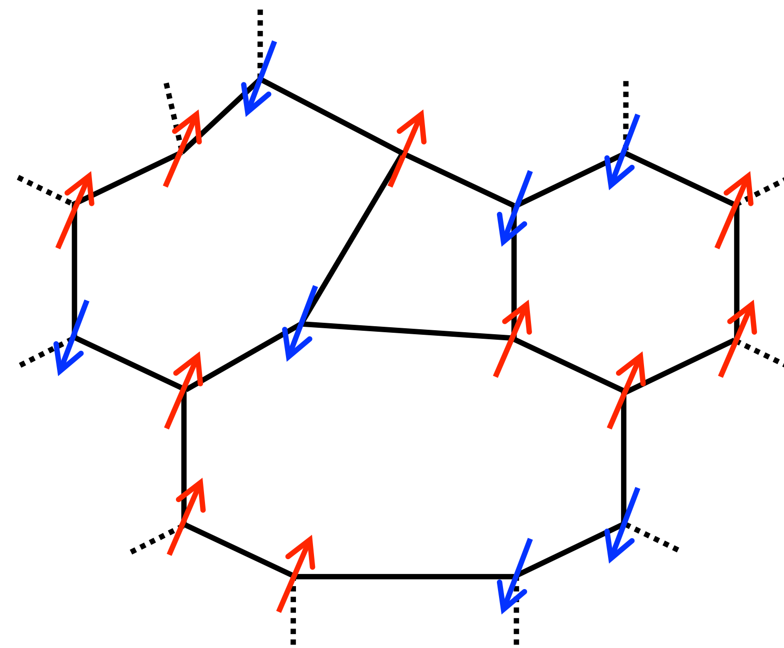
$$\beta = \beta_c \neq \infty$$

(2) Physics around β_c described by

2d Majorana fermion

Ising model on a planar graph (w/ coordination number = 3) (G'):

$$Z_{G'}(\beta) = \sum_{\sigma} \prod_{<i,j>} e^{\beta \sigma_i \sigma_j}$$



Summing all planar graphs (w/ coordination number = 3),

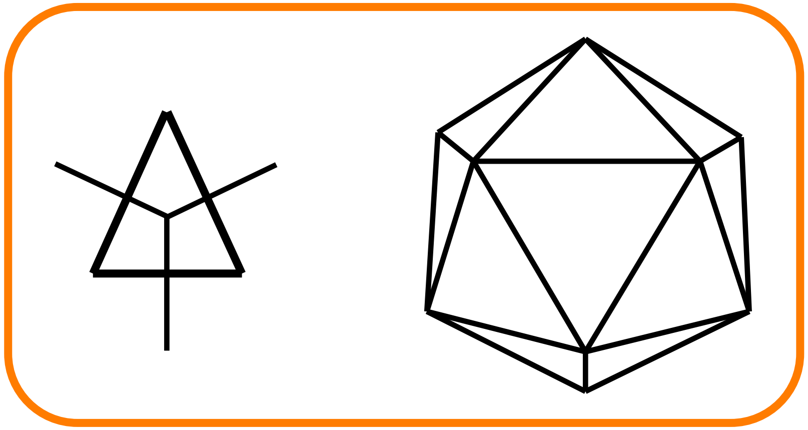
$$\{G, G', G'', \cdots\}$$

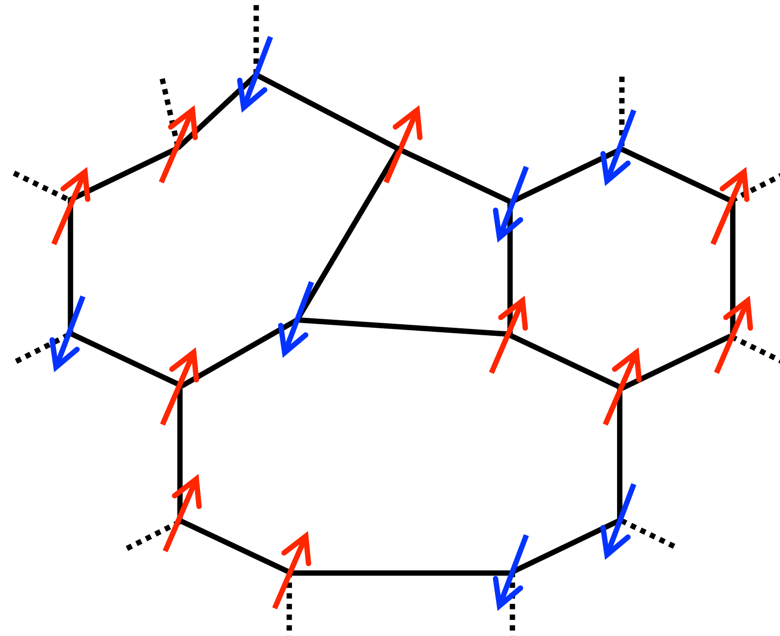
one can construct a solvable model (Ising model on 2d DT).

[Kazakov, 1986]

Ising model on a planar graph (w/ coordination number = 3) (G'):

$$Z_{G'}(\beta) = \sum_{\sigma} \prod_{<i,j>} e^{\beta \sigma_i \sigma_j}$$





Summing all planar graphs (w/ coordination number = 3),

$$\{G, G', G'', \cdots\}$$

one can construct a solvable model (Ising model on 2d DT).

