Cosmological effective field theories: Sorting the good, the bad and the ugly

Scott Melville

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UNIVERSITY OF CAMBRIDGE

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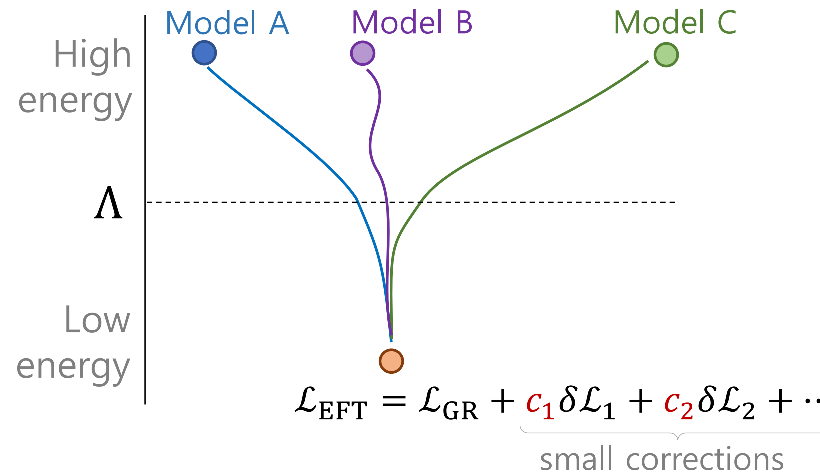
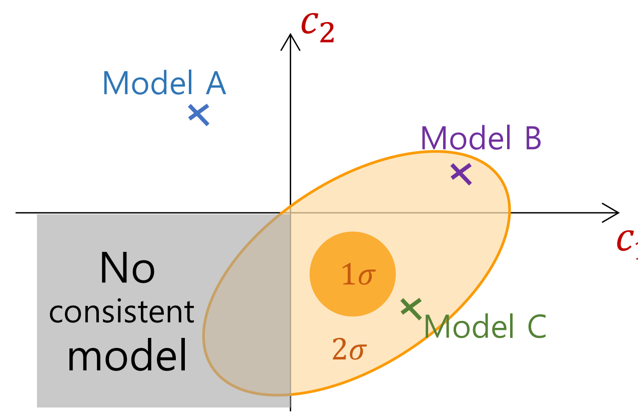
Effective field theory is a model-independent way to parameterize small corrections

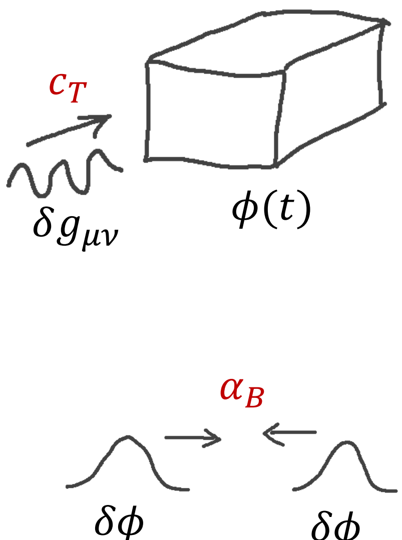


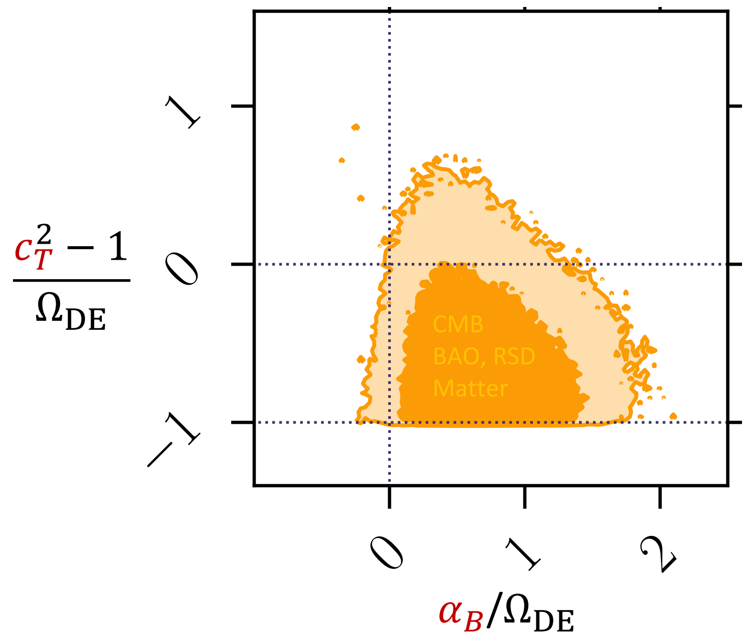
But not all EFT parameters can be realized in consistent models



Goal: Identify/remove unphysical regions of parameter space

For a particular Horndeski EFT of dark energy, ϕ, coupled to the metric $g_{\mu\nu}$,





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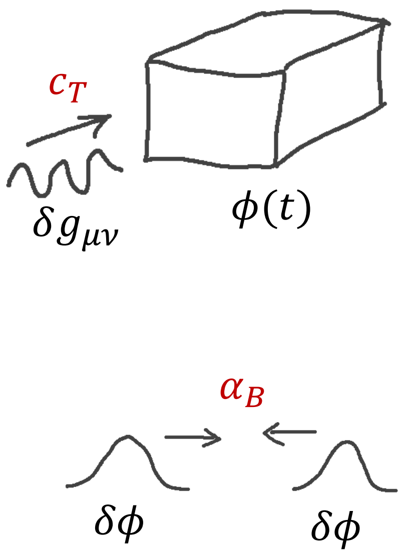

