gravity/CYBE correspondence

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based on work with T. Araujo, I. Bakhmatov, M. M. Sheikh-Jabbari, J. Sakamoto, Y. Sakatani, H. Yavartanoo, K .Yoshida

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Basic Idea

Matsumoto & Yoshida (2014+)

"...a possible classification of integrable deformations and the corresponding gravity solution in terms of solutions of CYBE..."

"...a correspondence between the classical r-matrices satisfying the CYBE and the deformed type IIB supergravity backgrounds..."

built on Yang-Baxter σ-model (Klimcik) & application to strings on $AdS_5 \times S^5$ (Delduc, Magro, Vicedo; Kawaguchi, Matsumoto, Yoshida)

Viewpoint

(Super)gravity appears to know about r-matrix solutions to the Classical Yang-Baxter Equation.

This it does through its equations of motion.

Concretely, through a simple matrix inversion one can define a deformation and the equations of motion fix the deformation to be an r-matrix solution to the CYBE.

At first sight, this is surprising: Einstein gravity is dynamical, but the CYBE is algebraic!

Classical Yang-Baxter

"Classical limit" of the QYB: simpler equation

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = -c^2[X, Y]$$

$$X, Y \in \mathfrak{g}, \quad c \in \mathbb{C}$$

$$R(X) = r^{ij}\, b_i\, \mathrm{Tr}[b_j X] \qquad\qquad r = \frac{1}{2} r^{ij} b_i \wedge b_j$$

$$\mathrm{Tr}[b_i X]\mathrm{Tr}[b_j Y] b_k \left(r^{l_1 i} r^{l_2 j} f_{l_1 l_2}{}^{k} + r^{l_1 j} r^{l_2 k} f_{l_1 l_2}{}^{i} + r^{l_1 k} r^{l_2 i} f_{l_1 l_2}{}^{j} \right) = -c^2[X, Y]$$

Generalized Supergravity

Arutyunov, Hoare, Frolov, Roiban, Tseytlin; Tseytlin, Wulff

$$R_{MN} - \frac{1}{4} H_{MKL} H_N{}^{KL} - T_{MN} + \nabla_M X_N + \nabla_N X_M = 0$$

$$\frac{1}{2} \nabla^K H_{KMN} + \frac{1}{2} \mathcal{F}^K \mathcal{F}_{KMN} + \frac{1}{12} \mathcal{F}_{MNKLP} \mathcal{F}^{KLP} = X^K H_{KMN} + \nabla_M X_N - \nabla_N X_M$$

$$R - \frac{1}{12} H^2 + 4 \nabla_M X^M - 4 X_M X^M = 0$$

$$T_{MN} \equiv \frac{1}{2} \mathcal{F}_M \mathcal{F}_N + \frac{1}{4} \mathcal{F}_{MKL} \mathcal{F}_N{}^{KL} + \frac{1}{96} \mathcal{F}_{MPQRS} \mathcal{F}_N{}^{PQRS} - \frac{1}{4} G_{MN} \left(\mathcal{F}_K \mathcal{F}^K + \frac{1}{6} \mathcal{F}_{PQR} \mathcal{F}^{PQR} \right)$$

$$X = \mathrm{d}\Phi + I + i_I B, \quad \mathcal{F} = e^{\Phi} F$$

Elitzur, Giveon, Rabinovici, Schwimmer, Veneziano (1994)

Generalized Supergravity

RR sector equations simpler when expressed in Page forms.

$$\mathrm{d}Q_{2n-1} = i_I Q_{2n+1}, \quad n = 1, 2, 3, 4$$

$$Q_1 = F_1, \quad Q_3 = F_3 + B \wedge F_1, \quad Q_5 = F_5 + B \wedge F_3 + \frac{1}{2} B^2 \wedge F_1,$$

$$Q_7 = - * F_3 + B \wedge F_5 + \frac{1}{2} B^2 \wedge F_3 + \frac{1}{3!} B^3 \wedge F_1,$$

$$Q_9 = *F_1 - B \wedge *F_3 + \frac{1}{2} B^2 \wedge F_5 + \frac{1}{3!} B^3 \wedge F_3 + \frac{1}{4!} B^4 \wedge F_1$$

Recipe I

Consider a supergravity solution with isometry group specified by metric g & two-form B.

Deform it by a bivector.

