Quantum Diffusion during Cosmic Inflation

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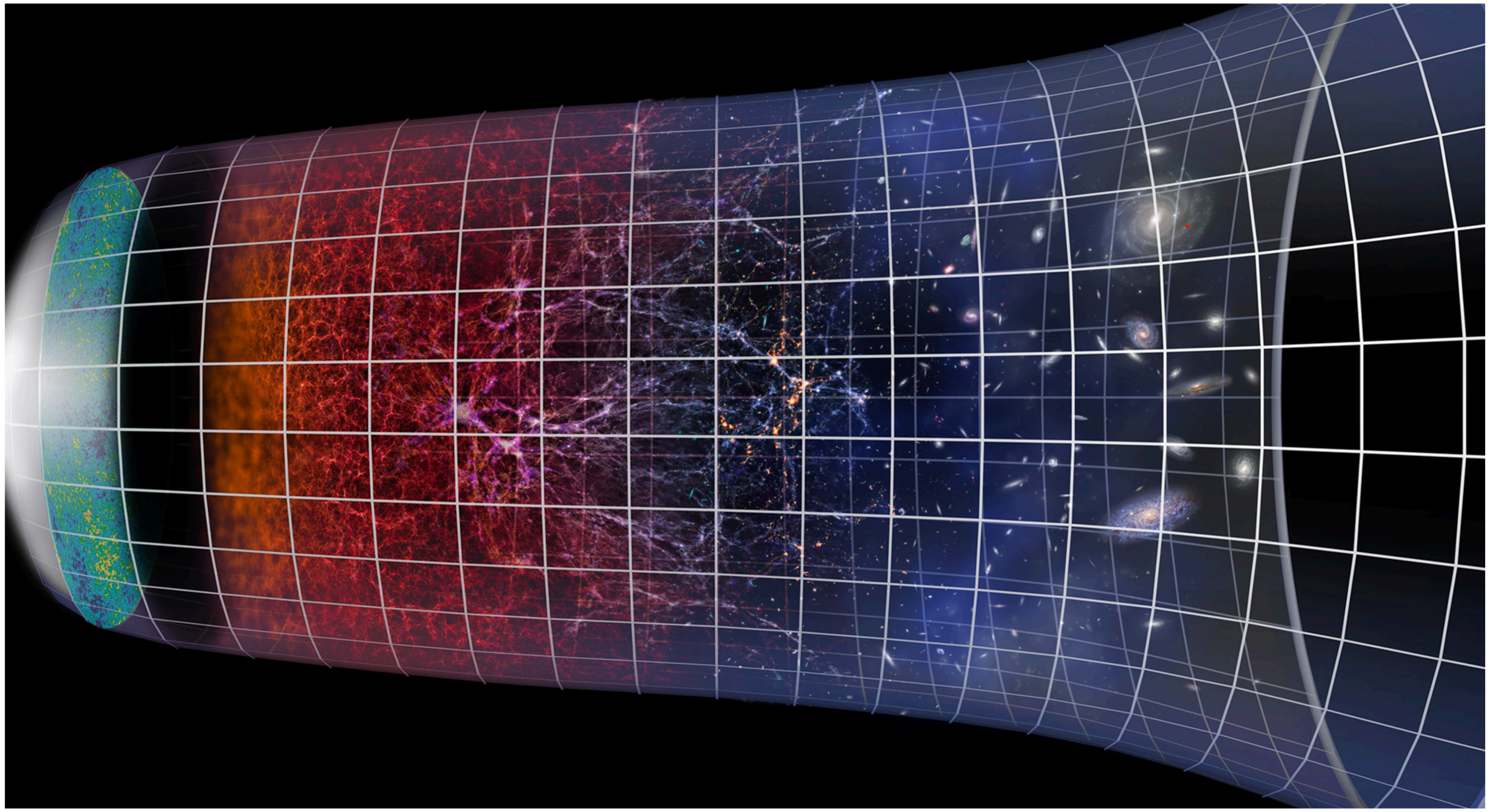
International Workshop "Dawn of Gravitational-wave Cosmology and Theory of Gravity"

Cosmic Inflation

Inflation is a **high-energy** phase of **accelerated expansion** in the early Universe

Starobinsky
Guth
Linde
Albrecht & Steinhardt
Mukhanov & Chibisov
Guth & Pi
Hawking
Bardeen, Steinhardt & Turner
(early 80's)



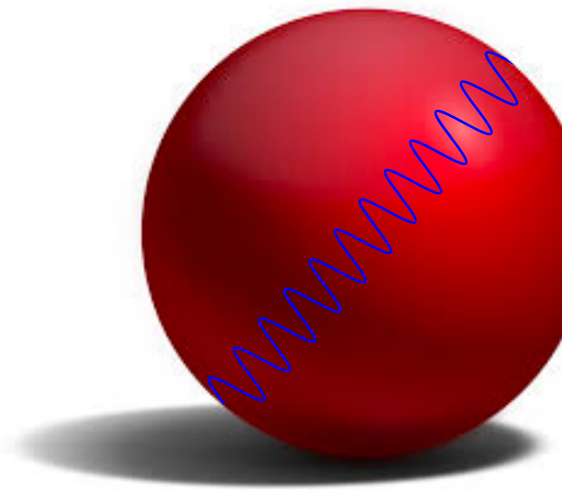
$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\vec{x}^2 \quad \text{with} \quad \ddot{a} > 0 \quad \text{and} \quad (10\,\mathrm{MeV})^4 < \rho < \left(10^{16}\,\mathrm{GeV}\right)^4$$

