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Weyl anomaly cancellation in string theories on generalized supergravity backgrounds

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0. Introduction

String Theory

A promising candidate of the unified theory of 4 forces in nature.

However, String Theory has not been completed yet!

In particular, String Theory is defined only perturbatively, and there are various approaches towards the non-perturbative formulation of String Theory.

EX String Field Theory, Matrix Model, Tensor Model etc.

Question

Is there anything to consider for perturbative string theory? YES!

3 well-known formulations of perturbative string theory

1. NS-R formulation (world-sheet fermions)
2. Green-Schwarz formulation (space-time fermions + kappa symmetry)
3. Pure Spinor formulation (space-time fermion + pure spinor condition)

The Green-Schwarz (GS) formulation of type IIB superstring

Space-time fermions contain 32 components (= 2 x 16 comps. of Majorana-Weyl spinor).

The on-shell condition reduces # of d.o.f. to 16, but # of physical comps. should be 8.

So it is necessary to impose an additional condition.

$\Rightarrow$ Kappa-symmetry (a fermionic gauge symmetry)

FACT: the on-shell condition of the standard type IIB SUGRA

$\Rightarrow$ kappa-invariant GS string theory

[Grisaru-Howe-Mezincescu -Nilsson-Townsend, 1985]

The inverse was conjectured, but not true.

New result: kappa-invariant GS string theory on an arbitrary background

$\Rightarrow$ the generalized type IIB SUGRA [Tseytlin-Wulf, 1605.04884]

This issue has been resolved after more than 30 years from the old work.

This is the recent fundamental progress in String Theory!

What does this result indicate?

Low Energy Effective Theory emerging from String Theory may be more general than the well-known SUGRA!

Generalized SUGRA = SUGRA + an extra vector field

(The detail of generalized SUGRA will be explained soon)

We may have missed an important ingredient in String Phenomenology for more than 30 years.

It is really significant to study the generalized SUGRA in more detail.

It may be possible to get a nice idea to solve the long-standing problems such as cosmological constant problem and stabilizing de Sitter vacuum.

The plan of my talk

1. What is the generalized type IIB SUGRA?

2. Weyl anomaly cancelation in string theories on generalized SUGRA backgrounds

3. Summary and Discussion

1. What is the generalized type IIB SUGRA?

The generalized eqns of type IIB SUGRA

[Arutyunov-Frolov-Hoare-Roiban-Tseytlin, 1511.05795]

$$R_{MN} - \frac{1}{4} H_{MKL} H_N{}^{KL} - T_{MN} + \boxed{D_M X_N} + \boxed{D_N X_M} = 0 \,,$$

$$\frac{1}{2} D^K H_{KMN} + \frac{1}{2} F^K F_{KMN} + \frac{1}{12} F_{MNKLP} F^{KLP} = \boxed{X^K} H_{KMN} + \boxed{D_M X_N} - \boxed{D_N X_M}$$

$$R - \frac{1}{12} H^2 + 4 \boxed{D_M X^M} - \boxed{4 X_M X^M} = 0 \,,$$

$$D^M \mathcal{F}_M - \boxed{Z^M} \mathcal{F}_M - \frac{1}{6} H^{MNK} \mathcal{F}_{MNK} = 0 \,, \qquad \boxed{I^M} \mathcal{F}_M = 0 \,, \qquad \mathcal{F}_{n_1 n_2 \ldots} = e^{\Phi} F_{n_1 n_2 \ldots}$$

$$D^K \mathcal{F}_{KMN} - \boxed{Z^K} \mathcal{F}_{KMN} - \frac{1}{6} H^{KPQ} \mathcal{F}_{KPQMN} - \boxed{(I \wedge} \mathcal{F}_1)_{MN} = 0 \,,$$

$$D^K \mathcal{F}_{KMNPQ} - \boxed{Z^K} \mathcal{F}_{KMNPQ} + \frac{1}{36} \epsilon_{MNPQRSTUVW} H^{RST} \mathcal{F}^{UVW} - \boxed{(I \wedge} \mathcal{F}_3)_{MNPQ} = 0$$

$$T_{MN} \equiv \frac{1}{2} \mathcal{F}_M \mathcal{F}_N + \frac{1}{4} \mathcal{F}_{MKL} \mathcal{F}_N{}^{KL} + \frac{1}{4 \times 4!} \mathcal{F}_{MPQRS} \mathcal{F}_N{}^{PQRS} - \frac{1}{4} G_{MN} \left(\mathcal{F}_K \mathcal{F}^K + \frac{1}{6} \mathcal{F}_{PQR} \mathcal{F}^{PQR} \right)$$