Seiberg, Witten (1999)

$$[(g+B)^{-1}+\Theta]=g'+B', \quad e^{-2\Phi}\sqrt{-g}=e^{-2\Phi'}\sqrt{-g'}$$

Sakamoto, Sakatani; Borsato, Wulff

Killing vector is determined: $I^{\mu}=\nabla^{(g)}_{\nu}\Theta^{\nu\mu}$

YB = open-closed string

This "open-closed string" map came out of efforts to understand AdS/CFT picture of Yang-Baxter deformations.

Earlier work by S. van Tongeren (Abelian twists) showing NC parameter is the r-matrix.

But this is more general.

In 1702.02861 we showed this extended to all YB deformations based on r-matrix solutions to hCYBE and the open string metric is undeformed.

In 1708.03163 we extended this to YB deformations based on r-matrix solutions to mCYBE.

Recipe II

Philosophy: "All information in the bivector"

How do we implement this in the RR sector?

Employ AdS/CFT logic

$$Q_{2(n-p)+1} = \frac{(-1)^p}{p!} \Theta^p \lrcorner \tilde{Q}_{2n+1}$$

Ex: $AdS_5 \times S^5$

$$Q_5 = 4[\mathrm{vol}(AdS_5) + \mathrm{vol}(S^5)]$$

$$Q_3 \propto *_5 \Theta \quad \Rightarrow \quad \mathrm{d}Q_3 \propto \mathrm{d} *_5 \Theta \propto i_I Q_5$$

Example I

Consider $AdS_2 \times S^2$

$$\mathrm{d}s^2 = \frac{(-\mathrm{d}t^2 + \mathrm{d}z^2)}{z^2} + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 + \mathrm{d}s^2(T^6),$$

$$F_5 = (1 + *_{10})\frac{1}{\sqrt{2}z^2}\mathrm{d}t \wedge \mathrm{d}z \wedge (\omega_r - \omega_i)$$

with the (yet unspecified) deformation

$$\Theta^{tz} = \Theta_1(t,z), \quad \Theta^{\theta\phi} = \Theta_2(\theta,\phi)$$

Example I

Follow recipe to get deformed NS sector

$$\mathrm{d}s^2 = \frac{z^2(-\mathrm{d}t^2 + \mathrm{d}z^2)}{z^4 - \Theta_1^2} + \frac{\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2}{1 + \Theta_2^2 \sin^2\theta},$$

$$B = \frac{\Theta_1}{z^4 - \Theta_1^2} \mathrm{d}t \wedge \mathrm{d}z - \frac{\Theta_2 \sin^2\theta}{1 + \Theta_2^2 \sin^2\theta} \mathrm{d}\theta \wedge \mathrm{d}\phi,$$

$$e^{2\Phi} = \frac{e^{2\Phi_0} z^4}{(z^4 - \Theta_1^2)(1 + \Theta_2^2 \sin^2\theta)},$$

$$I = -\frac{1}{z^2} \partial_z (z^2 \Theta_1) \partial_t + \partial_t \Theta_1 \partial_z - \partial_\phi \Theta_2 \partial_\theta + \frac{1}{\sin\theta} \partial_\theta (\sin\theta \Theta_2) \partial_\phi$$

Example I

Complete RR sector

$$F_3 = -\frac{\Theta_1}{\sqrt{2}}\frac{1}{z^2}(\omega_r - \omega_i) - \frac{\Theta_2}{\sqrt{2}}\sin\theta(\omega_r + \omega_i),$$

$$F_5 = \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^4-\Theta_1^2}\,\mathrm{d}t\wedge\mathrm{d}z\wedge(\omega_r-\omega_i) + \frac{\sin\theta}{1+\Theta_2^2\sin^2\theta}\,\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge(\omega_r+\omega_i) + \frac{\Theta_1\Theta_2\sin\theta}{z^4-\Theta_1^2}\,\mathrm{d}t\wedge\mathrm{d}z\wedge(\omega_r+\omega_i) \right.$$
$$\left. - \frac{\Theta_1\Theta_2\sin^2\theta}{z^2(1+\Theta_2^2\sin^2\theta)}\,\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge(\omega_r-\omega_i)\right]$$

Moving parts have yet to be determined.

We will solve for these terms.