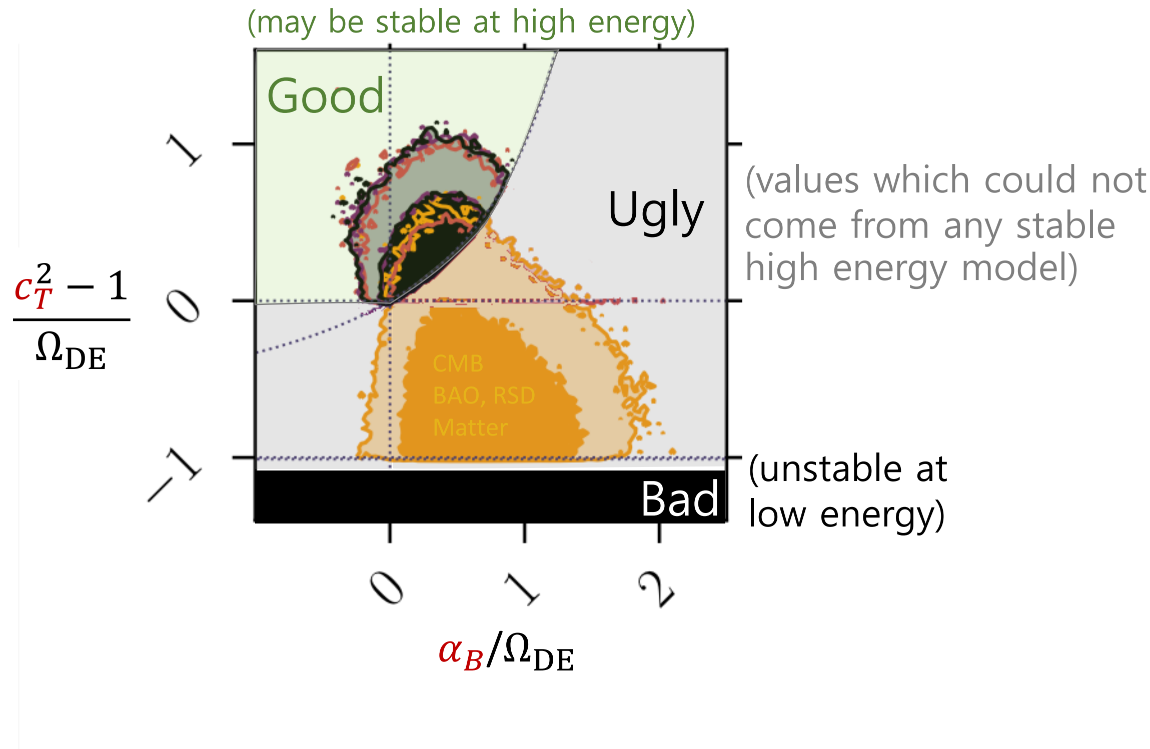

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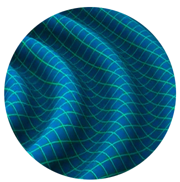
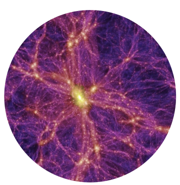
Outline

Stability of classical oscillators — [Kramers+Kronig, 1927], ...

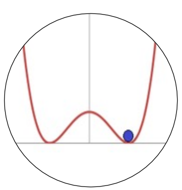
Positivity of QFTs — [Adams++, 2006] [de Rham+SM+Tolley+Zhou, 2017], ...



Gravitational wave speed — [de Rham+SM, 2018] [de Rham+SM+Noller, 2021]



Dark energy clustering — [SM+Noller, 2019]



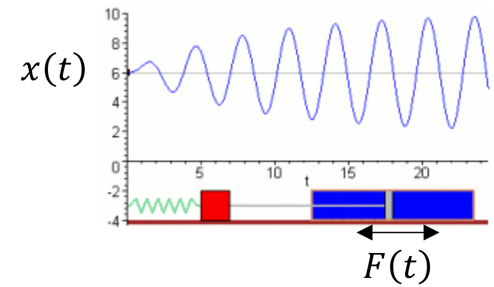
Exploring different vacua — [Grall+SM, 2020] [Davis+SM, 2021] [SM+Noller, 2022]

A Simple Example

Consider a classical oscillator,

$$x(t) = \int dt' \; G(t - t') F(t')$$

position — response/Green's fn — force



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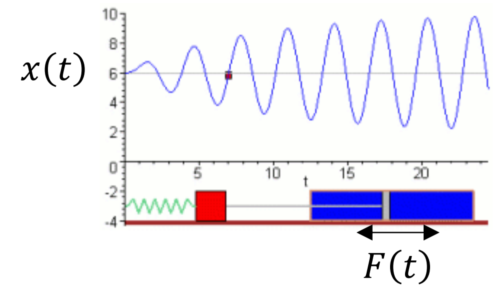

G is found by solving the equations of motion,

e.g. $F = \omega_0^2 x + \gamma\dot{x} + \ddot{x} + \text{interactions}$

$$\Rightarrow \; G(\omega) = \left(\omega_0^2 - i\gamma\omega - \omega^2 + \ldots\right)^{-1}$$

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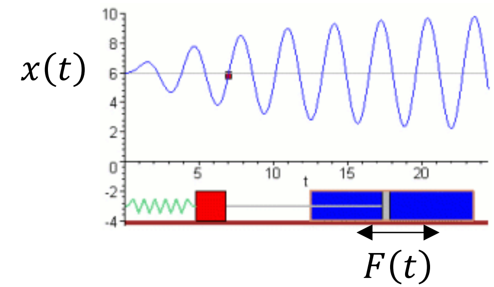