Modified Bianchi identities

$$(d\mathcal{F}_1 - \boxed{Z \wedge} \mathcal{F}_1)_{MN} - \boxed{I^K} \mathcal{F}_{MNK} = 0 \,,$$

$$(d\mathcal{F}_3 - \boxed{Z \wedge} \mathcal{F}_3 + H_3 \wedge \mathcal{F}_1)_{MNPQ} - \boxed{I^K} \mathcal{F}_{MNPQK} = 0 \,,$$

$$(d\mathcal{F}_5 - \boxed{Z \wedge} \mathcal{F}_5 + H_3 \wedge \mathcal{F}_3)_{MNPQRS} + \frac{1}{6} \epsilon_{MNPQRSTUVW} \boxed{I^T} \mathcal{F}^{UVW} = 0$$

Modified parts: $X,\quad I,\quad Z$ 3 vector fields

But $X_M \equiv I_M + Z_M$, so two of them are independent.

Then I & Z satisfy the relations given by

$$\boxed{D_M I_N + D_N I_M = 0\,, \quad D_M Z_N - D_N Z_M + I^K H_{KMN} = 0\,, \quad I^M Z_M = 0}$$

Supposing that I is chosen such that the Lie derivative

$$(\mathcal{L}_I B)_{MN} = I^K \partial_K B_{MN} + B_{KN}\partial_M I^K - B_{KM}\partial_N I^K$$

vanishes, the 2nd equation above can be solved by

$$Z_M = \partial_M \Phi - B_{MN} I^N \quad .$$

Thus, after all, only I is independent.

Note When $I = 0$, the usual type IIB SUGRA is reproduced.

Characteristics of the generalized SUGRA

Talks by Yuho, Junichi, Eoin, Suguru, Falk

1. **Yang-Baxter deformation is a solution generation technique** [Klimcik]

   Classical r-matrices lead to sols. of the generalized SUGRA.

   [Delduc-Magro-Vicedo, 1309.5850] [Kawaguchi-Matsumoto-KY, 1401.4855]

2. **The extra vector field *I* can be seen as a non-geometric Q-flux**

   For example, T-folds can be realized as solutions of the generalized SUGRA.

   [Fernandez Melgarejo-Sakamoto-Sakatani-KY, 1710.06849]

3. **Generalized SUGRA can be obtained from DFT or EFT**

   A slightly modified section condition leads to generalized type IIB and IIA SUGRAs.

   [Sakatani-Uehara-KY, 1611.05856] [Baguet-Magro-Samtleben, 1612.07210] [Sakamoto-Sakatani-KY, 1703.09213]

4. **Generalized T-duality rule for the *I* direction**

   A modification is to ``add a linear-dependent part to dilaton depending on I"

   $$\Phi \longrightarrow \widetilde{\Phi} = \Phi + I \cdot \tilde{x}$$

   [Arutyunov-Frolov-Hoare-Roiban-Tseytlin, 1511.05795]

   Remarkably, the dual background is a solution of the usual SUGRA!

Influence of the appearance of the generalized SUGRA?

Recall that there are a lot of the so-called pathological backgrounds, which do not satisfy the on-shell condition of the standard SUGRA.

EX non-abelian T-dualities lead to such backgrounds

However, these backgrounds may be solutions of the generalized SUGRA.

In fact, there is an intimate relation between non-abelian T-duality and Yang-Baxter deformation

(A class of) non-abelian T-duality $\Longleftrightarrow$ Yang-Baxter defs. (with CYBE)

conjecture : [Hoare-Tseytlin, 1609.02550] proof : [Borsato-Wulff, 1609.09834]

$\Longrightarrow$ Solutions of the generalized SUGRA

How about non-abelian T-duality which cannot be described as YB deformations?