Example I

$$\Theta_1 = c_1 t z + c_2 z(t^2 - z^2) + c_3 z,$$
$$\Theta_2 = c_4 \cos\phi + c_5 \sin\phi + c_6 \cot\theta$$

$$\kappa^2 = -c_1^2 + 4c_2 c_3 = c_4^2 + c_5^2 + c_6^2, \quad e^{2\Phi_0} = 1 + \kappa^2$$

Conditions are precisely (modified) CYBE

$$b_1 = -t\partial_t - z\partial_z, \quad b_2 = -\partial_t, \quad b_3 = -(t^2 + z^2)\partial_t - 2tz\partial_z, \ldots$$

Works for η-deformations!

What did we learn?

Example is simple, but deformation is fixed by equations of motion to be an r-matrix solution to the CYBE.

Can repeat with other geometries, but solving for NC parameter is tricky in higher dimensions.

Can assume it is a product of Killing vectors:

$$\Theta^{\alpha\beta} = r^{ij} K_i^\alpha K_j^\beta, \quad \nabla_\mu K_{i\,\nu} + \nabla_\nu K_{i\,\mu} = 0$$

Same structure as an r-matrix with arbitrary coefficients.

Bi-Killing bivector

Assuming bivector is bi-Killing, nice things happen.

$$I^{\mu} = \frac{1}{2} r^{ij} f_{ij}{}^{k} K^{\mu}_{k}$$

Jacobi identity from NC is simply the CYBE

$$\Theta^{[\alpha\rho} \nabla_{\rho} \Theta^{\beta\gamma]} = K^{\alpha}_{i} K^{\beta}_{j} K^{\gamma}_{k} f_{l_1 l_2}{}^{[i} r^{j l_1} r^{k] l_2} = 0$$

Perturbative Proof

Expand in the bivector, plug into equations of motion

$$
\begin{aligned}
g_{\mu\nu} &= G_{\mu\nu} + \Theta_{\mu}{}^{\alpha}\Theta_{\alpha\nu} + \mathcal{O}(\Theta^4), \\
B_{\mu\nu} &= -\Theta_{\mu\nu} - \Theta_{\mu\alpha}\Theta^{\alpha\beta}\Theta_{\beta\nu} + \mathcal{O}(\Theta^5), \\
\phi &= \Phi + \frac{1}{4}\Theta_{\rho\sigma}\Theta^{\rho\sigma} + \mathcal{O}(\Theta^4)
\end{aligned}
$$

Scalar equation at second order

$$
K_i^{\alpha} K_k^{\beta} \nabla_\alpha K_{\beta m} \left(f_{l_1 l_2}{}^{m} r^{il_1} r^{kl_2} + f_{l_1 l_2}{}^{k} r^{ml_1} r^{il_2} + f_{l_1 l_2}{}^{i} r^{kl_1} r^{ml_2} \right) + \left(\Theta^{\beta\gamma}\Theta^{\alpha\lambda} + \Theta^{\alpha\beta}\Theta^{\gamma\lambda} + \Theta^{\gamma\alpha}\Theta^{\beta\lambda} \right) R_{\beta\gamma\alpha\lambda} = 0.
$$

Can prove using $\boldsymbol{\beta}$ supergravity **Bakhmatov, Musaev**

Example II

TsT transformation simple in this language.

$$\Theta = \kappa \partial_{\varphi_1} \wedge \partial_{\varphi_2}$$

Easy to embed this into O(d,d) transformation, so it is clearly a type of T-duality transformation.

$$h = \begin{pmatrix} 1 & 0 \\ \Theta & 1 \end{pmatrix} \qquad \mathcal{H}' = h\mathcal{H}h^T$$

Lunin-Maldacena deformations can be rewritten.

Aybike Ozer; Sakamoto, Sakatani, Yoshida,...

Example III

Can choose non-integrable, non-coset example.

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

$$\Theta = \alpha T_1 \wedge T_2 + \beta T_2 \wedge T_3 + \gamma T_3 \wedge T_1 + \delta T_4 \wedge T_1 + \epsilon T_4 \wedge T_2 + \lambda T_4 \wedge T_3$$

$$[T_1, T_2] = T_3 + \text{cyclic}$$

$$0 = \beta\epsilon - \delta\gamma = \alpha\epsilon - \gamma\lambda = \alpha\delta - \lambda\beta,$$
$$0 = \alpha^2 + \beta^2 + \gamma^2$$

Information from equations of motion same as CYBE.

modified CYBE

We have shown perturbatively that the inversion plus bi-Killing ansatz reduces the supergravity equations to the homogeneous CYBE.