Turns the primordial universe into an ultra-high energy laboratory

Cosmic Inflation

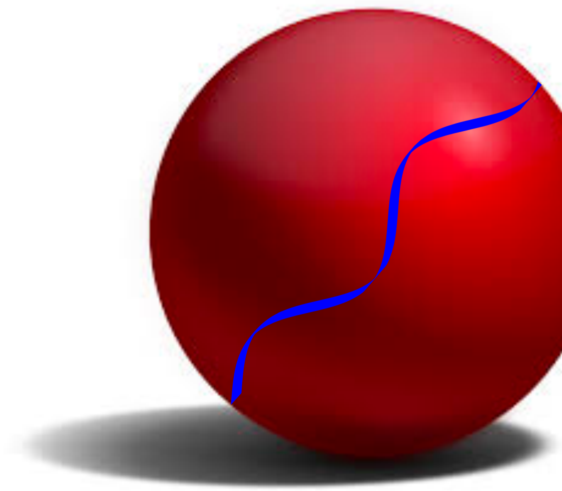
Hubble parameter $H = \dot{a}/a$

H^{-1} : characteristic time scale, or length scale ($c = 1$), of the expansion



$\lambda \ll H^{-1}$

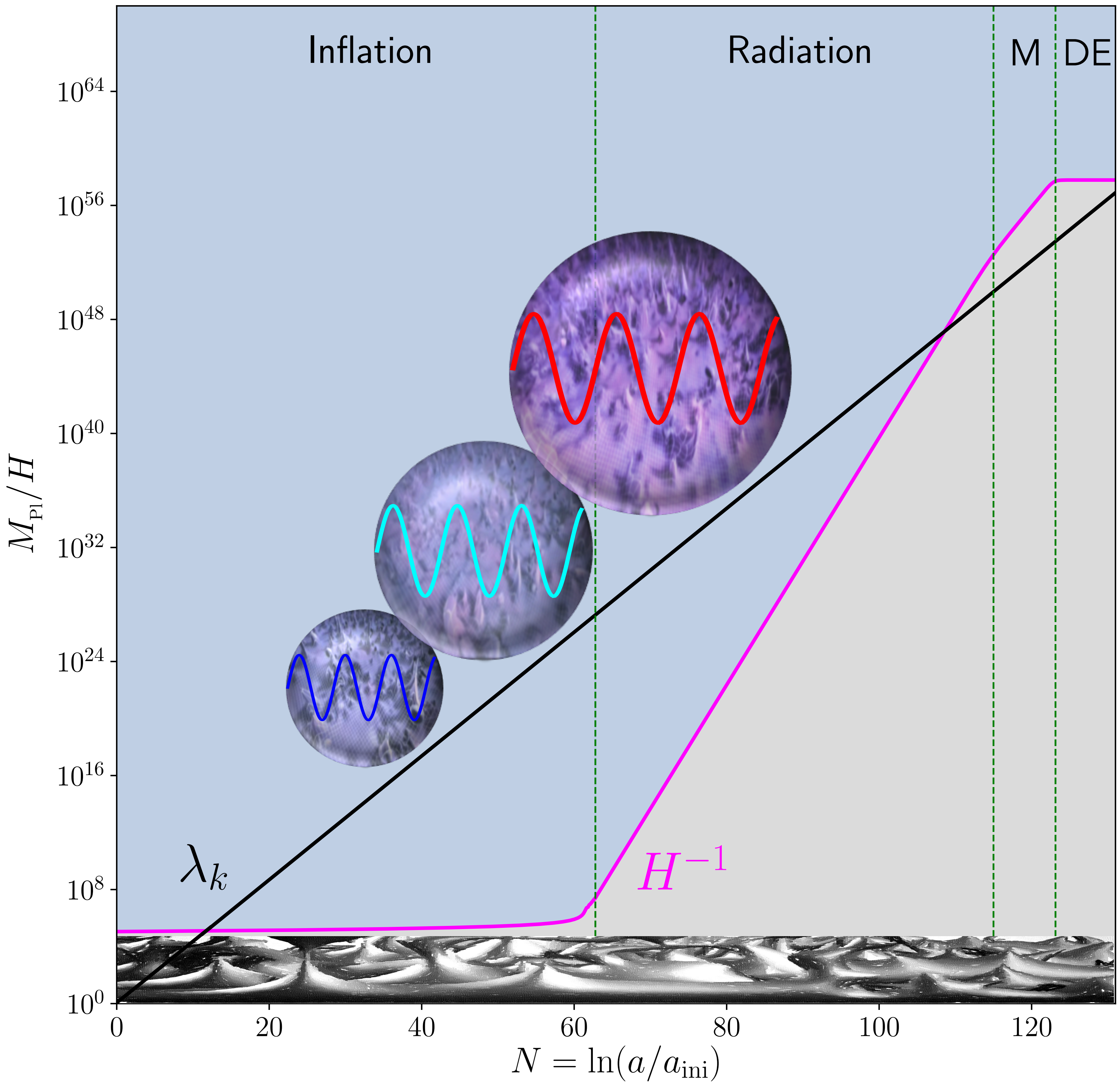
Insensitive to space-time curvature



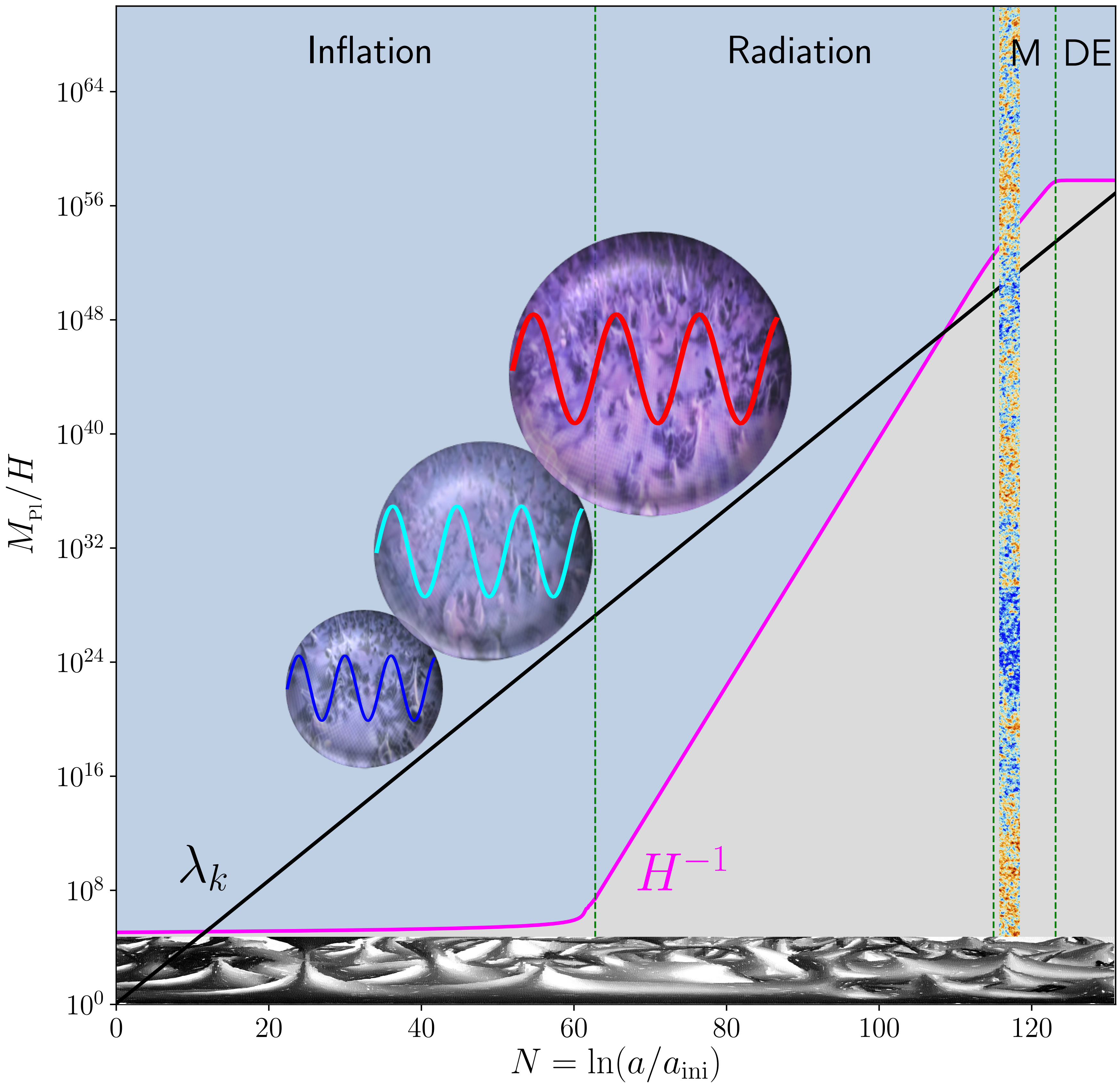
$\lambda \gtrsim H^{-1}$

Feels space-time curvature

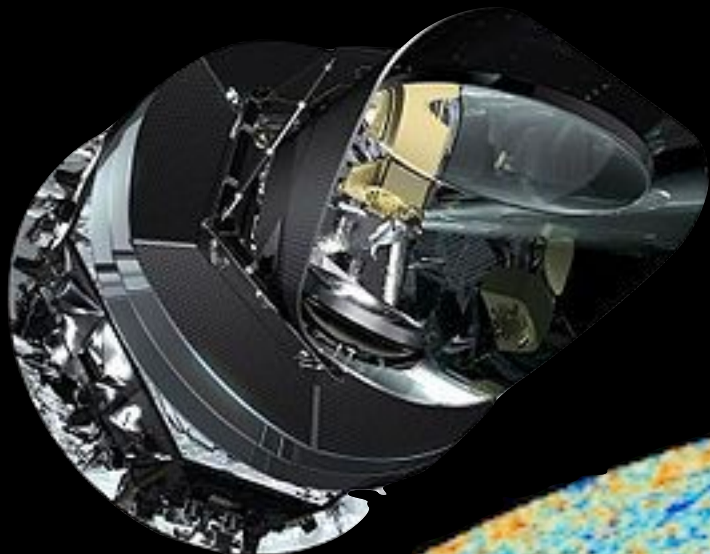
Cosmic Inflation

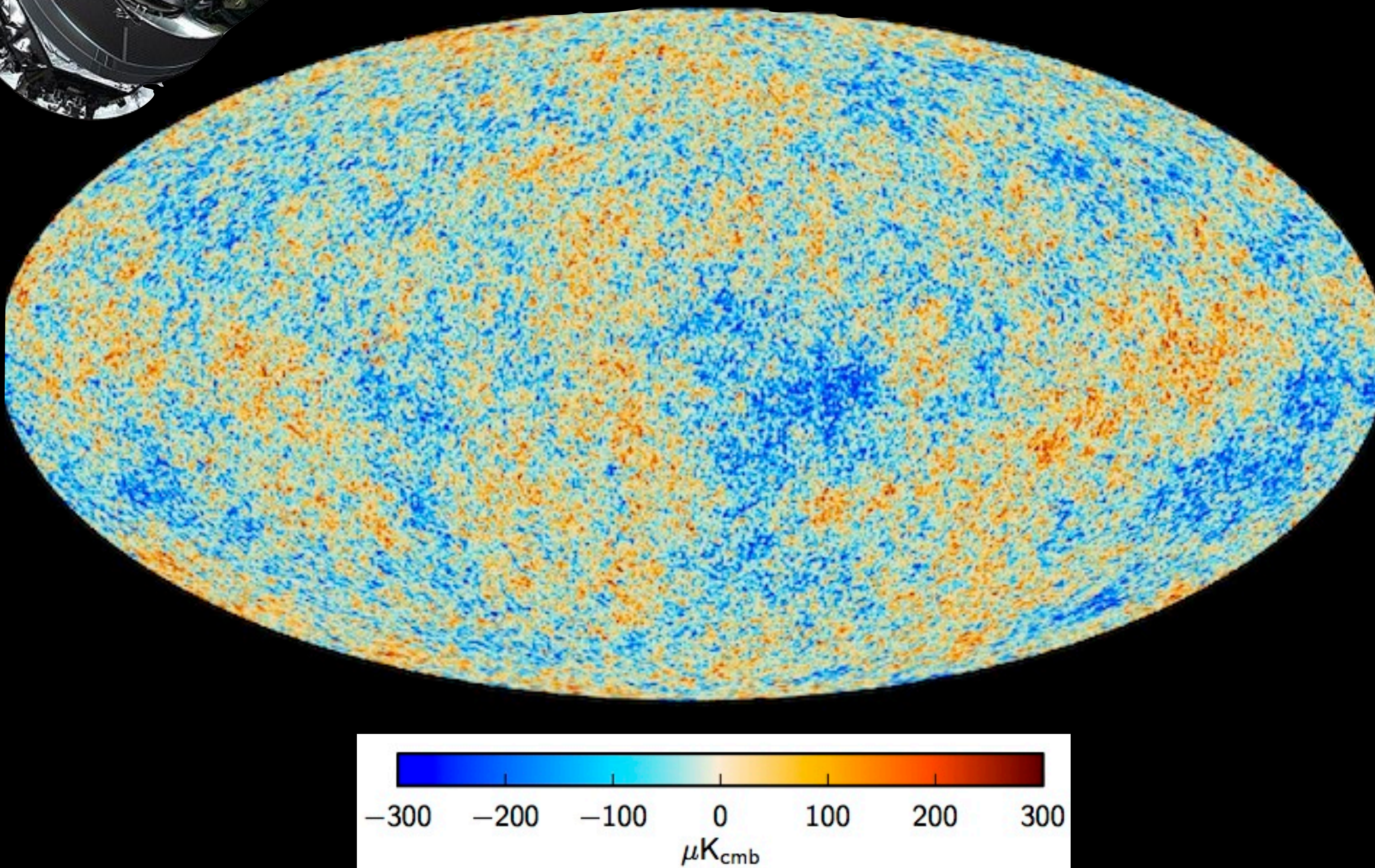


Cosmic Inflation

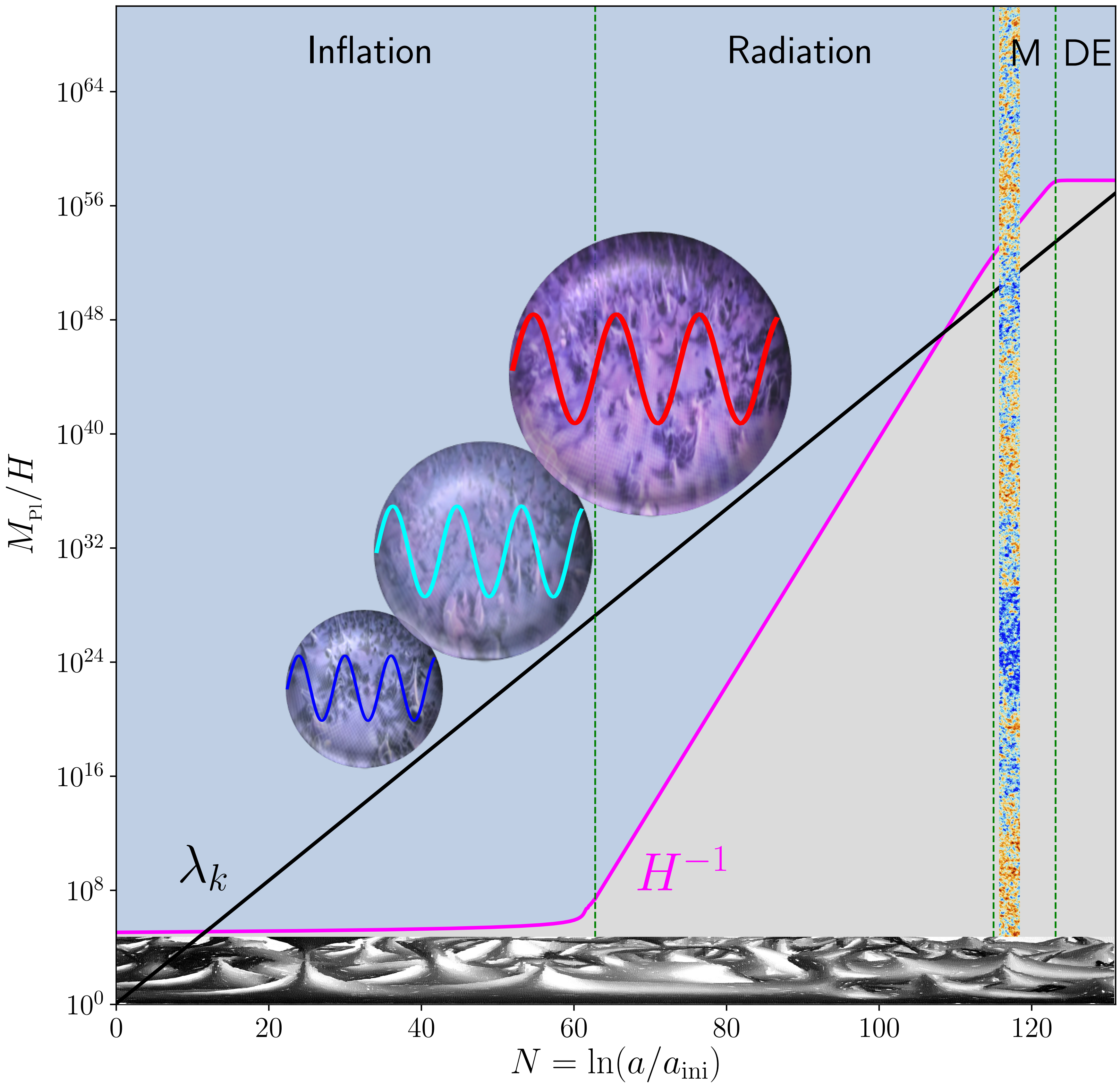


Planck satellite

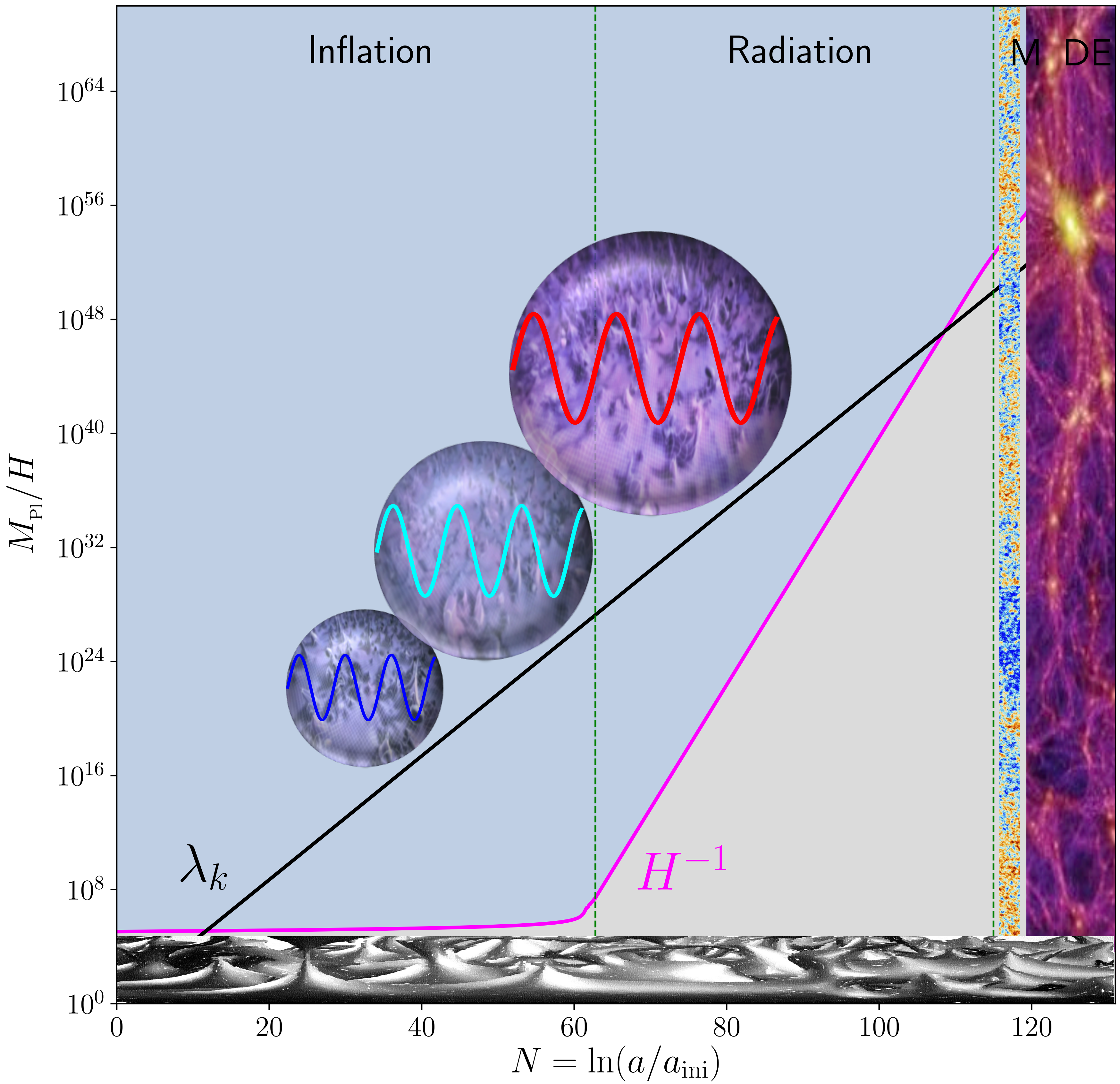




Cosmic Inflation



Cosmic Inflation



Cosmological Perturbation Theory

Density fluctuations are small at CMB scales ⟶ Perturbation Theory

$$g_{\mu\nu}(\vec{x},t) = \bar{g}_{\mu\nu}(t) + \widehat{\delta g}_{\mu\nu}(\vec{x},t)$$

$$\phi(\vec{x},t) = \bar{\phi}(t) + \widehat{\delta\phi}(\vec{x},t)$$

Homogeneous and isotropic solution of the classical problem

Quantised fluctuation

—> Quantum-field-theory on curved space-time

Strong assumption: universe is quasi homogeneous and isotropic at all scales

This may be broken at:

- Larger scales: space-time structure beyond the observable universe
- Smaller scales: formation of compact objects such as primordial black holes