An example of such non-abelian T-dualized backgrounds

The Gasperini-Ricci-Veneziano background [hep-th/9308112]

$$ds^2 = -dt^2 + \frac{(t^4+y^2)\,dx^2 - 2\,x\,y\,dx\,dy + (t^4+x^2)\,dy^2 + t^4\,dz^2}{t^2\,(t^4+x^2+y^2)} + ds^2_{T^6}\,,$$

$$B_2 = \frac{(x\,dx + y\,dy)\wedge dz}{t^4+x^2+y^2}\,, \qquad \Phi = \frac{1}{2}\ln\left[\frac{1}{t^2\,(t^4+x^2+y^2)}\right],$$

This is not a solution of the usual SUGRA.

However, by adding the following extra vector field

$$\boxed{I^z = -2}$$

[Fernandez Melgarejo-Sakamoto-Sakatani-KY, 1710.06849]

this background becomes a solution of the generalized SUGRA.

Further confirmation : [M. Hong-Y. Kim-O Colgain, 1801.09567]

More ``pathological'' backgrounds may be sols. of the generalized SUGRA.

So what kinds of pathologies have been resolved?

Classical level : kappa-symmetry is ensured, so acceptable as GS string theory.

Quantum level : Weyl invariance on the world-sheet?

Recall that the problem in the non-abelian T-duality argument is that the beta function does not vanish because the background is not a solution of the usual SUGRA.

The original claim [Arutyunov-Frolov-Hoare-Roiban-Tseytlin, 1511.05795]

When the background is given by a solution of the generalized SUGRA, scale invariance is ensured, but Weyl invariance would not be.

Actually, an appropriate counter-term has not been discussed so much.

2. Weyl anomaly cancelation in string theories on generalized SUGRA backgrounds

[Sakamoto-Sakatani-KY, 1703.09213]

[Fernandez Melgarejo-Sakamoto-Sakatani-KY, 1811.10600]
accepted in PRL.

For simplicity, we will focus upon the bosonic string case hereafter.

Weyl invariance of the bosonic string theory (D=26)

<u>The classical action</u>

$$a, b = \tau, \sigma, \quad \varepsilon^{\tau\sigma} = 1/\sqrt{-\gamma}, \quad \varepsilon_{\tau\sigma} = -\sqrt{-\gamma}$$

$$S = -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \sqrt{-\gamma} \left[G_{mn}\gamma^{ab} - B_{mn}\varepsilon^{ab} \right] \partial_a X^m \partial_b X^n$$

At classical level,

$$T^a_{\ a} \equiv \frac{4\pi}{\sqrt{-\gamma}} \gamma^{ab} \frac{\delta S}{\delta \gamma^{ab}} = 0 \quad \Rightarrow \quad \text{Weyl invariant}$$

But at quantum level, the trace anomaly appears [Callan-Friedan-Martinec-Perry, '85]

$$2\alpha' \langle T^a_{\ a} \rangle = \left(\beta^G_{mn} \gamma^{ab} - \beta^B_{mn} \varepsilon^{ab} \right) \partial_a X^m \partial_b X^n$$

where

$$\beta^G_{mn} = \alpha' \left(R_{mn} - \frac{1}{4} H_{mpq} H_n^{\ pq} \right), \quad \beta^B_{mn} = \alpha' \left(-\frac{1}{2} D^k H_{kmn} \right)$$

Quantum scale invariance

[Hull-Townsend, '86]

Suppose that the beta functions take the following forms:

$$\beta^G_{mn} = -2\alpha' D_{(m}Z_{n)}, \quad \beta^B_{mn} = -2\alpha'(Z^k H_{kmn} + 2D_{[m}I_{n]}).$$

Then scale invariance is preserved at quantum level.

In fact, the trace anomaly can be rewritten into a total derivative form:

$$\langle T^a{}_a \rangle = -\mathcal{D}_a \left[(Z_n \gamma^{ab} - I_n \varepsilon^{ab}) \partial_b X^n \right]$$

where the eom of X has been utilized.

NOTE

This supposition is satisfied for the solutions of the generalized SUGRA.

Here *Z* and *I* are arbitrary vector fields, and of course these are nothing but those in the generalized SUGRA!