[Kazakov, 1986]

Ising model on 2d dynamical triangulations (DT): [Kazakov, 1986]

$$Z(\beta, g_3) = \sum_G \frac{1}{|Aut(G)|} g_3^{n(G)} \boxed{Z_G(\beta)}$$

Ising model on G

g_3 : weight of a vertex

$n(G)$:= #(vertices in G)

Ising model on 2d dynamical triangulations (DT): [Kazakov, 1986]

$$Z(\beta, g_3) = \sum_G \frac{1}{|Aut(G)|} g_3^{n(G)} Z_G(\beta)$$

$$=: \sum_n g_3^n \boxed{Z_n(\beta)}$$

Partition function

Ising model on 2d dynamical triangulations (DT): [Kazakov, 1986]

$$Z(\beta, g_3) = \sum_G \frac{1}{|Aut(G)|} g_3^{n(G)} Z_G(\beta)$$

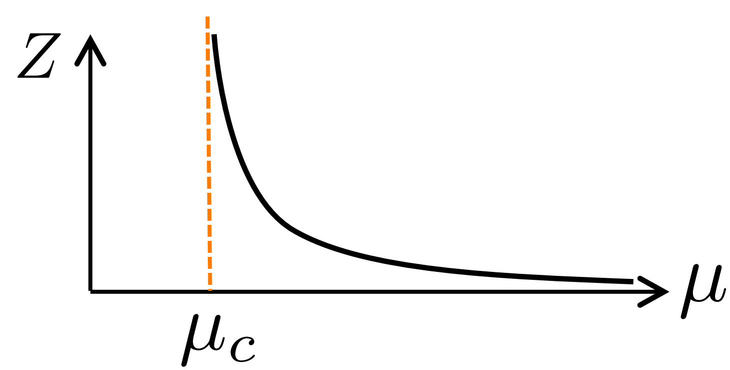
$$=: \sum_n g_3^n Z_n(\beta)$$

$$g_3 = e^{-\mu} \qquad F_n(\beta) := \log Z_n(\beta)$$

$$= \sum_n e^{-n(\mu - \frac{1}{n} F_n(\beta))}$$

Radius of convergence, $(g_3)_c = e^{-\mu_c}$

$$\mu_c := \lim_{n\to\infty} \frac{1}{n} F_n(\beta)$$



Average of #(vertices) goes to infinity as $\mu \to \mu_c$:

$$\langle n \rangle := g_3 \frac{\partial}{\partial g_3} \log Z(\beta, g_3) \Big|_{\mu=\mu_c} = \infty \qquad (g_3 = e^{-\mu})$$

Thermodynamic limit of dynamical triangulations (DT):

$$\mu \to \mu_c \quad \text{or} \quad g_3 \to (g_3)_c = e^{-\mu_c}$$

Free energy (per vertex) of Ising model dressed by DT:

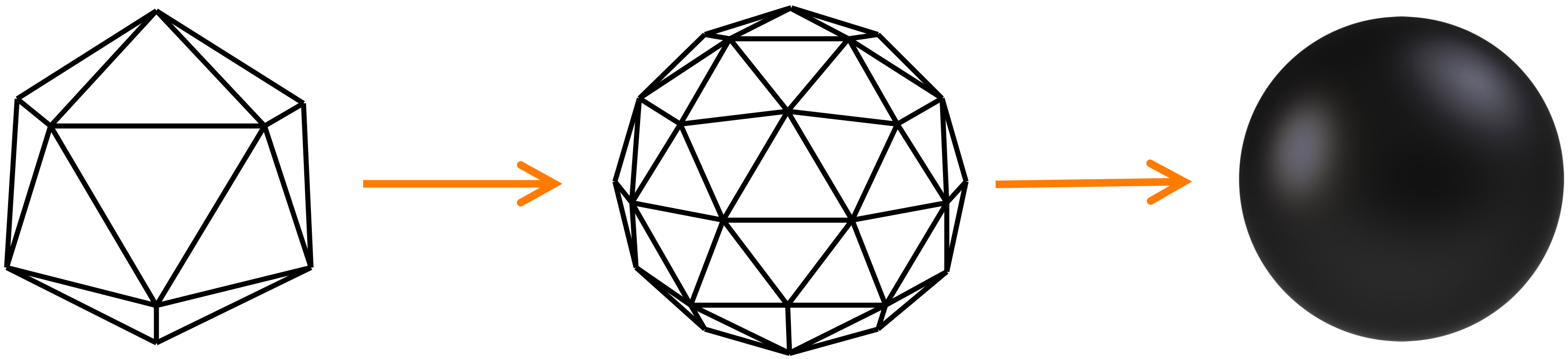
$$f(\beta) = -\frac{1}{\beta}\mu_c(\beta) = -\frac{1}{\beta} \lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta)$$

f(β) becomes singular at $\beta = \beta_c$

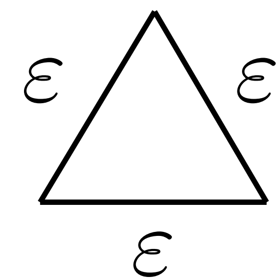
⟶ Critical point of Ising model on 2d DT

Continuum limit of dynamical triangulations (DT):

$$\mu \to \mu_c \quad \text{and} \quad \varepsilon \to 0 \quad \text{w/} \quad A = \langle n \rangle \varepsilon^2 \quad \text{kept fixed}$$



where ε is the lattice spacing and A is a physical area.



Continuum theories

(1) 2d pure gravity at $\beta \neq \beta_c$

(2) 2d gravity coupled to fermions at $\beta = \beta_c (\neq \infty)$

Definition via Hermitian NxN two-matrix model: [Kazakov, 1986]

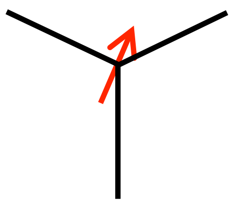
$$Z_N(\beta, g_3) = \int D\varphi_+ D\varphi_- e^{-N\mathrm{tr}U(\varphi_+,\varphi_-)}$$