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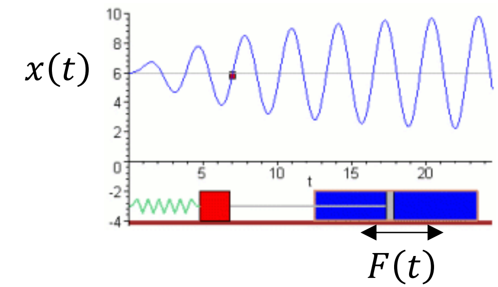

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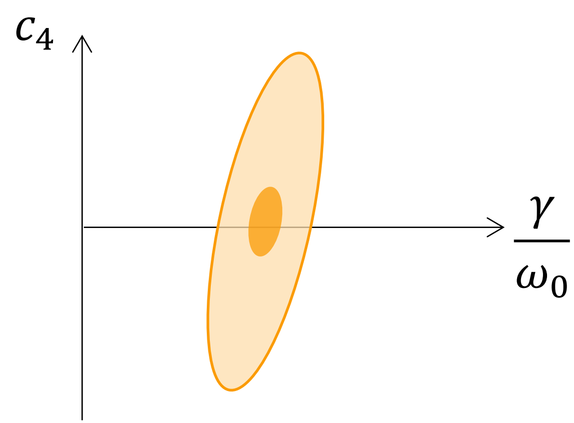
Can now fit EFT coefficients $\{\omega_0, \gamma, c_3, c_4\}$ to data.

But are all parameter values equally good?



A Simple Example

Suppose we perturb the oscillator with $F(t) = \mathrm{Re}\left[\ F_0\, e^{-i\omega t}\ \right]$

$$\Rightarrow\ x(t) = \mathrm{Re}\left[\, G(\omega) F_0\, e^{-i\omega t}\right]$$

The average change in energy is, $\dot{E} = \langle F(t)\dot{x}(t)\rangle_{\text{one cycle}} = F_0^2\,\omega\, \underline{\mathrm{Im}\, G(\omega)}$

$\dot{E} < 0$ signals an instability

The system is stable iff $\mathrm{Im}\, G(\omega) > 0$

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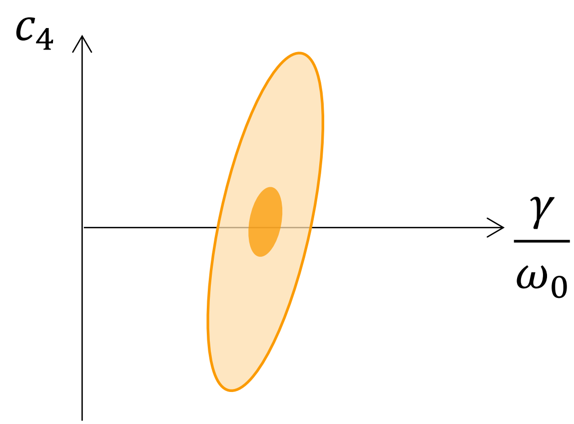

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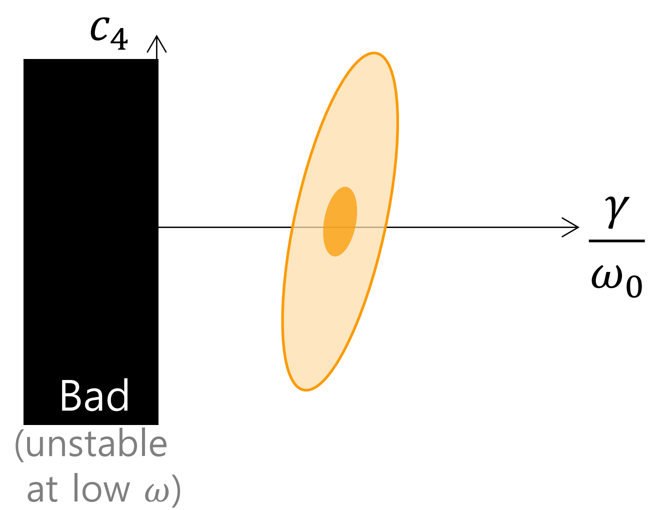
Stability at low frequencies requires,

$$\mathrm{Im}\, G(\omega) = \frac{\omega}{\omega_0^4}\left(\gamma + \frac{c_3\omega^2}{\Lambda} + \cdots\right) > 0$$

$$\Rightarrow \quad \gamma > 0$$

What about at high frequencies?



A Simple Example

We do not know $G(\omega)$ at high frequencies... **however,** since $x(t)$ depends only on the past,

$$x(t) = \int dt' \; G(t-t')F(t') \quad \Rightarrow \quad G(t-t') = 0 \text{ for } t' > t \quad \text{(Causality)}$$

the high and low frequency regimes are related, [Kramers+Kronig 1927], ...

$$\frac{2}{\pi}\int_0^\infty \frac{d\omega'\omega'}{\omega'^2 - \omega^2} \, \mathrm{Im}\, G(\omega') = G(\omega)$$

Proof:

$$\frac{2}{\pi}\int_0^\infty \frac{d\omega'\omega'}{\omega'^2 - \omega^2} \sin(\omega' t) G(t) = e^{i\omega t}\, \mathrm{sign}(t) G(t) = e^{i\omega t} G(t)$$