However, the modified CYBE is special and more enigmatic.

Through a dilaton shift, our recipe appears to work more generally, but necessitates an RR sector.

This shift takes one outside of T-duality.

η-deformation

As stressed, method works for modified CYBE with "displacement" of NS and RR sectors

$$
\begin{aligned}
g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu =& -\frac{(1+\rho^2)\mathrm{d}t^2}{1-\kappa^2\rho^2} + \frac{\mathrm{d}\rho^2}{(1+\rho^2)(1-\kappa^2\rho^2)} + \frac{\rho^2 \mathrm{d}\zeta^2}{1+\kappa^2\rho^4\sin^2\zeta} \\
&+ \frac{\rho^2\cos^2\zeta \mathrm{d}\psi_1^2}{1+\kappa^2\rho^4\sin^2\zeta} + \rho^2\sin^2\zeta \mathrm{d}\psi_2^2 \\
B =& -\frac{\kappa\rho^4\sin 2\zeta}{2(1+\kappa^2\rho^4\sin^2\zeta)}\mathrm{d}\zeta\wedge\mathrm{d}\psi_1 + \frac{\kappa\rho}{1-\kappa^2\rho^2}\mathrm{d}t\wedge\mathrm{d}\rho
\end{aligned}
$$

$$
\Theta^{\zeta\psi_1} = \kappa\tan\zeta, \quad \Theta^{t\rho} = \kappa\rho
$$

η-deformation

Can embed the conformal algebra in superalgebra gl(4,4).

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj}, \quad i, j = 1, \dots 4$$

$$D = \frac{1}{2}(E_{\lambda\lambda} - E_{\dot{\lambda}\dot{\lambda}}), \quad P_{\alpha\dot{\beta}} = E_{\alpha\dot{\beta}}, \quad K_{\dot{\alpha}\beta} = E_{\dot{\alpha}\beta},$$

$$L_{\alpha\beta} = E_{\alpha\beta} - \frac{1}{2}\delta_{\alpha\beta} E_{\lambda\lambda}, \quad \bar{L}_{\dot{\alpha}\dot{\beta}} = E_{\dot{\alpha}\dot{\beta}} - \frac{1}{2}\delta_{\dot{\alpha}\dot{\beta}} E_{\dot{\lambda}\dot{\lambda}}$$

$$\alpha, \beta, \lambda = 1, 2, \quad \dot{\alpha}, \dot{\beta}, \dot{\lambda} = 3, 4$$

$$r = c(E_{12} \wedge E_{21} + E_{13} \wedge E_{31} + E_{14} \wedge E_{41} + E_{23} \wedge E_{32} + E_{24} \wedge E_{42} + E_{34} \wedge E_{43})$$

η-deformation

Can understand our bivector through Killing vectors of undeformed geometry.

$$D = -i\partial_t, \quad L_{11} = -\frac{i}{2}(\partial_{\psi_1} + \partial_{\psi_2}), \quad \bar{L}_{33} = -\frac{i}{2}(\partial_{\psi_1} - \partial_{\psi_2}),$$
$$L_{12} = e^{i(\psi_1+\psi_2)}\left(\tan\zeta\partial_{\psi_1} + i\partial_\zeta - \cot\zeta\partial_{\psi_2}\right),$$
$$P_{14} = e^{i(t+\psi_1)}\frac{\sqrt{1+\rho^2}}{\rho}\left(\rho\cos\zeta\partial_\rho + \frac{i\rho^2}{1+\rho^2}\cos\zeta\partial_t - \sin\zeta\partial_\zeta + i\sec\zeta\partial_{\psi_1}\right),$$
$$K_{41} = e^{-i(t+\psi_1)}\frac{\sqrt{1+\rho^2}}{\rho}\left(\rho\cos\zeta\partial_\rho - \frac{i\rho^2}{1+\rho^2}\cos\zeta\partial_t - \sin\zeta\partial_\zeta - i\sec\zeta\partial_{\psi_1}\right)$$

$$r = 4ic(\rho\partial_t \wedge \partial_\rho + \tan\zeta\partial_\zeta \wedge \partial_{\psi_1})$$

bi-Yang-Baxter

Consider now AdS_3 x S^3 - can be described by PCM

$$\mathrm{d}s^2 = -(1+\rho^2)\mathrm{d}t^2 + \frac{\mathrm{d}\rho^2}{(1+\rho^2)} + \rho^2\mathrm{d}\psi_1^2 + (1-r^2)\mathrm{d}\varphi^2 + \frac{\mathrm{d}r^2}{(1-r^2)} + r^2\mathrm{d}\phi_1^2$$