The origin of the generalized SUGRA

Quantum Weyl invariance

As a special case of Hull and Townsend, one may take

$$Z_m = \partial_m \Phi, \qquad I_m = 0 \quad .$$

Then the trace anomaly is given by

$$\boxed{\langle T^a{}_a \rangle = -\mathcal{D}^a \partial_a \Phi \ .}$$

This anomaly can be cancelled out by adding the Fradkin-Tseytlin (FT) term:

$$S_{\mathrm{FT}} \equiv \frac{1}{4\pi} \int \mathrm{d}^2\sigma \sqrt{-\gamma}\, R^{(\gamma)} \Phi \qquad \longleftarrow \quad \boxed{\text{alpha' is not contained!}}$$

because

$$\boxed{\langle T^a{}_a \rangle_{\mathrm{FT}} = \frac{4\pi}{\sqrt{-\gamma}} \gamma^{ab} \frac{\delta S_{\mathrm{FT}}}{\delta \gamma^{ab}} = \mathcal{D}^a \partial_a \Phi \, .}$$

Note: the FT term itself should be regarded as quantum contribution.

A generalization of the FT term

Question: Can one generalize the FT term for the case with $I_m \neq 0$?

Generalized FT term: [Sakamoto-Sakatani-KY, 1703.09213]

Dual coordinates

$$S_{\mathrm{FT}}^{(*)} = \frac{1}{4\pi}\int \mathrm{d}^2\sigma\sqrt{-\gamma}\,R^{(\gamma)}\Phi_*\,, \qquad \Phi_* = \Phi + I^i\underline{\widetilde{Y}_i}$$

$$\langle T^a{}_a\rangle_{\mathrm{FT}}^{(*)} = \frac{4\pi}{\sqrt{-\gamma}}\gamma^{ab}\frac{\delta S_{\mathrm{FT}}^{(*)}}{\delta\gamma^{ab}} = \underline{+\mathcal{D}_a\left[\left(Z_n\gamma^{ab} - I_n\varepsilon^{ab}\right)\partial_b X^n\right]}$$

Exactly cancels out Hull-Townsend's trace anomaly!

Here we have used the eom of Hull's double sigma model,

$$\boxed{\partial_a\widetilde{Y}_i - G_{in}\varepsilon^b{}_a\partial_b X^n - B_{in}\partial_a X^n = 0}$$

Note that 2D Ricci scalar is locally a total derivative, so this cancellation is local.

A two-dim. Ricci scalar can be written locally as

$$\sqrt{-\gamma}\, R^{(\gamma)} = \partial_a \alpha^a$$

Then by integrating by part, the counter-term can be rewritten as

$$\boxed{S_{\rm FT} \;\equiv\; \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{-\gamma}\,R^{(\gamma)}\,\Phi - \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\varepsilon_{ab}\,\alpha^a\,J^b}$$

where *J* is a Noether current associated with the Killing vector *I*,

$$J^c \equiv \varepsilon^{ca} I^n \left(B_{nm}\,\partial_a X^m - g_{nm}\,\varepsilon_a{}^b\,\partial_b X^m\right)$$

The integrand of the 2nd term is local, but the integrated one would not be…

(Be careful for the global topology)

Also, be careful for diffeomorphism invariance of the 2nd term.

What is an appropriate α ? [Fernandez Melgarejo-Sakamoto-Sakatani-KY, 1811.10600]

- An obstacle [Deser-Jackiw, hep-th/9510145]

If α^a is constructed only from the metric, it is not covariant.

- A possible resolution [Yale-Padmanabhan, 1008.5154]

$$\boxed{\sqrt{-\gamma}R^{(2)} = 2\,\sigma\,\partial_a\left[\sqrt{-\gamma}(n^b\mathcal{D}_b n^a - n^a\mathcal{D}_b n^b)\right]}$$

n^a : a normalized vector field $\left(\gamma_{ab}n^a n^b = \sigma\ , \quad \sigma = \pm 1\right)$

A remarkable point:

By introducing n^a, one can avoid Deser-Jackiw's argument.

NOTE: n^a does not appear in $\partial_a \alpha^a$.