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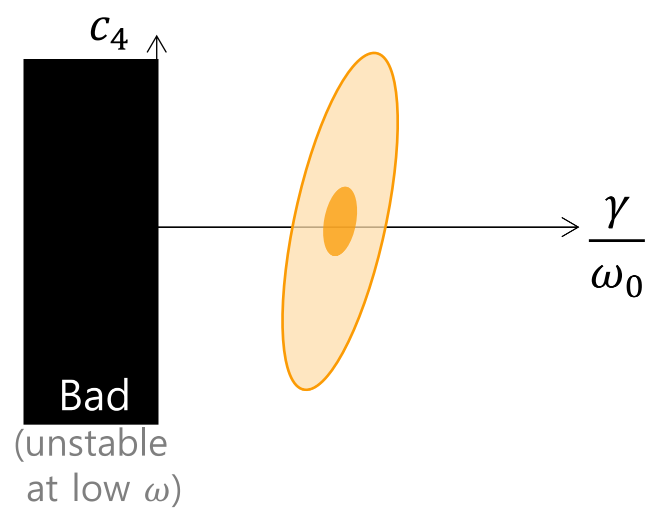

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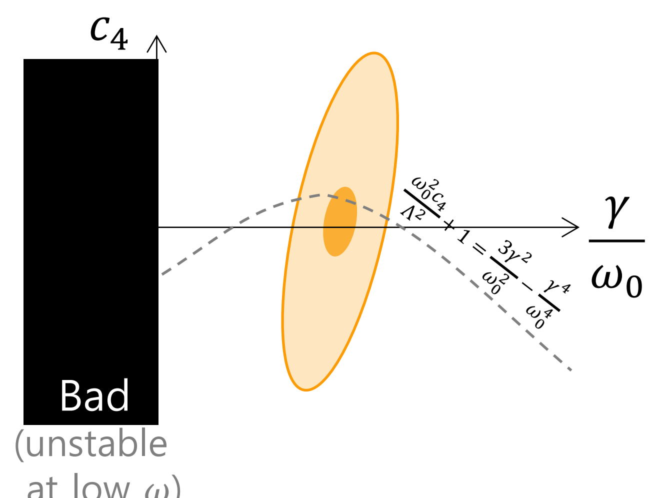
$$\frac{1}{4!}\partial_\omega^4 G(\omega)\Big|_{\omega=0} = \frac{2}{\pi}\int_0^\infty \frac{d\omega'}{\omega'^5} \mathrm{Im}\, G(\omega') > 0$$

$$\Rightarrow \quad \frac{\omega_0^2 c_4}{\Lambda^2} + 1 > \frac{3\gamma^2}{\omega_0^2} - \frac{\gamma^4}{\omega_0^4}$$

``Positivity'' bound



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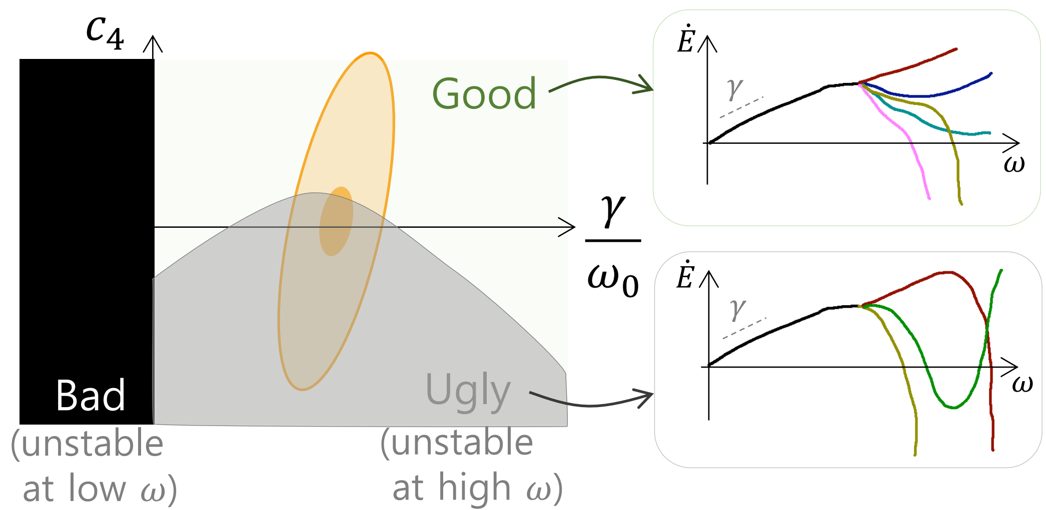
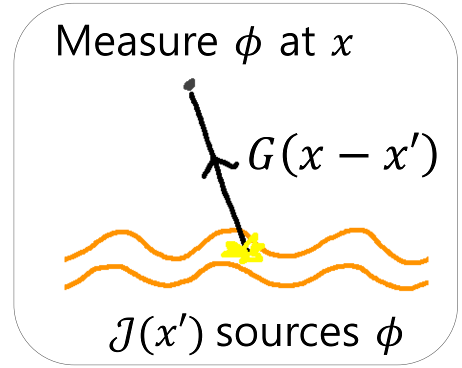


Positivity bounds

Now consider a QFT with scalar field operator $\phi(x)$. In response to a source $\mathcal{J}(x)$,

$$\langle \phi(x) \rangle_{\mathcal{J}} = \int d^4x' \; G(x - x') \mathcal{J}(x')$$