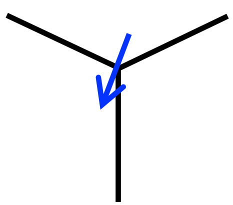
where

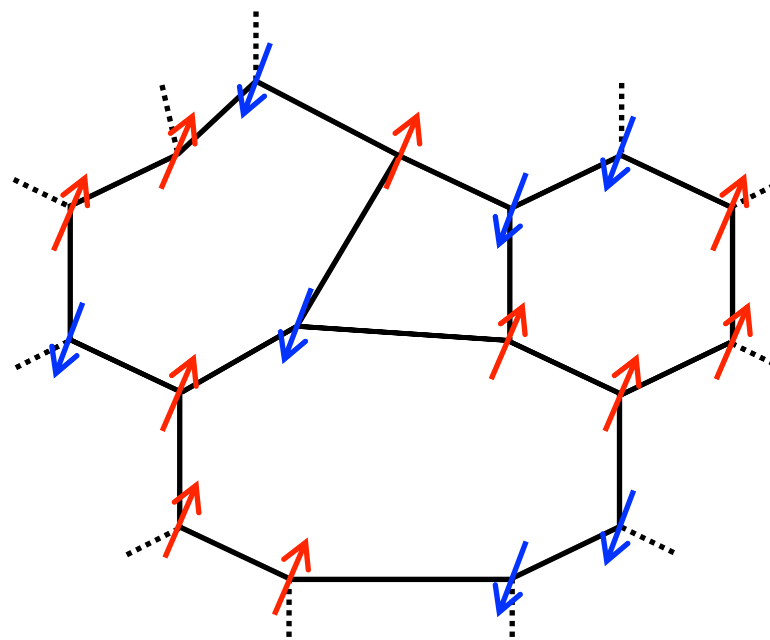
weight of vertex

$$U(\varphi_+, \varphi_-) = \frac{e^\beta}{\sinh(2\beta)}(\varphi_+^2 + \varphi_-^2 - 2e^{-2\beta}\varphi_+\varphi_-) - \frac{g_3}{3}(\varphi_+^3 + \varphi_-^3)$$

Vertices:

$$\mathrm{tr}\left(\varphi_+^3\right) \sim$$ 

$$\mathrm{tr}\left(\varphi_-^3\right) \sim$$ 



Definition via Hermitian NxN two-matrix model: [Kazakov, 1986]

$$Z_N(\beta, g_3) = \int D\varphi_+ D\varphi_- e^{-N\mathrm{tr}U(\varphi_+,\varphi_-)}$$

where nearest-neighbor interactions

$$U(\varphi_+,\varphi_-) = \frac{e^{\beta}}{\sinh(2\beta)}(\varphi_+^2 + \varphi_-^2 - 2e^{-2\beta}\varphi_+\varphi_-) - \frac{g_3}{3}(\varphi_+^3 + \varphi_-^3)$$

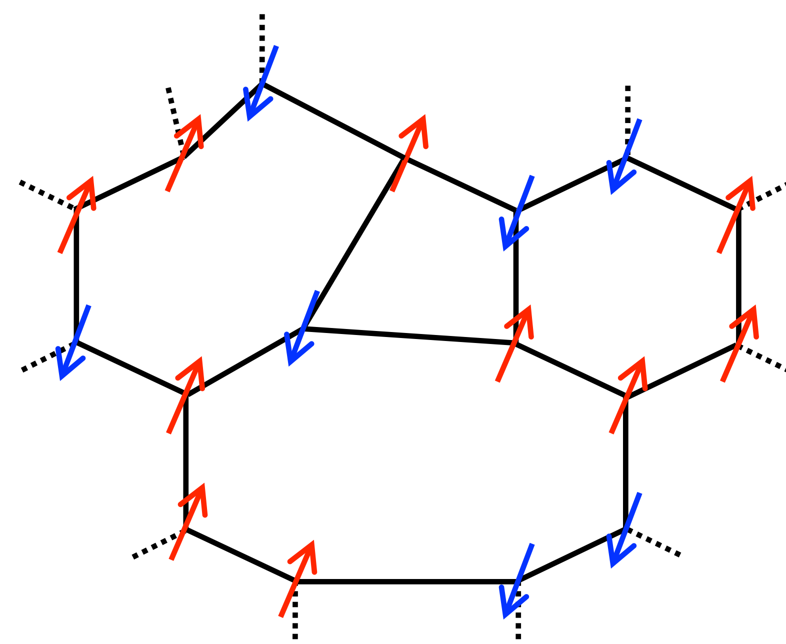
Propagators:

$$\langle\varphi_+\varphi_+\rangle = \frac{1}{N}e^{\beta\sigma_+\sigma_+} \sim$$

$$\langle\varphi_-\varphi_+\rangle = \frac{1}{N}e^{\beta\sigma_-\sigma_+} \sim$$

$$\langle\varphi_+\varphi_-\rangle = \frac{1}{N}e^{\beta\sigma_+\sigma_-} \sim$$

$$\langle\varphi_-\varphi_-\rangle = \frac{1}{N}e^{\beta\sigma_-\sigma_-} \sim$$

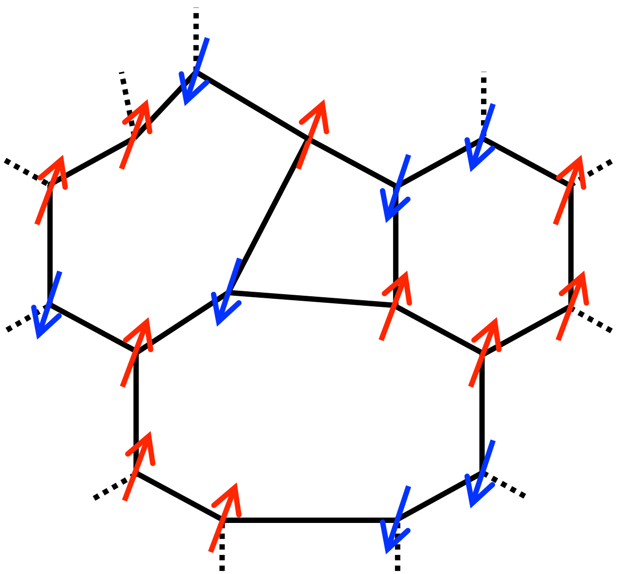


Definition via Hermitian NxN two-matrix model: [Kazakov, 1986]

$$Z_N(\beta, g_3) = \int D\varphi_+ D\varphi_- e^{-N\mathrm{tr}U(\varphi_+,\varphi_-)}$$

where

$$U(\varphi_+,\varphi_-) = \frac{e^{\beta}}{\sinh(2\beta)}(\varphi_+^2 + \varphi_-^2 - 2e^{-2\beta}\varphi_+\varphi_-) - \frac{g_3}{3}(\varphi_+^3 + \varphi_-^3)$$



$\sim N^{2-2h}$

h=0 for planar
h>0 for non-planar

In the large-N limit, planar graphs survive.