(∵)
$$\begin{aligned} \partial_a \alpha^a &= \sqrt{-\gamma}\, R^{(2)} + 2\sigma\sqrt{-\gamma}\, \left(\mathcal{D}_a n^b \mathcal{D}_b n^a - \mathcal{D}_a n^a \mathcal{D}_b n^b\right) \\ &= \sqrt{-\gamma}\, R^{(2)} \end{aligned}$$

where we have used the following identity intrinsic to 2D:

$$\mathcal{D}_a n^b \mathcal{D}_b n^a = \mathcal{D}_a n^a \mathcal{D}_b n^b$$

A key observation: [Fernandez Melgarejo-Sakamoto-Sakatani-KY, 1811.10600]

α^a may depend on γ^{ab} and X^m through n^a,

while $\partial_a \alpha^a$ does not contain X^m .

A naive choice of n^a is the Noether current: J^a

$$n^a = \frac{1}{\sqrt{\sigma \gamma_{cd} J^c J^d}} J^a \quad .$$

By adopting this choice, we can express α^a as

$$\alpha^a := 2\,\sigma \sqrt{-\gamma}(n^b \mathcal{D}_b n^a - n^a \mathcal{D}_b n^b) \ .$$

This is manifestly *local* and *covariant.*

A reasoning of this choice:

The Killing direction *l* is specialized in target space-time and hence

the world-sheet counter-part J should be taken into account.

Another derivation of the modified counter-term

Let us start from the classical action of a gauged NLSM:

$$S = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\sqrt{-\gamma}\,\Big[(g_{mn}\gamma^{ab} - B_{mn}\varepsilon^{ab})D_aX^mD_bX^n - \widetilde{Z}\varepsilon^{ab}F_{ab}\Big]$$
$$+\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{-\gamma}\,R^{(2)}(\Phi+\widetilde{Z})$$

where $\quad D_aX^m \equiv \partial_aX^m - I^mA_a\,, \quad F_{ab} \equiv \partial_aA_b - \partial_bA_a$

This action is invariant under the gauge transformation:

$$X^m(\sigma) \longrightarrow X^m(\sigma) + I^mv(\sigma)\,, \quad A_a(\sigma) \longrightarrow A_a(\sigma) + \partial_av(\sigma)$$

Then the e.o.m. for an auxiliary field $\widetilde{Z}$ is given by

$$\epsilon^{ab}F_{ab} = -\alpha'\sqrt{-\gamma}\,R^{(2)}$$

By using the gauge transformation, F_{ab} vanishes to the leading order of α' .

So one can take the following gauge: $A_a = 0 + \alpha' \mathcal{A}_a$

Hence $\epsilon^{ab}(\partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a) = -\sqrt{-\gamma}\, R^{(2)}$

By using the e.o.m. for A_a ,

$$\partial_a \widetilde{Z} = \varepsilon^b{}_a J_b - |I|^2 \varepsilon^b{}_a A_b$$

one can eliminate $\widetilde{Z}$ from the action.

The resulting action is given by

$$\begin{aligned} S = &-\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \sqrt{-\gamma}\, \left[(g_{mn}\gamma^{ab} - B_{mn}\varepsilon^{ab}) D_a X^m D_b X^n\right] \\ &+\frac{1}{4\pi}\int \mathrm{d}^2\sigma \sqrt{-\gamma}\, \left[R^{(2)}\Phi + \epsilon_{ab}\underbrace{(-2\epsilon^{ac}\mathcal{A}_c)}_{\equiv\, \alpha^a} J^b \cancel{- \alpha'|I|^2\gamma^{ab}\mathcal{A}_a\mathcal{A}_b}\right] \end{aligned}$$

The previous action has been reproduced from a gauged NLSM!

3. Summary and Discussion

Summary & discussion

Generalized SUGRA and Weyl anomaly cancellation

Kappa-symmetry of GS superstring $\Rightarrow$ generalized SUGRA

A modified Fradkin-Tseytlin term $\Rightarrow$ Weyl anomaly cancellation

[Sakamoto-Sakatani-KY, 1703.09213]

However, for Weyl invariance, need to check a lot of things, e.g. higher genus cases.