The EFT expansion at low momentum is,

$$G(p) = \left(m^2 - p^2 - \frac{c_4 p^4}{\Lambda^2} + \cdots \right)^{-1}$$

(from eqn of motion, $\mathcal{J} = m^2 \phi + \partial_\mu \partial^\mu \phi +$ **interactions** with characteristic energy Λ)

Repeating the same steps,

Causality + Stability at high energies $\Rightarrow \; \partial_p^4 G(p)|_{p=0} = \int_{m^2}^{\infty} \frac{dp'^2}{p'^6} \text{Im}\, G(p') > 0$

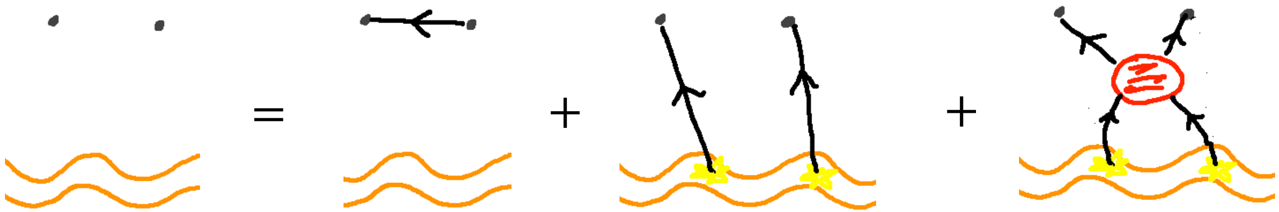
$\Rightarrow \; \underline{c_4 > -\Lambda^2 / m^2}$

but this is a weak constraint since often $m \ll \Lambda$.

Positivity bounds

Stronger bounds come from the non-linear response function,

$$\langle \phi(x_2)\phi(x_1)\rangle_{\mathcal{J}} = G(x_2 - x_1) + \langle \phi(x_2)\rangle_{\mathcal{J}}\langle \phi(x_1)\rangle_{\mathcal{J}} + \int d^4x_1' d^4x_2' \; F(x_1, x_2, x_1', x_2') \, \frac{\mathcal{J}(x_1')\mathcal{J}(x_2')}{2}$$



In momentum space, $F(p_1, p_2, p_3, p_4) = i\, A(s,t) \underbrace{G(p_1) \dots G(p_4)}_{\text{external legs}} \underbrace{\delta^4(p_1 + p_2 - p_3 - p_4)}_{\text{momentum conservation}}$

The amplitude $A(s,t)$ is a function of $16 - 4 - 4 - 6 = 2$ variables, e.g. $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$

$A(s,t) =$  interactions (with characteristic energy Λ)

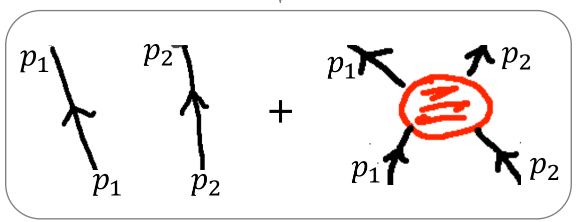
$$= c_0 + c_s \frac{s}{\Lambda^2} + c_t \frac{t}{\Lambda^2} + c_{ss} \frac{s^2}{\Lambda^4} + \cdots$$ at low $s, t \ll \Lambda^2$

Positivity bounds

Before fitting to data, we should ask: Can the EFT coefficients c_{ab} really have any values?

Suppose the ϕ fluctuations were created in the past with momenta p_1 and p_2 , what is the probability that we measure different momenta at late times?

$$P_{\text{different}} = 1 - P_{\text{same}} = 1 - |1 + i\, A(s,0)|^2 = 2\,\text{Im}\, A(s,0) - |A(s,0)|^2$$



Stability (positive probabilities) $\Rightarrow$ $2\,\text{Im}\, A(s,0) > |A(s,0)|^2 > 0$

Positivity bounds

Causality (response = 0 when $(ct_1 - ct_2) < |\boldsymbol{x}_1 - \boldsymbol{x}_2|$) relates high and low energy,

$$\frac{1}{\pi}\int_{4m^2}^{\infty} ds' \left(\frac{1}{s'-s} + \frac{1}{s'-u}\right) \text{Im}\, A(s',t) = A(s,t)$$

$(u = 4m^2 - s - t)$

Note: At large s, $\text{Im}\, A(s,t)/s^2 \to 0$ so integral guaranteed to converge as long as we take at least two ∂_s

Stability at high energies $\Rightarrow$ $\partial_s^2 A(s,t)\Big|_{\substack{s=0\\t=0}} = \frac{2}{\pi}\int_{4m^2}^{\infty} \frac{ds'}{s'^3} \text{Im}\, A(s',0) > 0$

$\Rightarrow$ $c_{ss} > 0$ [Adams++ 2006], ...

Using causality to translate high energy conditions into EFT bounds has recently led to a number of new constraints on all $\partial_s^a \partial_t^b A(s,t)$ with $a \geq 2$,

e.g. $c_{sst} > -\frac{3}{2} c_{ss}$ [de Rham+SM+Tolley+Zhou, 2017], ...

both with and without boost invariance. [Baumann++, 2016], [Grall+SM, 2020], ...

A Cosmological Example

Gravity/dark energy is difficult, but at low energies can expand action in fields and their derivatives.

Assuming diffeomorphism invariance & approximate symmetries $\phi \to \phi + c + c_\mu x^\mu$ and $\phi \to -\phi$ the leading interactions between $g_{\mu\nu}$ and ϕ are given by,

$$\mathcal{L}_{EFT} = G_4(X)R + P(X) + G_4{}'(X)\left(\left(\nabla_\mu \nabla_\nu \phi\right)^2 - \left(\nabla_\mu \nabla^\mu \phi\right)^2\right) + \cdots$$

with $X = -\frac{1}{2}(\nabla_\mu \phi)^2$

Roughly, $P(X)$ determines the background evolution, while $G_4(X)$ controls perturbations.