Definition via Hermitian NxN two-matrix model: [Kazakov, 1986]

$$Z_N(\beta, g_3) = \int D\varphi_+ D\varphi_- e^{-N\mathrm{tr}U(\varphi_+,\varphi_-)}$$

where

$$U(\varphi_+,\varphi_-) = \frac{e^\beta}{\sinh(2\beta)}(\varphi_+^2 + \varphi_-^2 - 2e^{-2\beta}\varphi_+\varphi_-) - \frac{g_3}{3}(\varphi_+^3 + \varphi_-^3)$$

Define Ising model on DT via the matrix model:

$$Z(\beta, g_3) = \lim_{N\to\infty} \frac{1}{N^2} \log\left(\frac{Z_N(\beta, g_3)}{Z_N(\beta, 0)}\right)$$

$$= \sum_G \frac{1}{|Aut(G)|} g_3^{n(G)} Z_G(\beta)$$

(G: a connected planar graph)

Zero-temperature

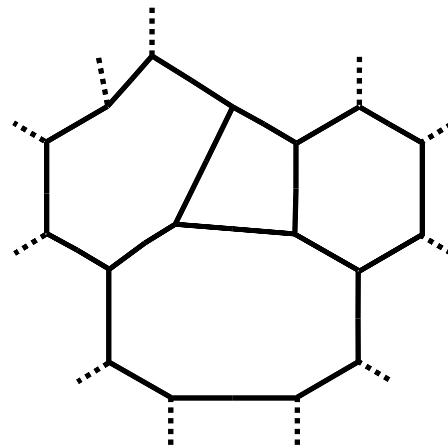
v.s

Finite temperature

Kazakov's potential: [Kazakov, 1986]

$$U^{(2)}(\psi_+,\psi_-) = \frac{1}{2}(\psi_+^2 + \psi_-^2 - 2c_{\mathrm{dt}}\psi_+\psi_-) - \frac{g_{\mathrm{dt}}}{3}(\psi_+^3 + \psi_-^3)$$

where $c_{\mathrm{dt}} = e^{-2\beta_{\mathrm{dt}}}$

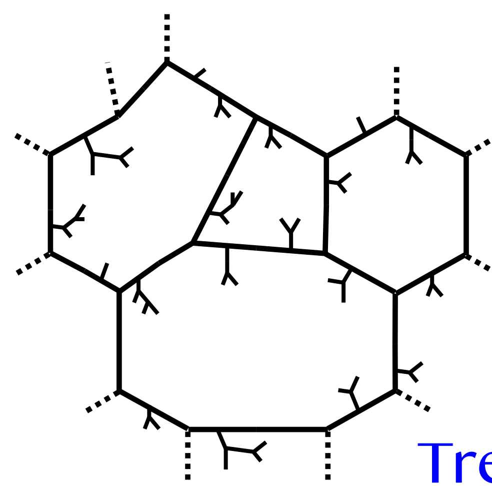


Skeleton graph

Our potential: [YS, Tanaka, 2017]

$$U^{(0)}(\varphi_+,\varphi_-) = \frac{1}{\theta}\left(\frac{1}{2}\left(\varphi_+^2 + \varphi_-^2 - 2c\varphi_+\varphi_-\right) - \underline{g(\varphi_- + \varphi_-)} - \frac{g}{3}\left(\varphi_+^3 + \varphi_-^3\right)\right)$$

where $c = e^{-2\beta}$

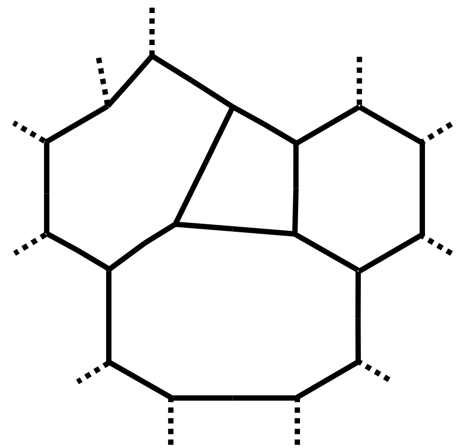


Trees are attached!

Kazakov's potential: [Kazakov, 1986]

$$U^{(2)}(\psi_+,\psi_-) = \frac{1}{2}(\psi_+^2 + \psi_-^2 - 2c_{\rm dt}\psi_+\psi_-) - \frac{g_{\rm dt}}{3}(\psi_+^3 + \psi_-^3)$$

where $c_{\rm dt} = e^{-2\beta_{\rm dt}}$

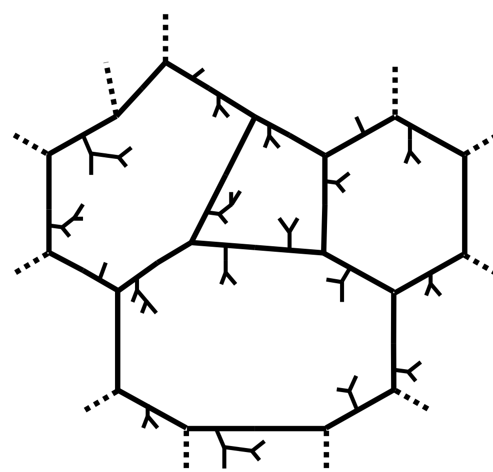


Skeleton graph

Our potential: [YS, Tanaka, 2017]

$$U^{(0)}(\varphi_+,\varphi_-) = \frac{1}{\theta}\left(\frac{1}{2}\left(\varphi_+^2 + \varphi_-^2 - 2c\varphi_+\varphi_-\right) - g(\varphi_- + \varphi_-) - \frac{g}{3}\left(\varphi_+^3 + \varphi_-^3\right)\right)$$

where $c = e^{-2\beta}$



$\sim \theta^{\#(loops)-2}$

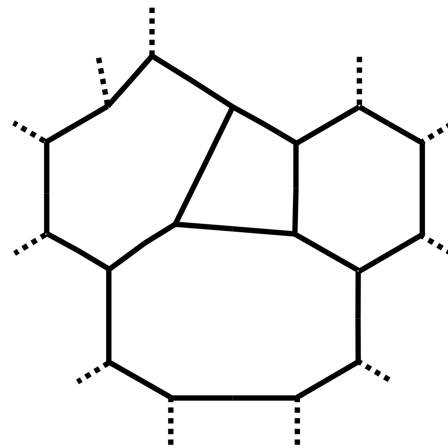
When $\theta << 1$,
trees become dominant