The integrand of the counter-term is local and covariant, but non-polynomial

The integration would be non-local. [Fernandez Melgarejo-Sakamoto-Sakatani-KY, 1811.10600]

The non-polynomial and non-local property might be compatible

with non-geometric b.g. like T-folds. (where the order of alpha' expansion is quite unclear)

Of course, it is useful to check the counter-term for simple exactly solvable backgrounds

[Sakatani-Tseytlin-KY, in progress]

Thank you!

Back up

An example of solutions of the generalized SUGRA

$$\begin{aligned} r &= E_{24}\wedge(c_1E_{22}-c_2E_{44}) \\ &= (p_0-p_3)\wedge\left[a_1\left(\frac{1}{2}\gamma_5-n_{03}\right)-a_2\left(n_{12}-\frac{i}{2}\mathbf{1}_4\right)\right] \end{aligned}$$

$$a_1\equiv\frac{c_1+c_2}{2}=\mathrm{Re}(c_1)\,,\qquad a_2\equiv\frac{c_1-c_2}{2\,i}=\mathrm{Im}(c_1)$$

The deformed background: [Kyono-KY, 1605.02519]

$$\begin{aligned} ds^2 &= \frac{-2dx^+dx^-+d\rho^2+\rho^2d\phi^2+dz^2}{z^2}-4\eta^2\left[(a_1^2+a_2^2)\frac{\rho^2}{z^6}+\frac{a_1^2}{z^4}\right](dx^+)^2+ds^2_{\mathrm{S}^5}\,, \\ B_2 &= 8\eta\left[\frac{a_1x^1+a_2x^2}{z^4}dx^+\wedge dx^1+\frac{a_1x^2-a_2x^1}{z^4}dx^+\wedge dx^2+a_1\frac{1}{z^3}dx^+\wedge dz\right], \\ F_3 &= 8\eta\left[\frac{a_2x^1-a_1x^2}{z^5}dx^+\wedge dx^1\wedge dz+\frac{a_1x^1+a_2x^2}{z^5}dx^+\wedge dx^2\wedge dz+\frac{a_1}{z^4}dx^+\wedge dx^1\wedge dx^2\right], \\ F_5 &= \text{undeformed}\,,\qquad \Phi=\text{const} \end{aligned}$$

$$\boxed{dF_3=16\eta\,\frac{a_1}{z^5}\,dx^+\wedge dx^1\wedge dx^2\wedge dz\neq 0}$$

The Bianchi identity is broken.

The e.o.m. of B_2 is not satisfied as well.

Some comments

- Special case   When a1=0, the classical r-matrix becomes unimodular.

  The background is a solution of type IIB SUGRA.   [Hubeney-Rangamani-Ross, hep-th/0504034]

- General case   The resulting background satisfies the generalized equations with

$$\boxed{I = -\frac{2\eta a_1}{z^2}\, dx^+}$$

[Kyono-KY, 1605.02519]

It is more interesting to perform ``generalized T-dualities'' for this solution.

A modification is to ``add a linear-dependent part to dilaton depending on I''

$$\boxed{\Phi \longrightarrow \widetilde{\Phi} = \Phi + I \cdot \tilde{x}}$$

[Arutyunov-Frolov-Hoare
-Roiban-Tseytlin, 1511.05795]

$\Rightarrow$ a solution of the usual type IIB SUGRA

Furthermore, this ``T-dualized'' background is locally equivalent to $AdS_5 \times S^5$

via a non-linear field redefinition.   [Orlando-Reffert-Sakamoto-KY, 1607.00795]

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The classical action

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At classical level,

$$T^a{}_a \equiv \frac{4\pi}{\sqrt{-\gamma}} \gamma^{ab} \frac{\delta S}{\delta \gamma^{ab}} = 0 \quad \Rightarrow \quad \text{Weyl invariant}$$

But at quantum level, the trace anomaly appears [Callan-Friedan-Martinec-Perry, '85]

$$2\alpha' \langle T^a{}_a \rangle = \left(\beta^G_{mn} \gamma^{ab} - \beta^B_{mn} \varepsilon^{ab} \right) \partial_a X^m \partial_b X^n$$

where

$$\beta^G_{mn} = \alpha' \left(R_{mn} - \frac{1}{4} H_{mpq} H_n{}^{pq} \right), \quad \beta^B_{mn} = \alpha' \left(-\frac{1}{2} D^k H_{kmn} \right)$$