[Bellini+Sawicki 2014], ...

We will focus on two observables:

Low-frequency GW speed: $c_T^2 = \left(1 - \frac{2X\partial_X G_4}{G_4}\right)^{-1} \Leftarrow \partial_X G_4$

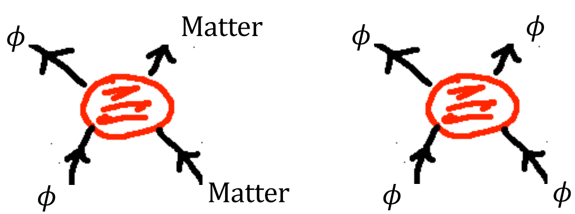
Dark energy clustering: $\alpha_B = 8\, c_T^2 \frac{X^2 \partial_X^2 G_4}{G_4} + 2(c_T^2 - 1) \Leftarrow \partial_X^2 G_4$

A Cosmological Example

Focus on theories in which $P(X)$ has both flat and cosmological vacua,

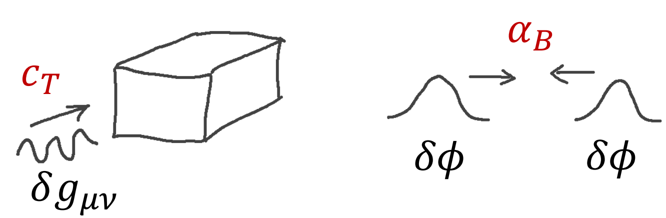
$$\mathcal{L}_{\text{EFT}}[\phi]$$

$\phi = 0 + \delta\phi$ | $\phi = \phi_{\text{cosmo}}(t) + \delta\phi$



$G_4' \geq 0$ | ${G_4'}^2 \leq G_4 G_4''$

Import → Positivity

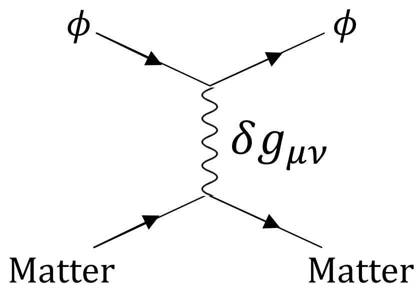


$c_T > 1$ | $\frac{1}{2}\alpha_B < 1 - c_T^{-2}$

Speed of gravitational waves

The EFT coupling between metric and ϕ affects the speed of gravitational waves.

Can constrain this by scattering ϕ with any matter field, [de Rham, SM, Noller 2019]



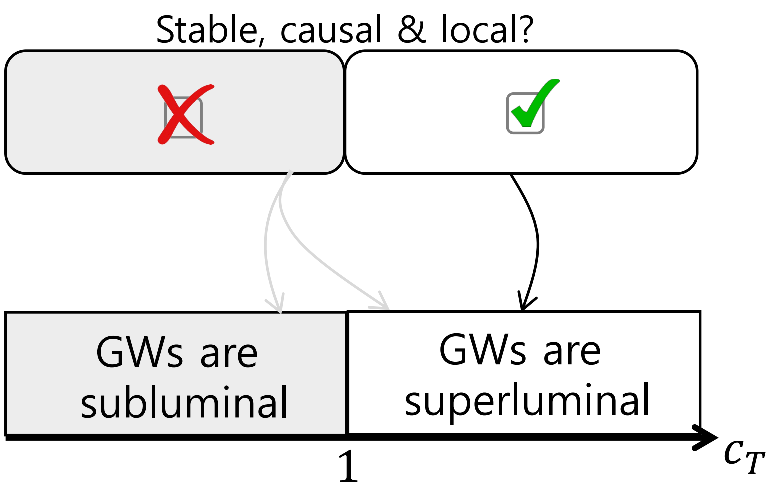
$$= \partial_X G_4 \, s^2 + \cdots$$

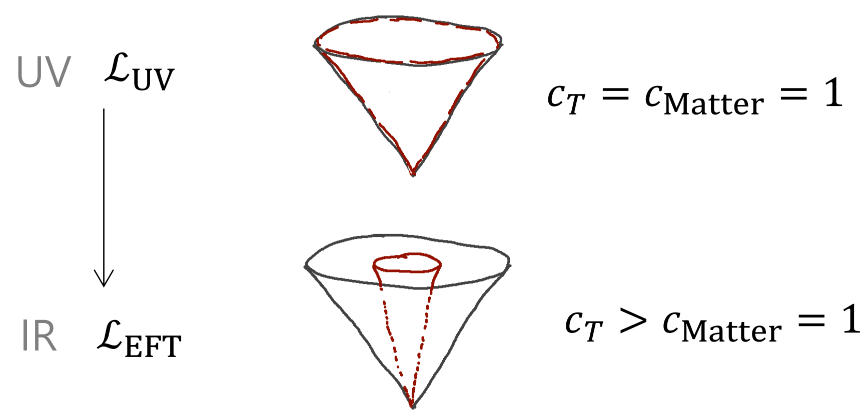
Causality+Stability at high energies

$\Downarrow$

Positivity of $\partial_s^2 A \;\Rightarrow\; \partial_X G_4 > 0$

$\Rightarrow \quad c_T^2 > 1$





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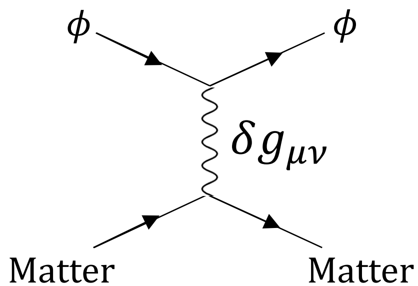