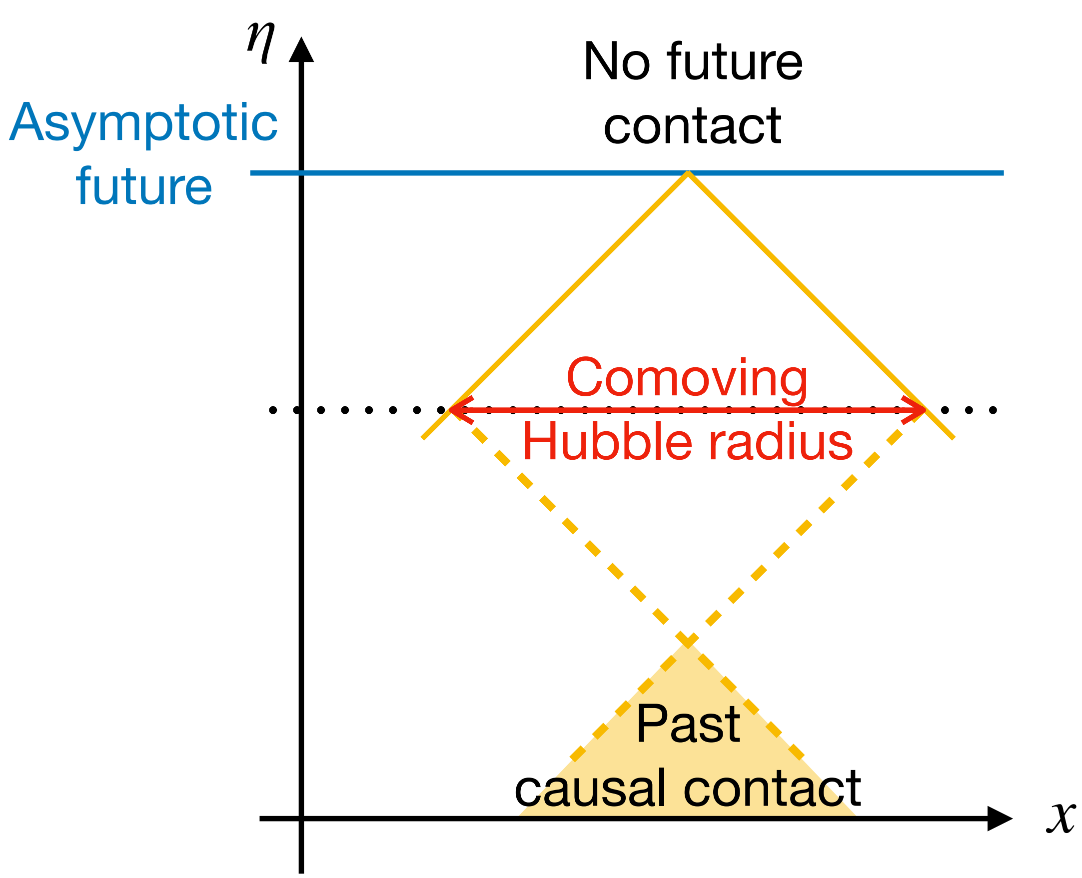
Outline

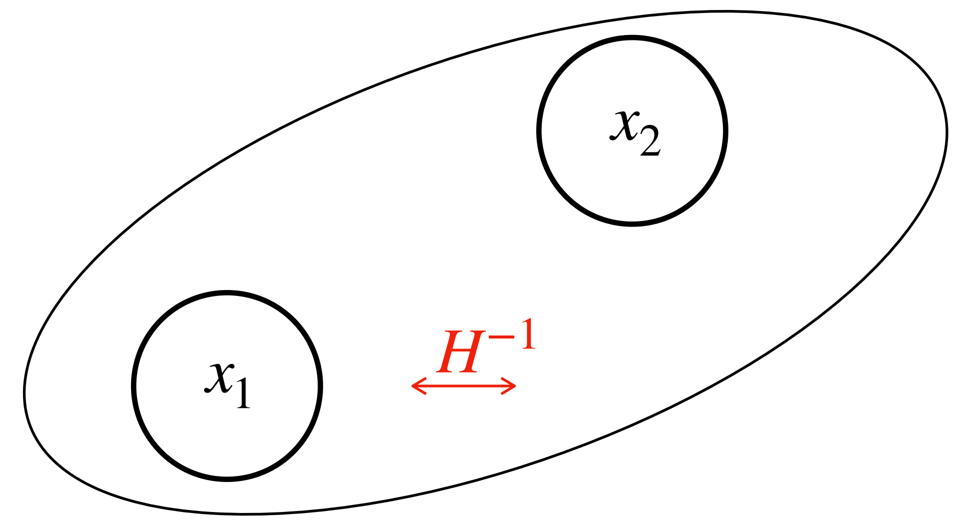
- Stochastic δN formalism
- Statistics of cosmological fluctuations
- Primordial black holes

Separate Universe

$$\mathrm{d}s^2 = a^2\left(-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2\right)$$

de-Sitter universe: $a = -1/(H\eta), \quad -\infty < \eta < 0$



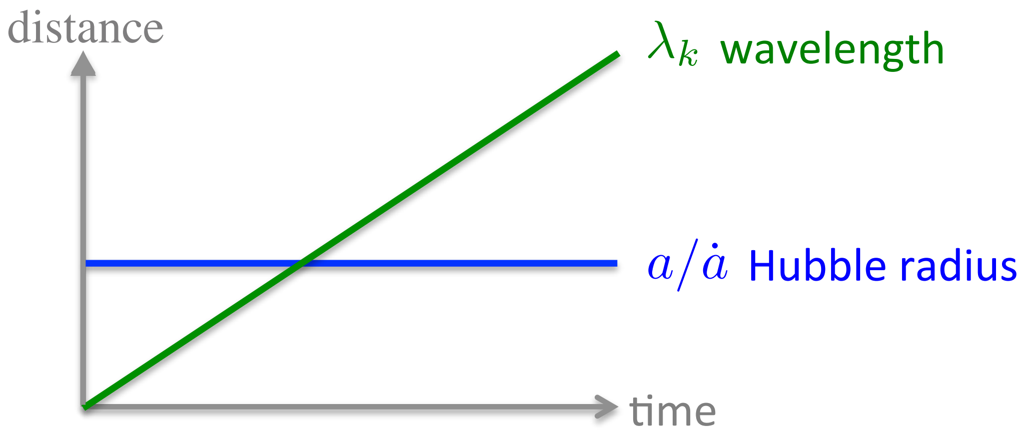


If a large fluctuation develops at $\boldsymbol{x}_1$, this cannot affect the local geometry at $\boldsymbol{x}_2$

Separate universe: On large scales, the universe can be described by an ensemble of independent, locally homogeneous and isotropic patches

Salopek & Bond; Sasaki & Stewart; Wands, Malik, Lyth & Liddle

Stochastic Inflation



coarse-grained fields —> stochastic inflation

sub-Hubble scales —> CPT

Coarse-grained field $\hat{\phi}_{\mathrm{cg}}(N, \vec{x}) = \int_{k<\sigma H a(N)} \mathrm{d}\vec{k} \left[\phi_{\vec{k}}(N) e^{-i\vec{k}\cdot\vec{x}} \hat{a}_{\vec{k}} + \phi^{\star}_{\vec{k}}(N) e^{i\vec{k}\cdot\vec{x}} \hat{a}^{\dagger}_{\vec{k}}\right]$

$N = \ln(a)$

Quantum fluctuations source the **background**

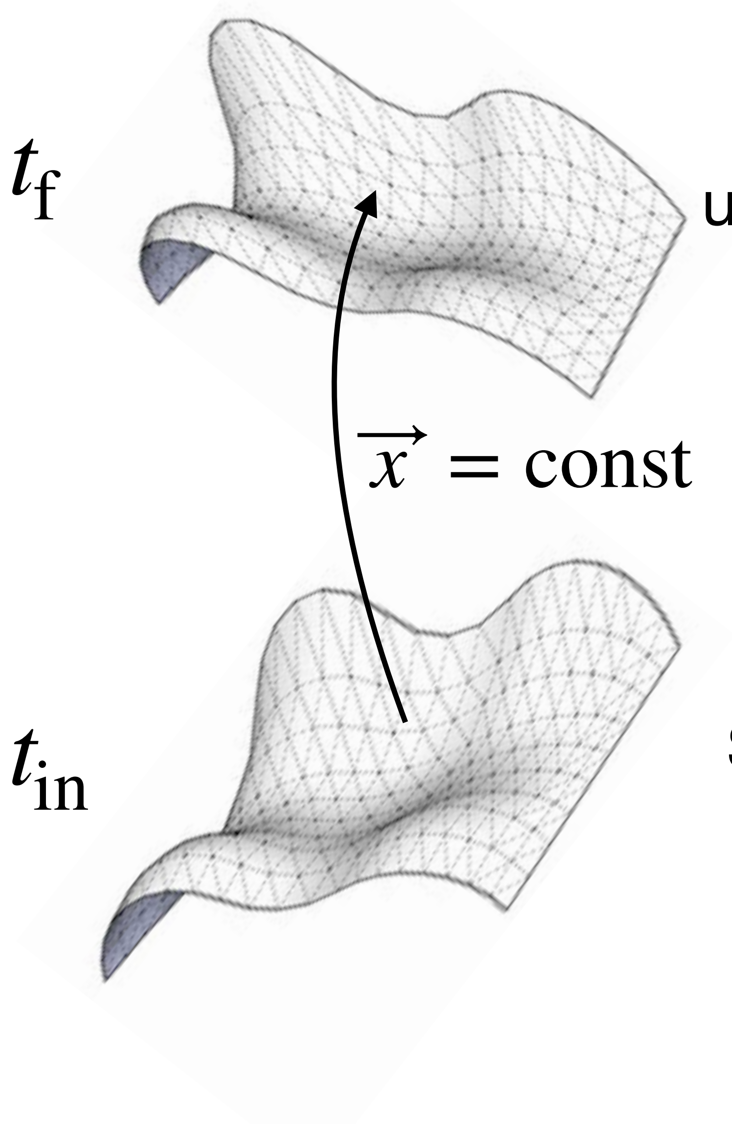
At leading order in slow roll: $\frac{\mathrm{d}}{\mathrm{d}N}\phi_{\mathrm{cg}} = \mathscr{D}_{\mathrm{background}}(\phi_{\mathrm{cg}}) + \xi$ Starobinsky, (1982) 1986

Over one e-fold: $\frac{\Delta\phi_{\mathrm{quant}}}{\Delta\phi_{\mathrm{classical}}} \sim \zeta_{\mathrm{classical}}$

What about far from the classical regime?

What about tail effects?