Kazakov's potential: [Kazakov, 1986]

$$U^{(2)}(\psi_+, \psi_-) = \frac{1}{2}(\psi_+^2 + \psi_-^2 - 2c_{\mathrm{dt}}\psi_+\psi_-) - \frac{g_{\mathrm{dt}}}{3}(\psi_+^3 + \psi_-^3)$$

finite temperature

$$(\beta_{\mathrm{dt}})_c^{-1} \neq 0$$

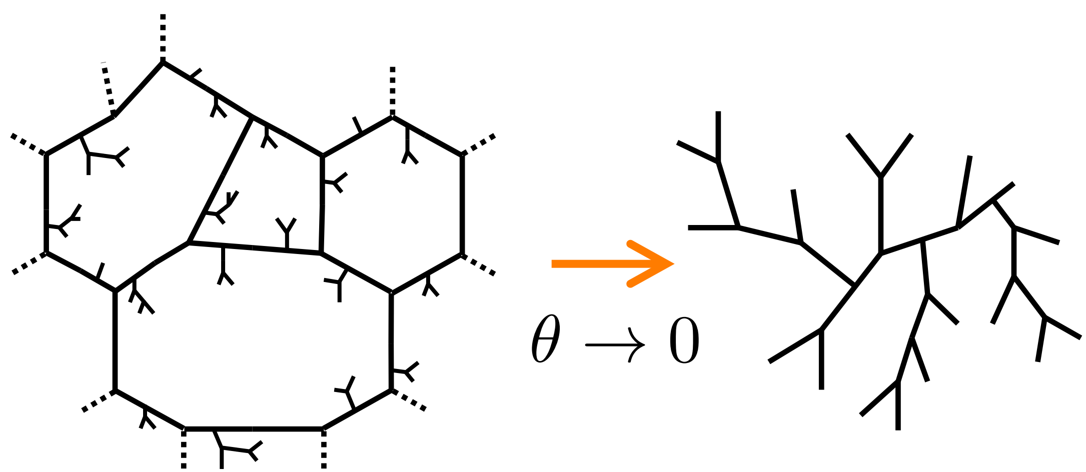


Our potential: [YS, Tanaka, 2017]

$$U^{(0)}(\varphi_+, \varphi_-) = \frac{1}{\theta}\left(\frac{1}{2}\left(\varphi_+^2 + \varphi_-^2 - 2c\varphi_+\varphi_-\right) - g(\varphi_- + \varphi_-) - \frac{g}{3}\left(\varphi_+^3 + \varphi_-^3\right)\right)$$

zero temperature

$$\lim_{\theta \to 0} \beta_c^{-1}(\theta) = 0$$



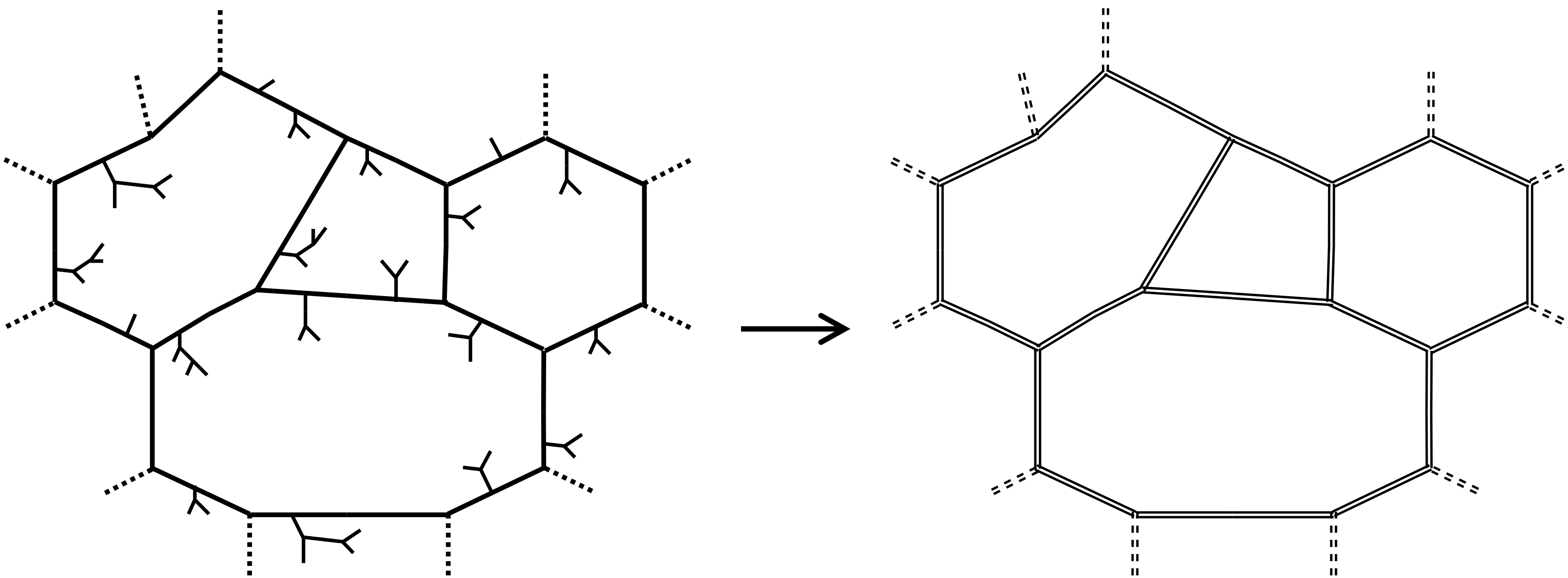
Remove the linear terms,

$$U^{(0)}(\varphi_+, \varphi_-) = \frac{1}{\theta}\left(\frac{1}{2}\left(\varphi_+^2 + \varphi_-^2 - 2c\varphi_+\varphi_-\right) - g(\varphi_- + \varphi_-) - \frac{g}{3}\left(\varphi_+^3 + \varphi_-^3\right)\right)$$