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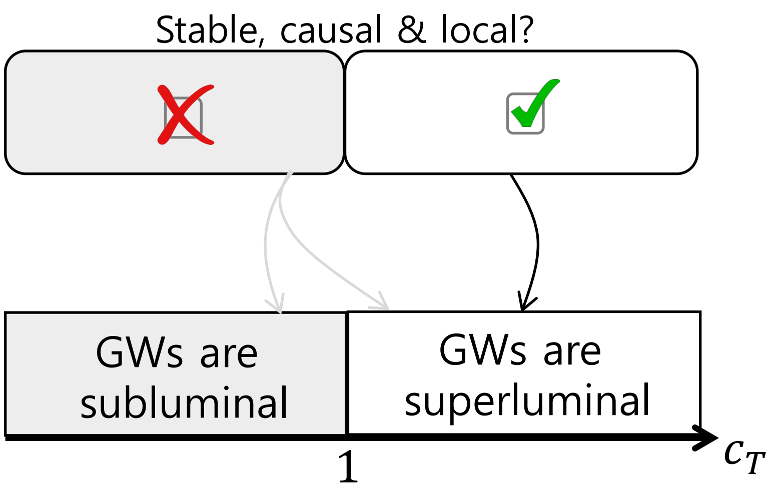
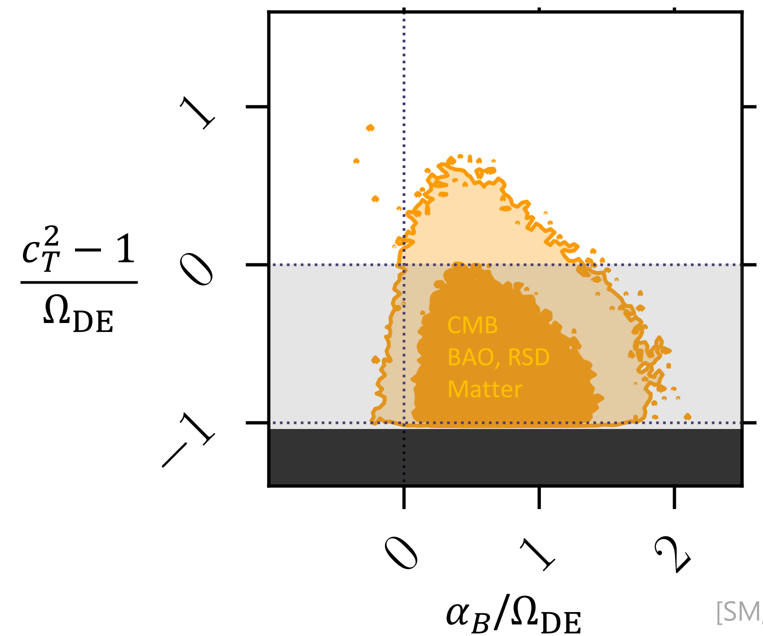


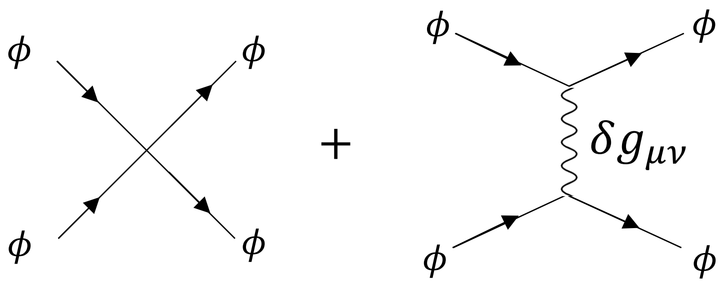


[SM, Noller 2019]

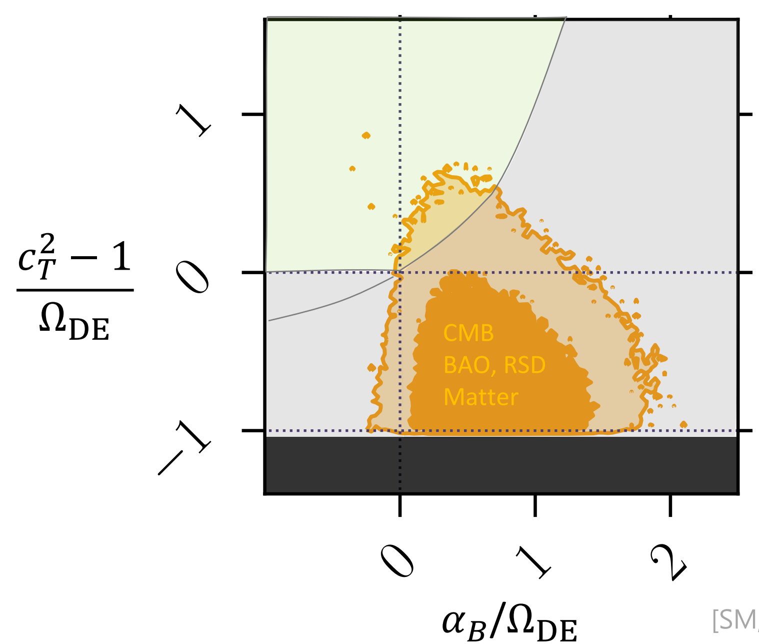
Dark energy clustering

ϕ self-interactions affect how dark energy clusters on large scales.