Stochastic-δN formalism



uniform density slice
$\delta\rho = 0$

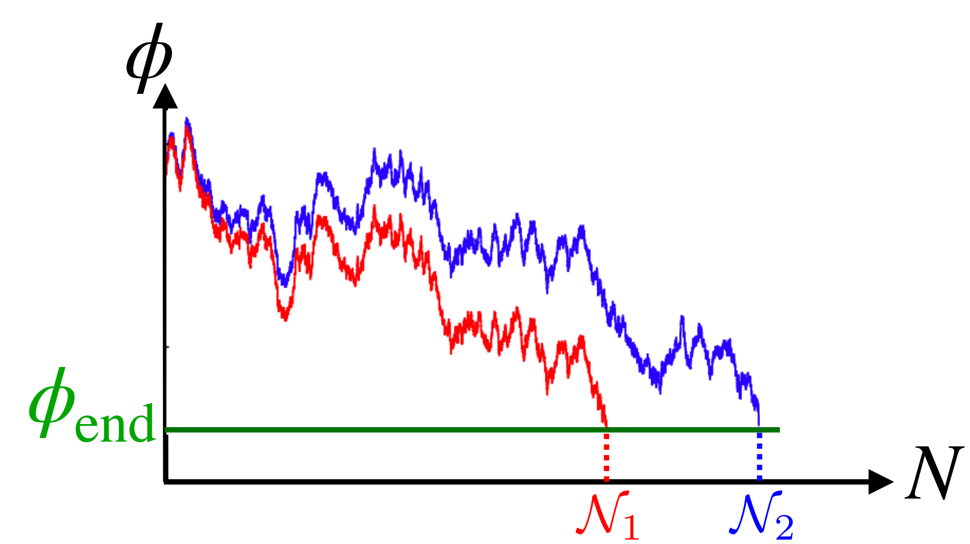
Spatially flat slice
$\psi = 0$

$$\zeta(t, x) = N(t, x) - N_0(t) \equiv \delta N$$

Lifshitz, Khalatnikov (1960)
Starobinsky (1983)
Wands, Malik, Lyth, Liddle (2000)

The realised number of e-folds
is a stochastic quantity:

$$\zeta_{\text{coarse grained}} = \mathcal{N} - \langle \mathcal{N} \rangle$$



Enqvist, Nurmi, Podolsky, Rigopoulos (2008) ; Fujita, Kawasaki, Tada, Takesako (2014); VV, Starobinsky (2015)

Stochastic-δN formalism

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = -\frac{V'}{3H^2} + \frac{H}{2\pi}\xi(N) \longrightarrow \frac{\mathrm{d}}{\mathrm{d}N}P(\phi, N) = \frac{\partial}{\partial\phi}\left(\frac{V'}{3H^2}P\right) + \frac{\partial^2}{\partial\phi^2}\left(\frac{H^2}{8\pi^2}P\right) = \mathscr{L}_\phi \cdot P$$

Langevin equation

Fokker-Planck equation

Equation for the PDF of the first passage time

VV, Starobinsky (2015)
Pattison, VV, Assadullahi, Wands (2017)

$$\frac{\mathrm{d}}{\mathrm{d}\mathcal{N}}\mathscr{P}(\mathcal{N}, \phi) = \mathscr{L}_\phi^\dagger \cdot \mathscr{P}$$

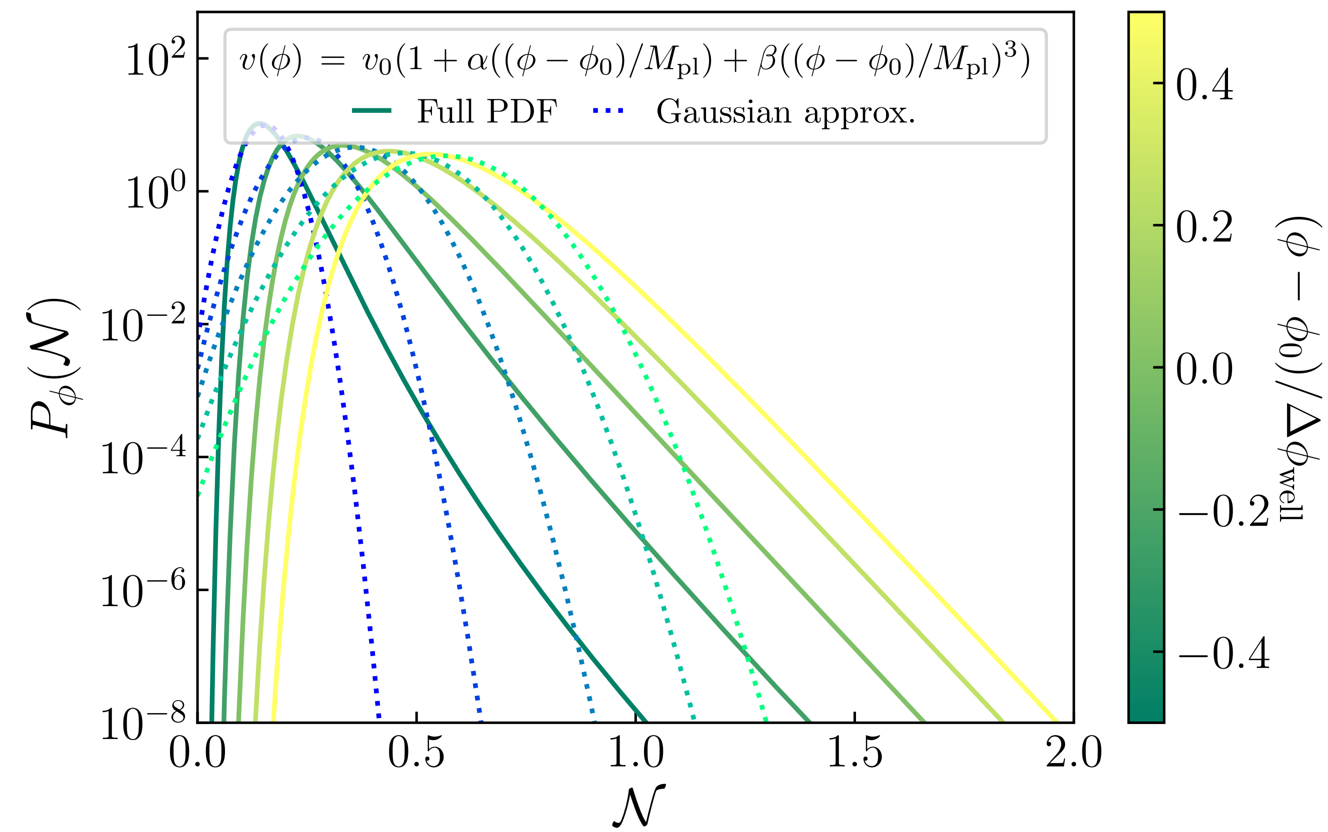
Computational program:

- Solve the first passage time problem
- This gives the one-point PDF of curvature perturbation coarse-grained at H_{end}
- Extract cosmologically relevant quantities (power spectrum, mass functions, etc)

Exponential tails

Pattison, VV, Assadullahi, Wands (2017)
Ezquiaga, Garcia-Bellido, VV (2020)

$$\mathscr{P}(\mathcal{N}, \phi) = \sum a_n(\phi) e^{-\Lambda_n \mathcal{N}}$$



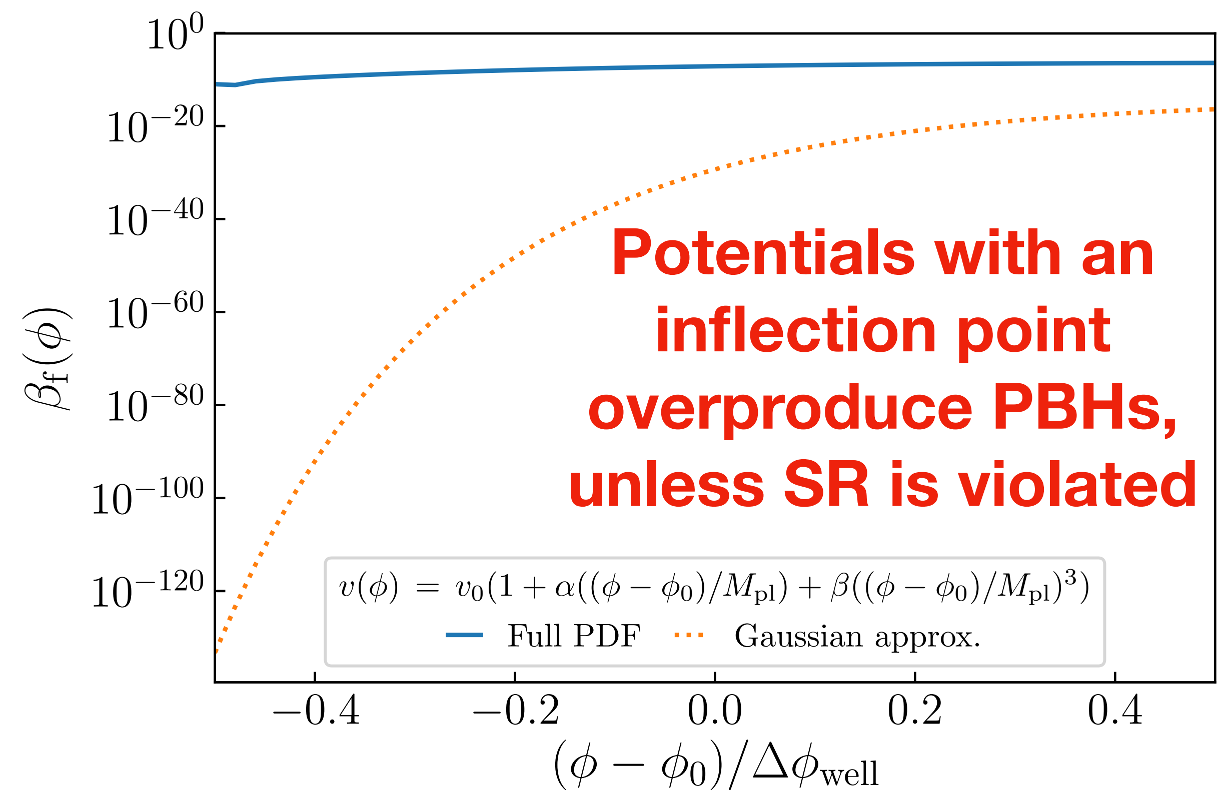
$v_0 = 10^{-3}$, $\alpha = 0.24$, $\beta = 9$, $\phi_{\rm end} = 9$