$$\varphi_\pm = \tilde{\varphi}_\pm + Z_{\text{tree}}(g, c) \qquad Z_{\text{tree}} = \frac{1 - c - \sqrt{(1-c)^2 - 4g^2}}{2g}$$

$$U^{(1)}(\tilde{\varphi}_+, \tilde{\varphi}_-) = \frac{1}{\theta}\left(\boxed{\frac{1 - 2gZ_{\text{tree}}}{2}}(\tilde{\varphi}_+^2 + \tilde{\varphi}_-^2) - c\tilde{\varphi}_+\tilde{\varphi}_- - \frac{g}{3}(\tilde{\varphi}_+^3 + \tilde{\varphi}_-^3)\right)$$

Trees are integrated out:



Normalize quadratic terms,

$$U^{(0)}(\varphi_+,\varphi_-) = \frac{1}{\theta}\left(\frac{1}{2}\left(\varphi_+^2 + \varphi_-^2 - 2c\varphi_+\varphi_-\right) - g(\varphi_- + \varphi_-) - \frac{g}{3}\left(\varphi_+^3 + \varphi_-^3\right)\right)$$

$$\Big\downarrow \quad \varphi_\pm = \tilde{\varphi}_\pm + Z_{\rm tree}(g,c)$$

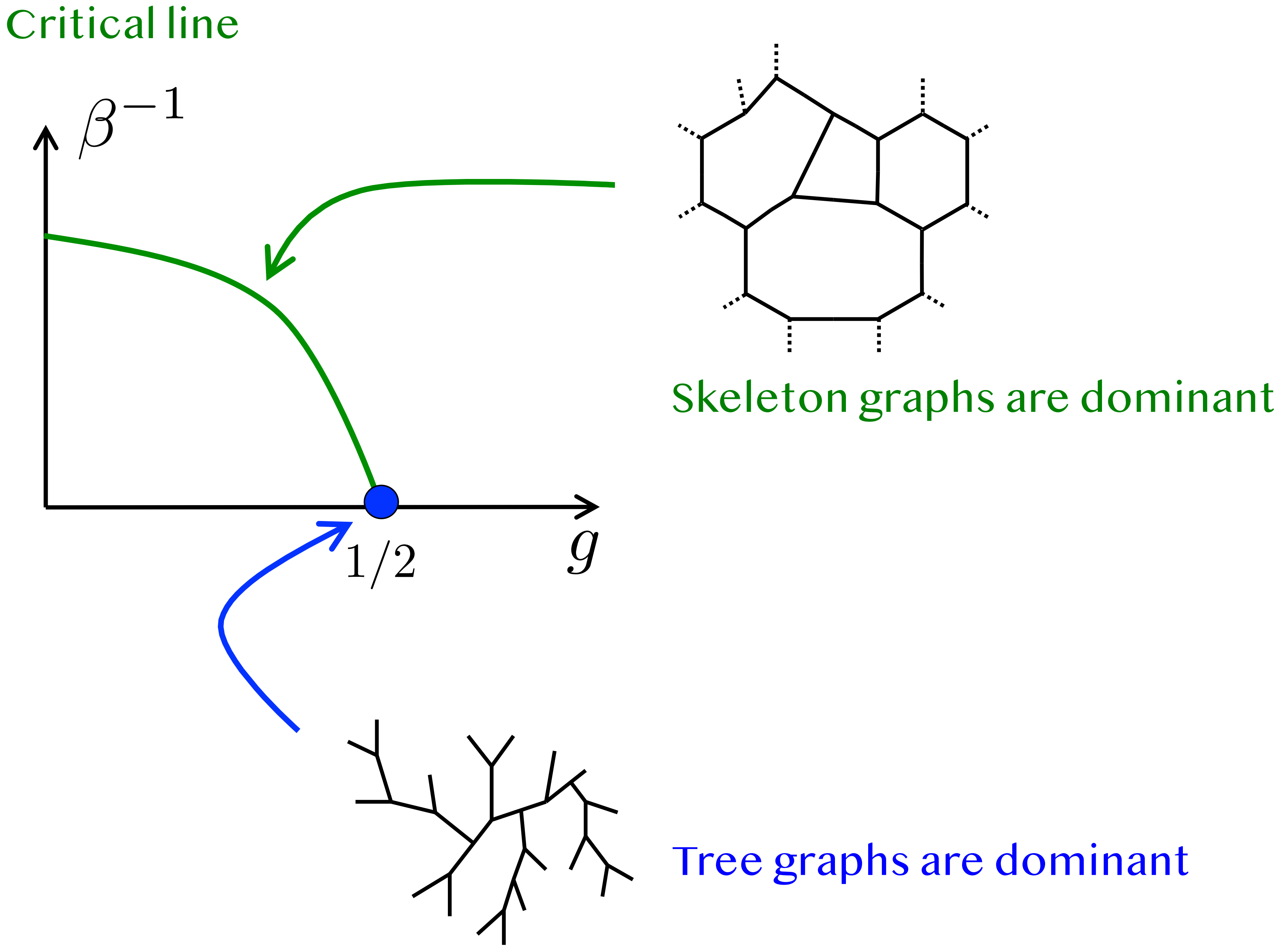
$$U^{(1)}(\tilde{\varphi}_+,\tilde{\varphi}_-) = \frac{1}{\theta}\left(\frac{1-2gZ_{\rm tree}}{2}(\tilde{\varphi}_+^2 + \tilde{\varphi}_-^2) - c\tilde{\varphi}_+\tilde{\varphi}_- - \frac{g}{3}(\tilde{\varphi}_+^3 + \tilde{\varphi}_-^3)\right)$$

$$\Big\downarrow \quad \tilde{\varphi}_\pm = \sqrt{\frac{\theta}{1-2gZ_{\rm tree}}}\psi_\pm$$