Can constrain this by scattering ϕ's,



$$= -\left(\partial_X^2 G_4 + \frac{(\partial_X G_4)^2}{G_4} \right) s^2 t + \cdots$$



[SM, Noller 2019]

Causality+Stability at high energies

$\Downarrow$

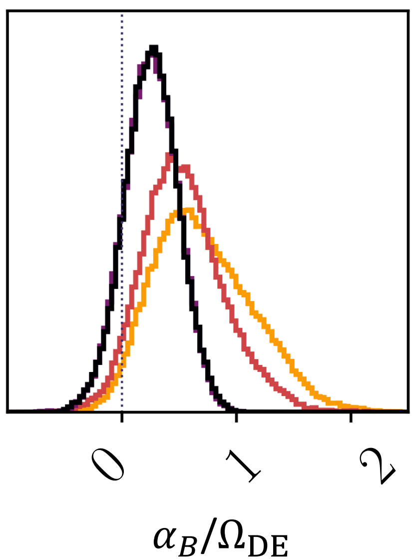
Positivity of $\partial_t \partial_s^2 A \;\Rightarrow \partial_X^2 G_4 < -\frac{(\partial_X G_4)^2}{G_4}$

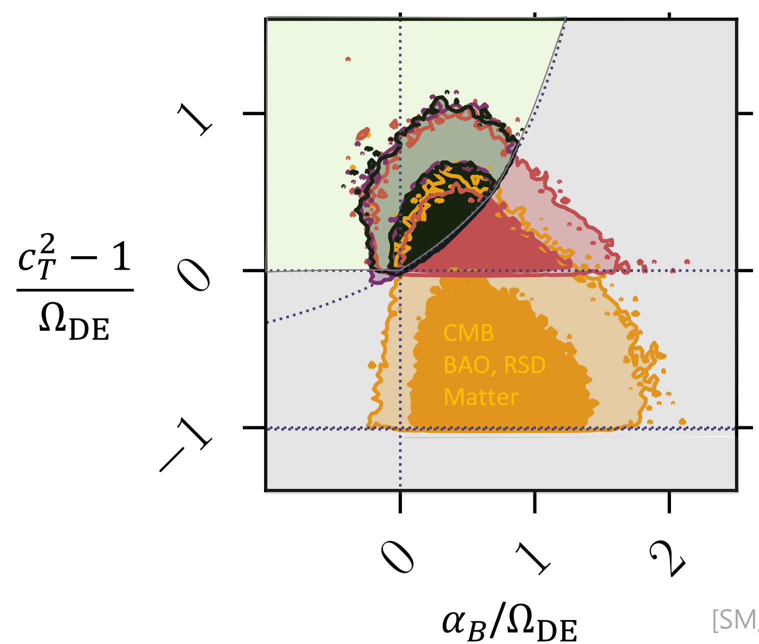
$\Rightarrow \; \frac{1}{2}\alpha_B < 1 - c_T^{-2}$

Dark energy clustering

Two important uses:

(i) IR priors from UV stability + causality,





(ii) UV features from IR data.

[SM, Noller 2019]

Some Caveats

This plot assumes:

(i) Particular IR dof/symmetries

See [Grall+SM 2021] for general EFT
of spontaneously broken time translations

(ii) $\phi = 0$ vacuum is stable

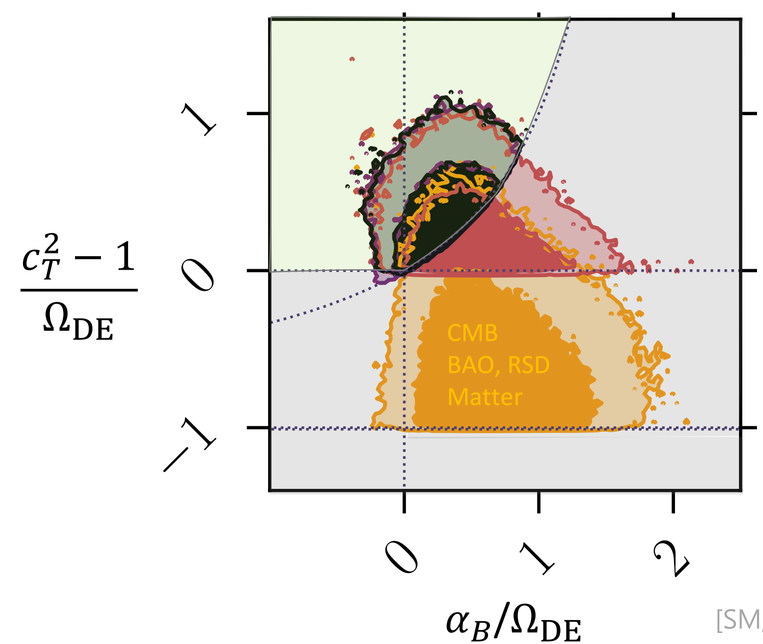
See [SM+Noller 2022] for different stable vacua

(iii) $\phi_{\rm cosmo}$ not too large $\left(|X| < \frac{\partial_X G_4}{\partial_X^2 G_4} \right)$

See [Davis+SM 2021] for large $|X|$

(iv) Time dependence $c_T(t), \alpha_B(t) \sim \Omega_{DE}(t)$

See [Noller+Nicola 2018], [Kennedy+Lombriser 2020]
for other parametrisations



[SM, Noller 2019]