Flat well

$$\Lambda_n = \frac{\pi^2}{\mu^2}\left(n + \frac{1}{2}\right)^2 \qquad \mu^2 = \frac{\Delta\phi_{\rm well}^2}{v_0 M_{\rm Pl}^2}$$

Constant slope well $\quad v = v_0\left(1 + \alpha\frac{\phi}{M_{\rm Pl}}\right)$

$$\Lambda_n \simeq \frac{\alpha^2}{4v_0} + \frac{\pi^2}{\mu^2}\left(n + \frac{1}{2}\right)^2$$

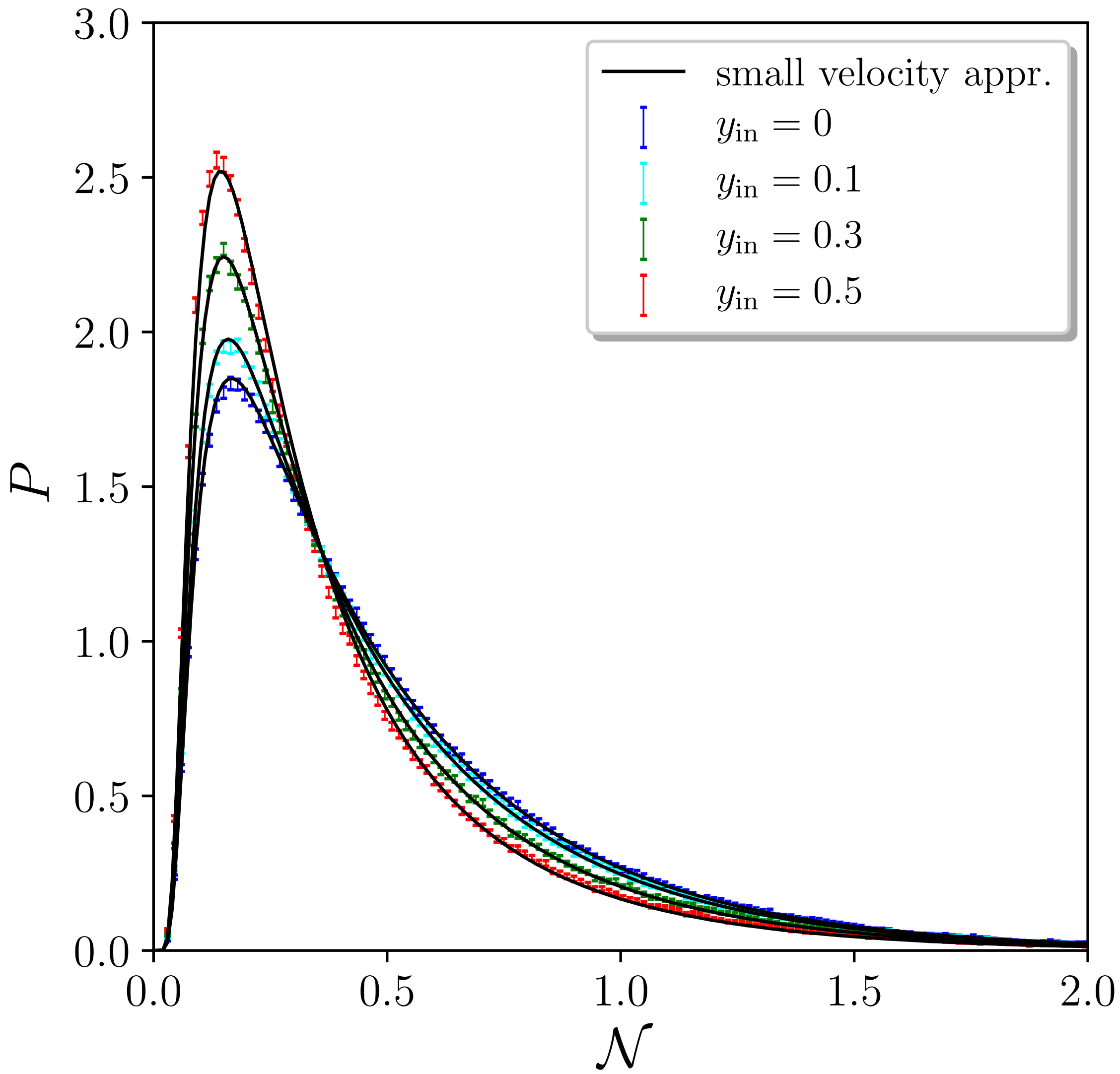


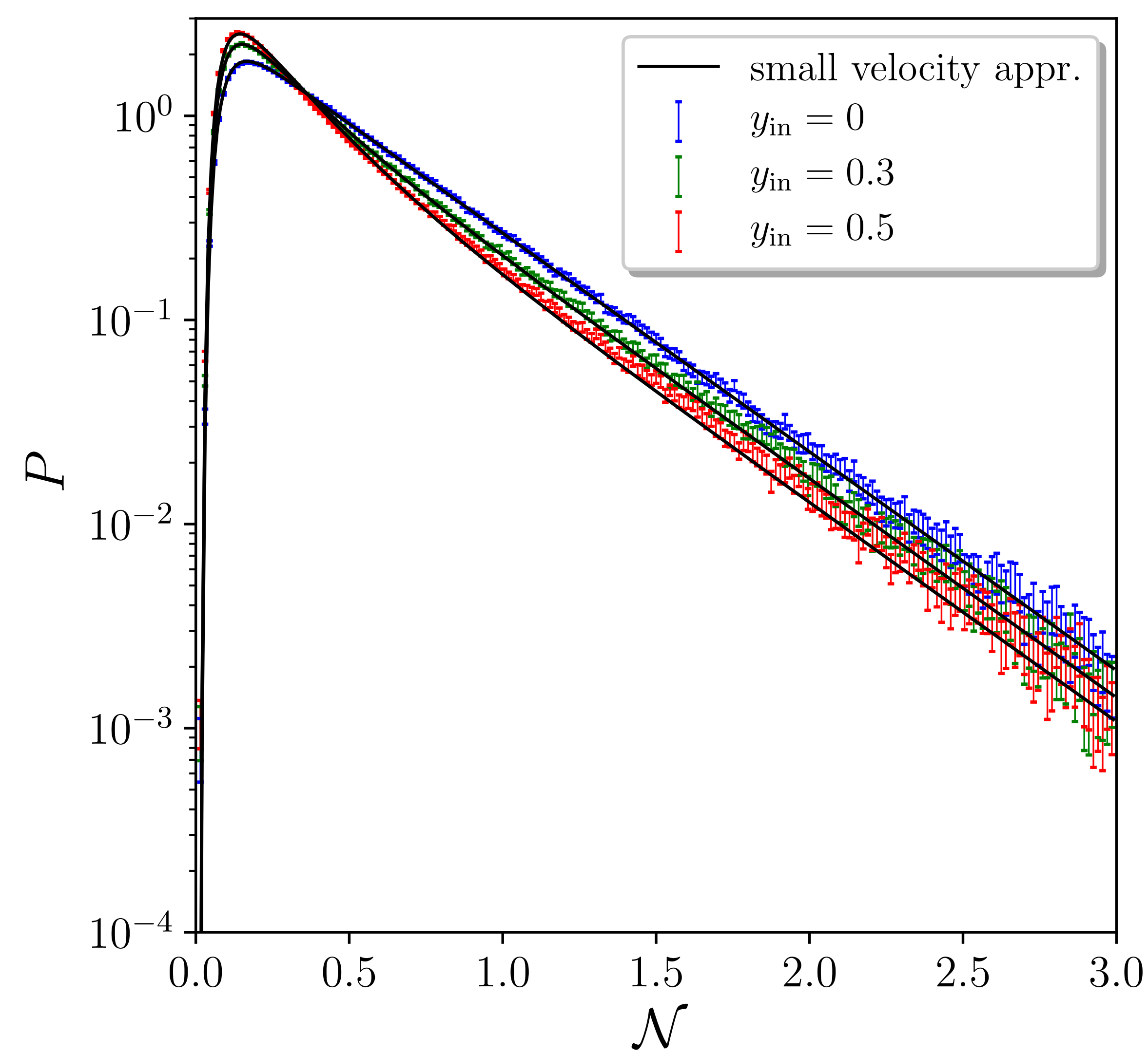
Cubic inflection point $\quad v = v_0\left(1 + \alpha\frac{\phi^3}{M_{\rm Pl}^3}\right)$

$$\Lambda_n \simeq \left(\frac{3}{2}\right)^{2/3} \pi^2 \left(v_0\alpha\right)^{1/3} \left(n + \frac{1}{2}\right)^2$$

Exponential tails
in ultra slow roll models

Pattison, Vennin, Wands, Assadullahi (2021)