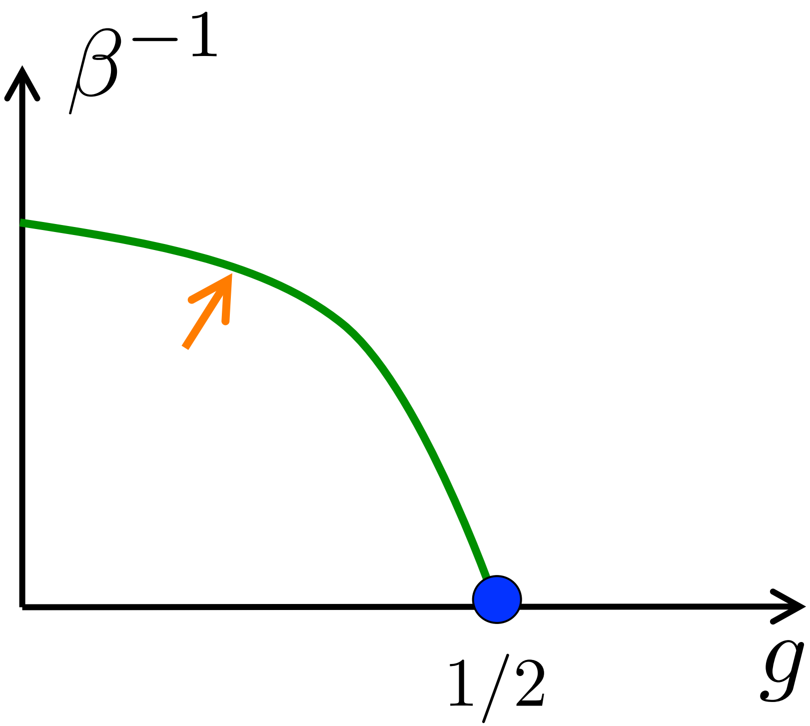
$$U^{(2)}(\psi_+,\psi_-) = \frac{1}{2}(\psi_+^2 + \psi_-^2 - 2c_{\rm dt}\psi_+\psi_-) - \frac{g_{\rm dt}}{3}(\psi_+^3 + \psi_-^3)$$

where

$$c_{\rm dt} := \frac{c}{1-2gZ_{\rm tree}(g,c)} \qquad g_{\rm dt} := \frac{\theta^{1/2}g}{(1-2gZ_{{\rm tree}(g,c)})^{3/2}}$$


Critical line
β^{-1}
$1/2$
g
Skeleton graphs are dominant
Tree graphs are dominant

Critical line



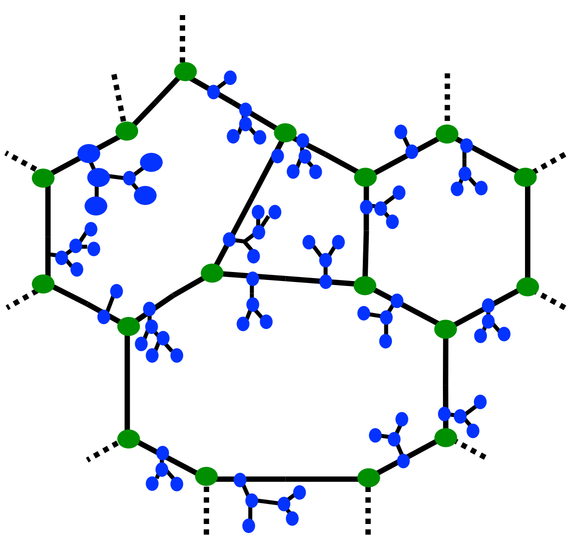
Continuum limit

$g \to g_c(\theta)$ & $\varepsilon \to 0$

w/ $\langle n \rangle \varepsilon^2$ kept fixed

$$\langle n \rangle \sim \frac{1}{g_c(\theta) - g}$$

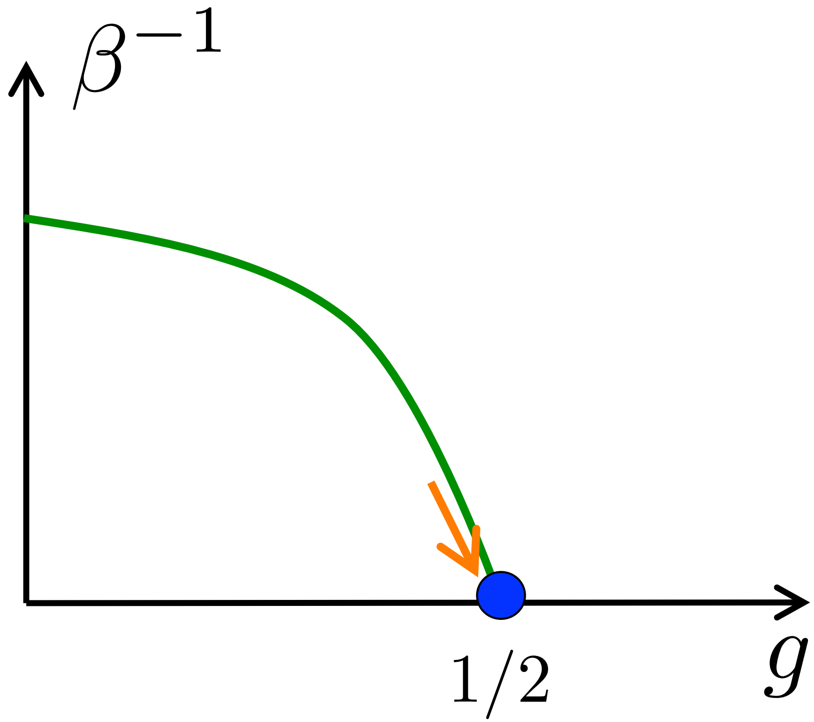
$$g = g_c(\theta)(1 - \Lambda \varepsilon^2)$$