Can get a two parameter integrable deformation:

$$\Theta^{t\rho} = \kappa_1\rho, \quad \Theta^{\varphi r} = \kappa_1 r, \quad \Theta^{\rho\psi_1} = -\kappa_2(\rho^{-1}+\rho), \quad \Theta^{r\phi_1} = \kappa_2(r^{-1}-r)$$

Additional TsTs & extension to AdS_3 x S^3 x S^3 are possible

$$\Theta^{t\psi_1} = \kappa_3, \quad \Theta^{\varphi\phi_1} = \kappa_4, \ldots$$

Generality?

How far can this gravity/CYBE correspondence be pushed?

Can we generate integrable deformation of $AdS_4 \times CP^3$?

Let us consider a warm-up: $AdS_5 \times T^{1,1}$

$$\mathrm{d}s^2 = \mathrm{d}s^2(AdS_5) + \mathrm{d}s^2(T^{1,1}), \quad F_5 = 4\left(\mathrm{vol}(AdS_5) + \mathrm{vol}(T^{1,1})\right)$$

$AdS_5 \times T^{1,1}$

In contrast to YB deformations based on homogeneous CYBE, deformations based on modified CYBE deform both AdS_5 and S^5.

The scalar equation couples these deformations.

So, a necessary condition for a supergravity solution is we get equal and opposite constant contributions to the equation.

In other words, coordinate dependence must drop out.

Does such a deformation of $T^{1,1}$ exist?

$T^{1,1}$

$$\mathrm{d}s^2(T^{1,1}) = \frac{1}{9}\left(\mathrm{d}\psi + \cos\theta_1 \mathrm{d}\phi_1 + \cos\theta_2 \mathrm{d}\phi_2\right)^2 + \frac{1}{6}\sum_{i=1}^{2}(\mathrm{d}\theta_i^2 + \sin^2\theta_i \mathrm{d}\phi_i^2)$$

Can identity seven Killing vectors:

$$K_1 = \partial_\psi, \quad K_2 = -\partial_{\phi_1}, \quad K_3 = -\partial_{\phi_2},$$
$$K_4 = \cos\phi_1 \partial_{\theta_1} - \cot\theta_1 \sin\phi_1 \partial_{\phi_1} + \frac{\sin\phi_1}{\sin\theta_1}\partial_\psi,$$
$$K_5 = \cos\phi_2 \partial_{\theta_2} - \cot\theta_2 \sin\phi_2 \partial_{\phi_2} + \frac{\sin\phi_2}{\sin\theta_2}\partial_\psi,$$
$$K_6 = \sin\phi_1 \partial_{\theta_1} + \cot\theta_1 \cos\phi_1 \partial_{\phi_1} - \frac{\cos\phi_1}{\sin\theta_1}\partial_\psi,$$
$$K_7 = \sin\phi_2 \partial_{\theta_2} + \cot\theta_2 \cos\phi_2 \partial_{\phi_2} - \frac{\cos\phi_2}{\sin\theta_2}\partial_\psi$$

$\mathrm{T}^{1,1}$

Candidate r-matrix $\quad SU(2) \times SU(2) \times U(1)$

$$\Theta^{t\rho} = \kappa\rho, \quad \Theta^{\zeta\psi_1} = \kappa \tan\zeta,$$

$$\Theta^{\theta_1\phi_1} = \eta \cot\theta_1, \quad \Theta^{\psi\theta_1} = \frac{\eta}{\sin\theta_1}, \quad \Theta^{\theta_2\phi_2} = \eta \cot\theta_2, \quad \Theta^{\psi\theta_2} = \frac{\eta}{\sin\theta_2}$$

dilaton equation $\quad 20\kappa^2 = \dfrac{\eta^2(288+\eta^2)}{324}$

BUT: can argue perturbatively against RR sector

Summary

The open-closed string map captures YB deformations based on homogeneous CYBE.

But with a small shift in dilaton (displacing sectors) it also works for the modified CYBE.

Clearly not a T-duality transformation.

Appear to be able to separate the CYBE from integrability.

Not clear if an η-deformation of AdS_4 x CP^3 exits.