See also Figueroa, Raatikainen, Rasanen, Tomberg 2021

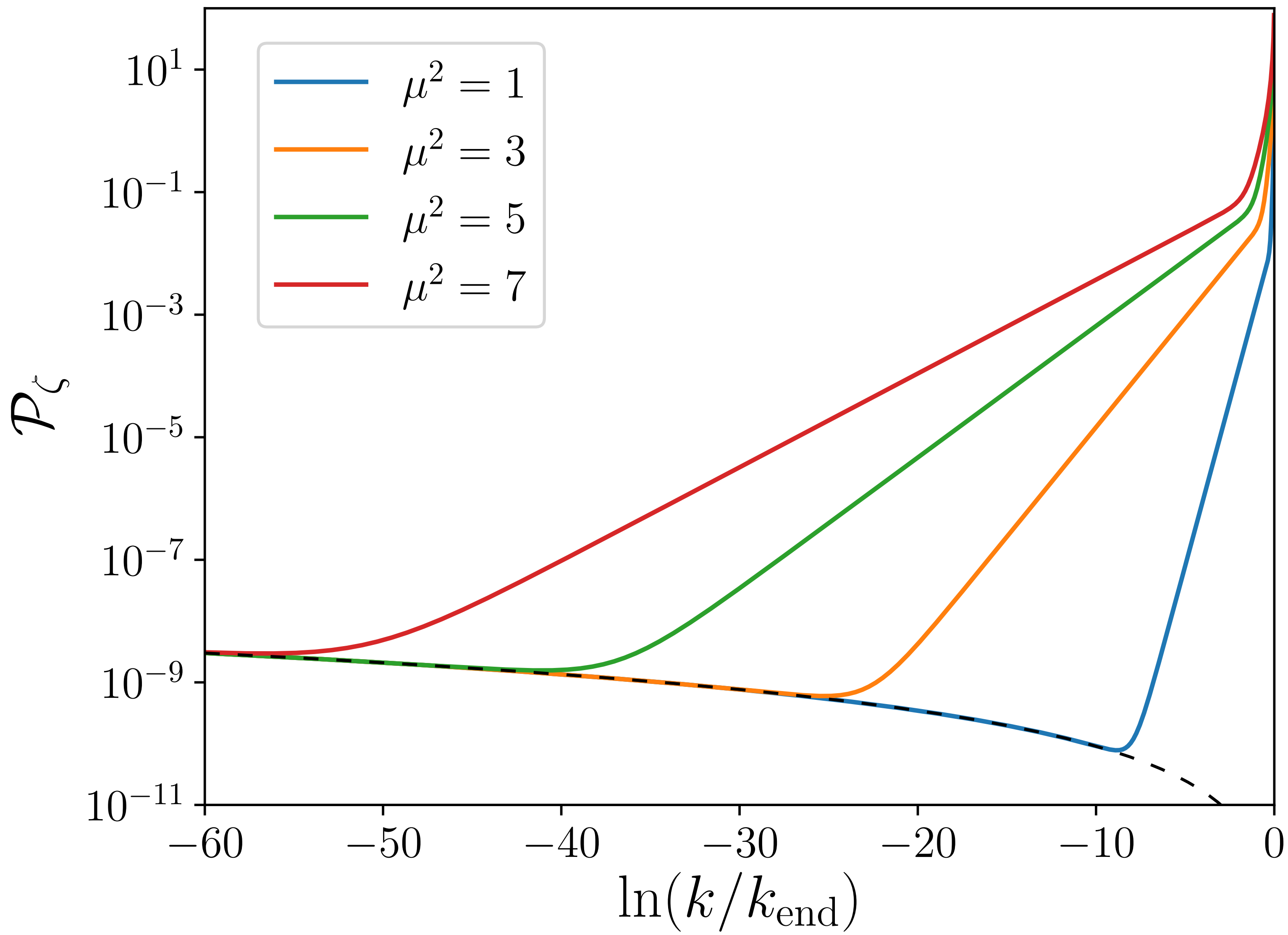
Extracting cosmological observables

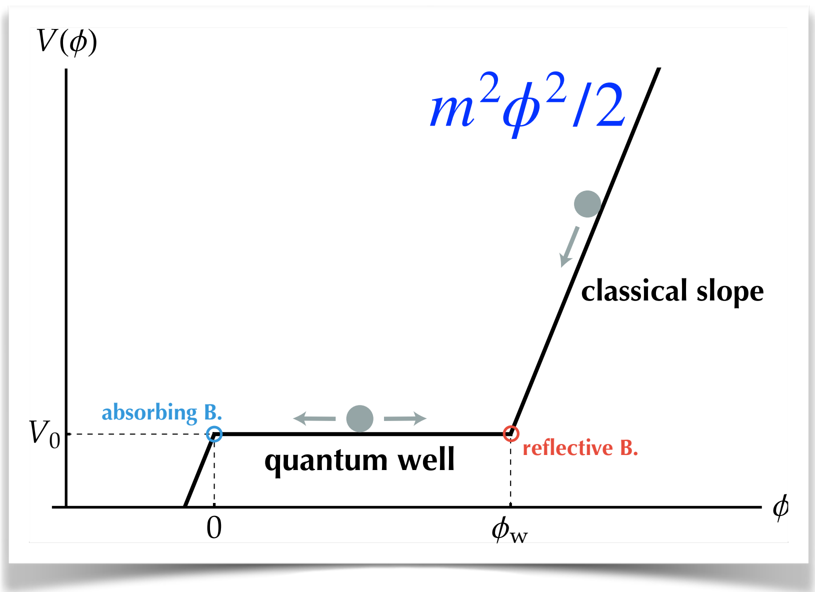
Power Spectrum

Kenta Ando, VV (2020)

$$\mathcal{P}_\zeta(k) = -\int_\Omega \mathrm{d}\boldsymbol{\Phi}_* \left.\frac{\partial P_{\mathrm{bw}}\left(\boldsymbol{\Phi}_* \mid N_{\mathrm{bw}}\right)}{\partial N_{\mathrm{bw}}}\right|_{N_{\mathrm{bw}}=-\ln(k/k_{\mathrm{f}})} \left\langle \delta\mathcal{N}^2\left(\boldsymbol{\Phi}_0 \to \boldsymbol{\Phi}_*\right)\right\rangle$$

Integration over the full inflating domain





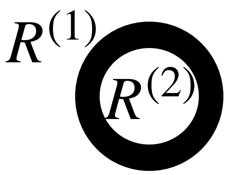
Extracting cosmological observables

One-point function at arbitrary scale

Yuichiro Tada, VV (2021)

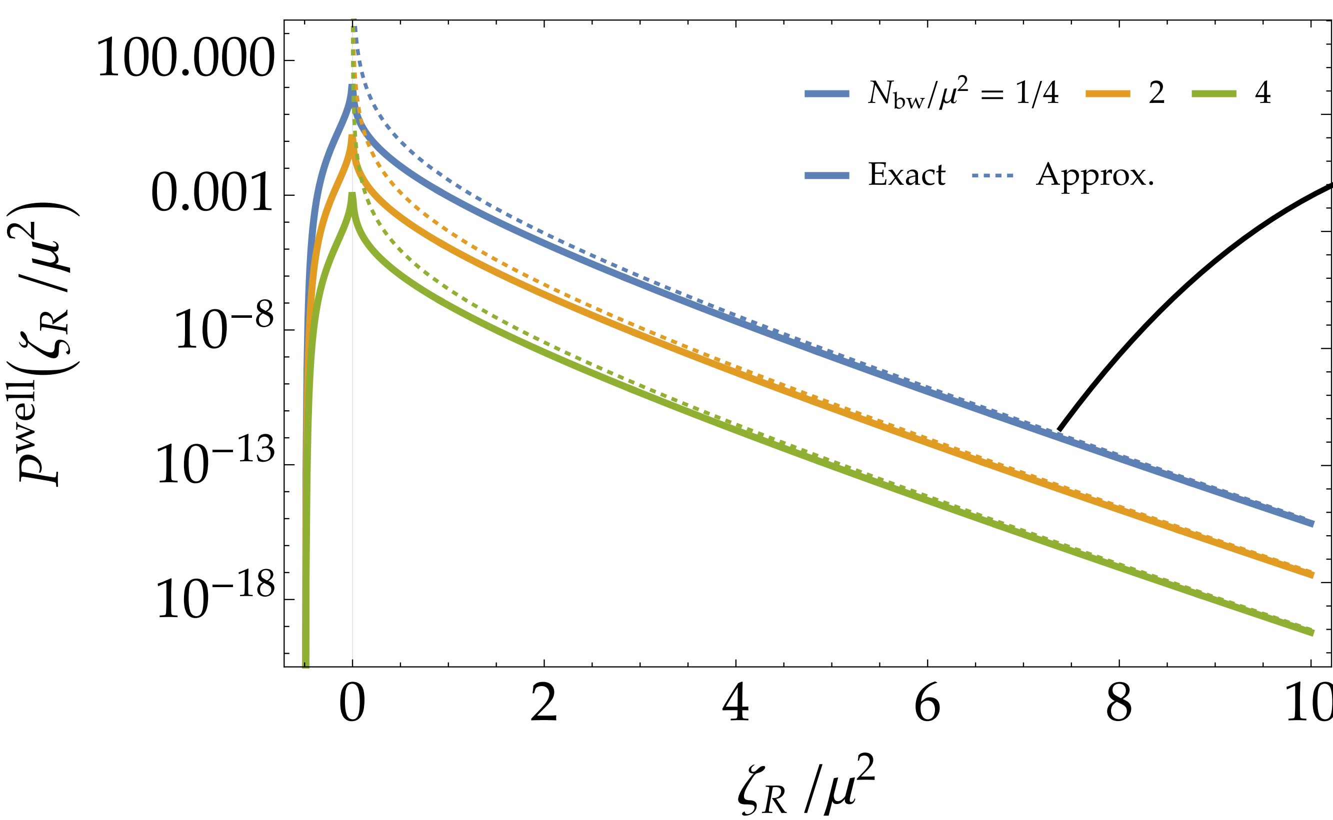
$$P\left(\zeta_R\right)=\boxed{\int_\Omega \mathrm{d}\boldsymbol{\Phi}_*} P_{\mathrm{bw}}\left[\boldsymbol{\Phi}_* \mid N_{\mathrm{bw}}(R)\right] P_{\mathrm{FPT}, \boldsymbol{\Phi}_0 \rightarrow \boldsymbol{\Phi}_*}\left[\zeta_R-\left\langle\mathcal{N}\left(\boldsymbol{\Phi}_*\right)\right\rangle+\left\langle\mathcal{N}\left(\boldsymbol{\Phi}_0\right)\right\rangle\right]$$

$$P(\Delta\zeta)=\boxed{\int_\Omega \mathrm{d}\boldsymbol{\Phi}_*^{(1)} \mathrm{d}\boldsymbol{\Phi}_*^{(2)}} P_{\mathrm{bw}}\left(\boldsymbol{\Phi}_*^{(1)}, \boldsymbol{\Phi}_*^{(2)} \middle| N_{\mathrm{bw}}^{(1)}, N_{\mathrm{bw}}^{(2)}\right) \delta\left[\Delta\zeta+\left\langle\mathcal{N}\left(\boldsymbol{\Phi}_*^{(1)}\right)\right\rangle-\left\langle\mathcal{N}\left(\boldsymbol{\Phi}_*^{(2)}\right)\right\rangle-\ln(1+\beta)\right]$$