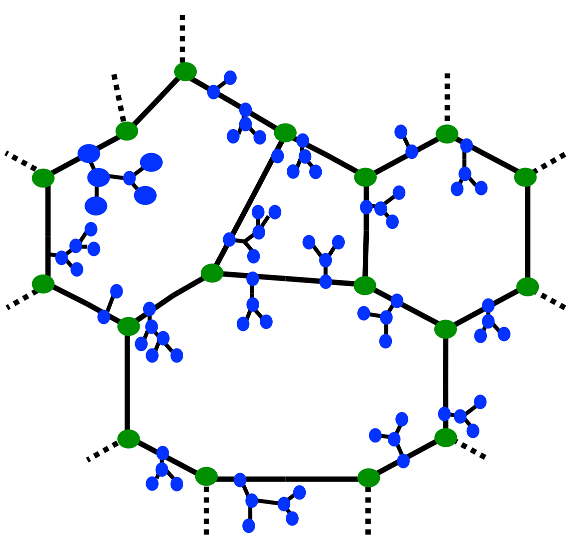


$$\langle n_{skel} \rangle \sim \frac{1}{\varepsilon^2}$$

$$\langle n_{tree} \rangle \sim \mathcal{O}(1)$$

Critical line





Continuum limit

$$g \to g_c(\theta) \quad \& \quad \varepsilon \to 0$$

w/ $\langle n \rangle \varepsilon^2$ kept fixed

$$\langle n \rangle \sim \frac{1}{g_c(\theta) - g}$$

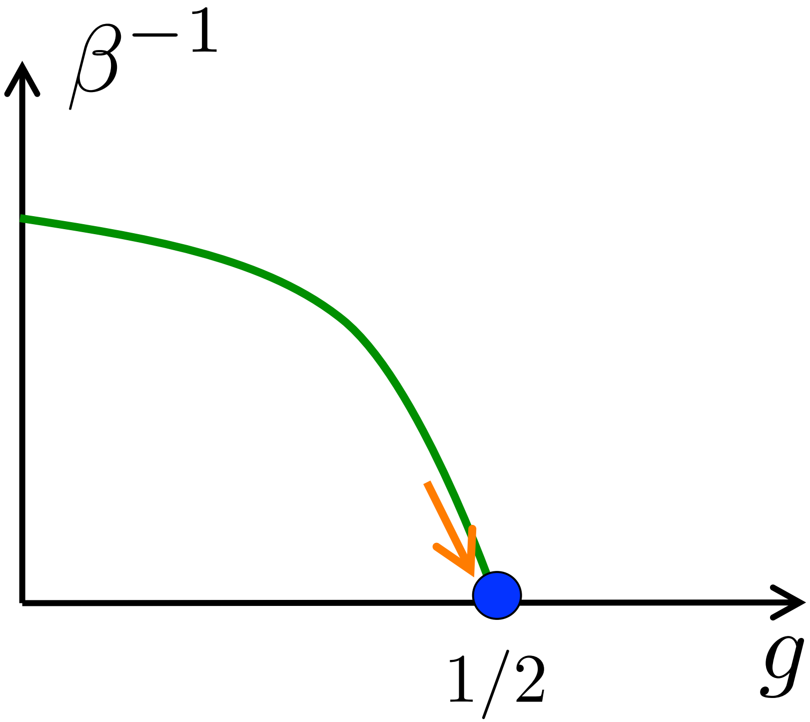
$$g = g_c(\theta)(1 - \Lambda\varepsilon^2)$$

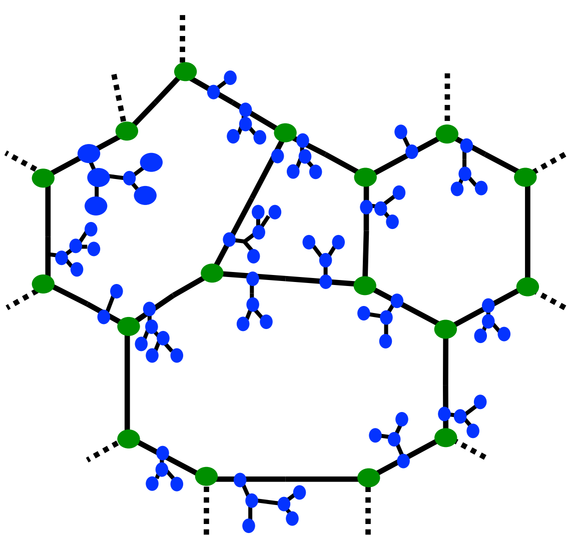
$$\theta = \Theta\varepsilon^3$$

$$\langle n_{skel} \rangle \sim \mathcal{O}(1)$$

$$\langle n_{tree} \rangle \sim \frac{1}{\varepsilon^2}$$

Critical line





Continuum limit

$$g \to g_c(\theta) \quad \& \quad \varepsilon \to 0$$

w/ $\langle n \rangle \varepsilon^2$ kept fixed

$$\langle n \rangle \sim \frac{1}{g_c(\theta) - g}$$

$$g = g_c(\theta)(1 - \Lambda \varepsilon^2)$$

$$\theta = \Theta_\alpha \varepsilon^\alpha \quad (0 < \alpha < 3)$$

$$\langle n_{skel} \rangle \sim \frac{1}{\varepsilon^{2 - \frac{2}{3}\alpha}}$$

$$\langle n_{tree} \rangle \sim \frac{1}{\varepsilon^{\frac{2}{3}\alpha}}$$

Using the relation between Kazakov's and our parametrisations, in the continuum limit

$$g = g_c(\theta)(1 - \Lambda\varepsilon^2) \qquad \theta = \Theta_\alpha \varepsilon^\alpha \quad (0 < \alpha < 3)$$

one can show

$$\frac{g_{\rm dt}}{(g_{\rm dt})_c} = \frac{g}{g_c}\left(1 - \frac{5^{2/3}}{14+\sqrt{7}}\frac{\Lambda}{\Theta_\alpha^{2/3}}\varepsilon^{2-\frac{2}{3}\alpha} + \cdots\right)^{3/2}$$

→ If $\alpha = 3$,
one cannot reach Kazakov's critical point

Discussion

Discussion part was intentionally deleted!!