→ Comoving density contrast

→ Compaction function



$$P\left(\zeta_R\right) \propto \frac{e^{-\frac{\pi^2}{4}\frac{\zeta_R}{\mu^2}}}{\left(\zeta_R/\mu^2\right)^3}$$

Quasi-exponential tail

Extracting cosmological observables

One-point function at arbitrary scale

Yuichiro Tada, VV (2021)

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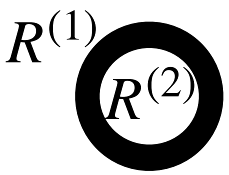
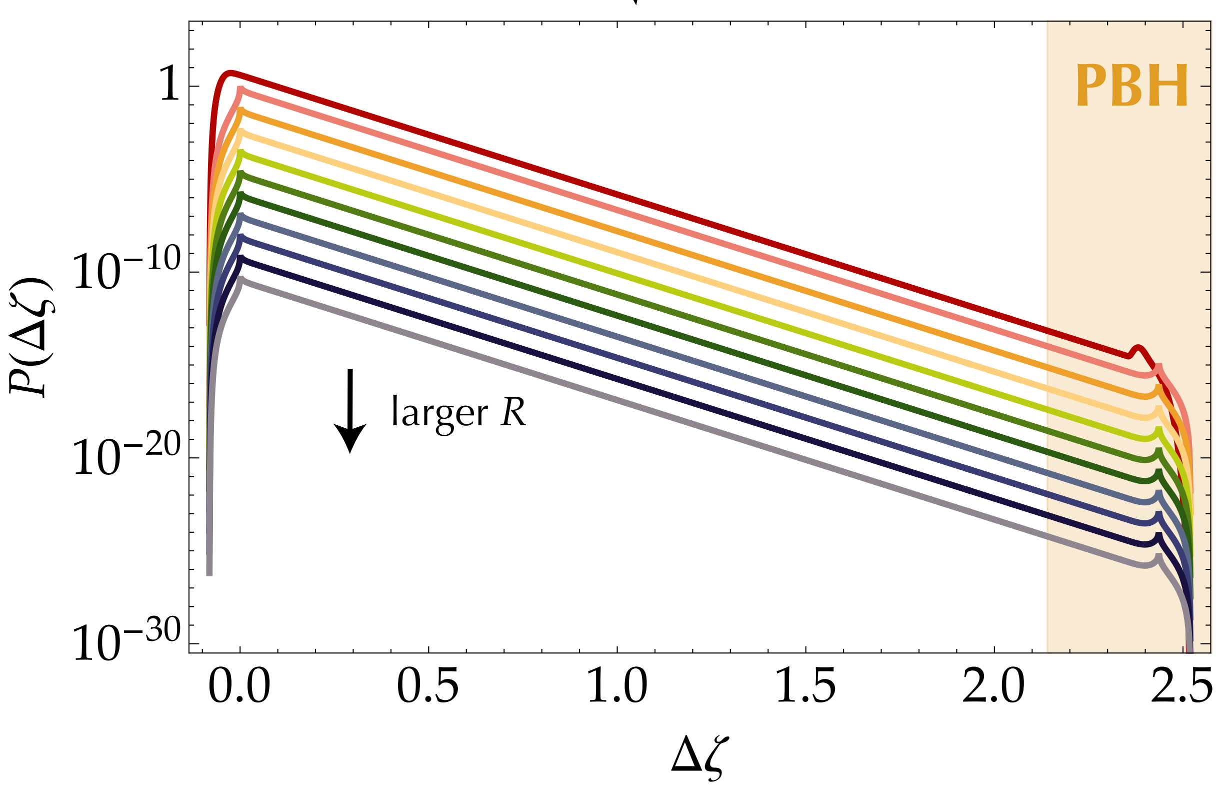
$$P(\Delta\zeta) = \int_\Omega \mathrm{d}\boldsymbol{\Phi}_*^{(1)}\mathrm{d}\boldsymbol{\Phi}_*^{(2)} P_{\mathrm{bw}}\left(\boldsymbol{\Phi}_*^{(1)}, \boldsymbol{\Phi}_*^{(2)} \middle| N_{\mathrm{bw}}^{(1)}, N_{\mathrm{bw}}^{(2)}\right) \delta\left[\Delta\zeta + \left\langle \mathcal{N}\left(\boldsymbol{\Phi}_*^{(1)}\right)\right\rangle - \left\langle \mathcal{N}\left(\boldsymbol{\Phi}_*^{(2)}\right)\right\rangle - \ln\left(1+\beta\right)\right]$$

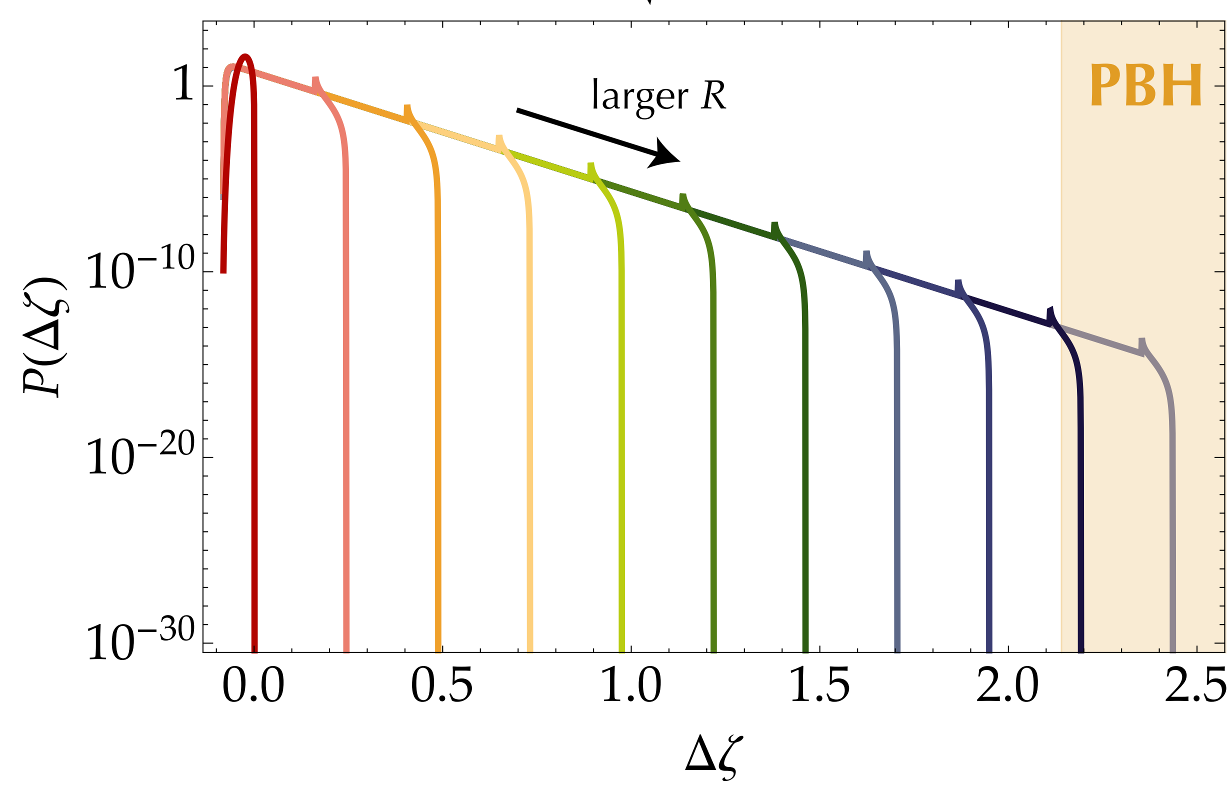


$\mu = \frac{1}{\sqrt{6}}, N_{\mathrm{bw}}^{(2)} > 0$



R_2 exits within the quantum well

$\mu = \frac{1}{\sqrt{6}}, N_{\mathrm{bw}}^{(2)} < 0$



R_2 exits below the quantum well

Extracting cosmological observables

One-point function at arbitrary scale

Yuichiro Tada, VV (2021)

$$P\left(\zeta_R\right) = \int_\Omega \mathrm{d}\boldsymbol{\Phi}_* P_{\mathrm{bw}}\left[\boldsymbol{\Phi}_* \mid N_{\mathrm{bw}}(R)\right] P_{\mathrm{FPT},\boldsymbol{\Phi}_0 \to \boldsymbol{\Phi}_*}\left[\zeta_R - \langle \mathcal{N}(\boldsymbol{\Phi}_*)\rangle + \langle \mathcal{N}(\boldsymbol{\Phi}_0)\rangle\right]$$

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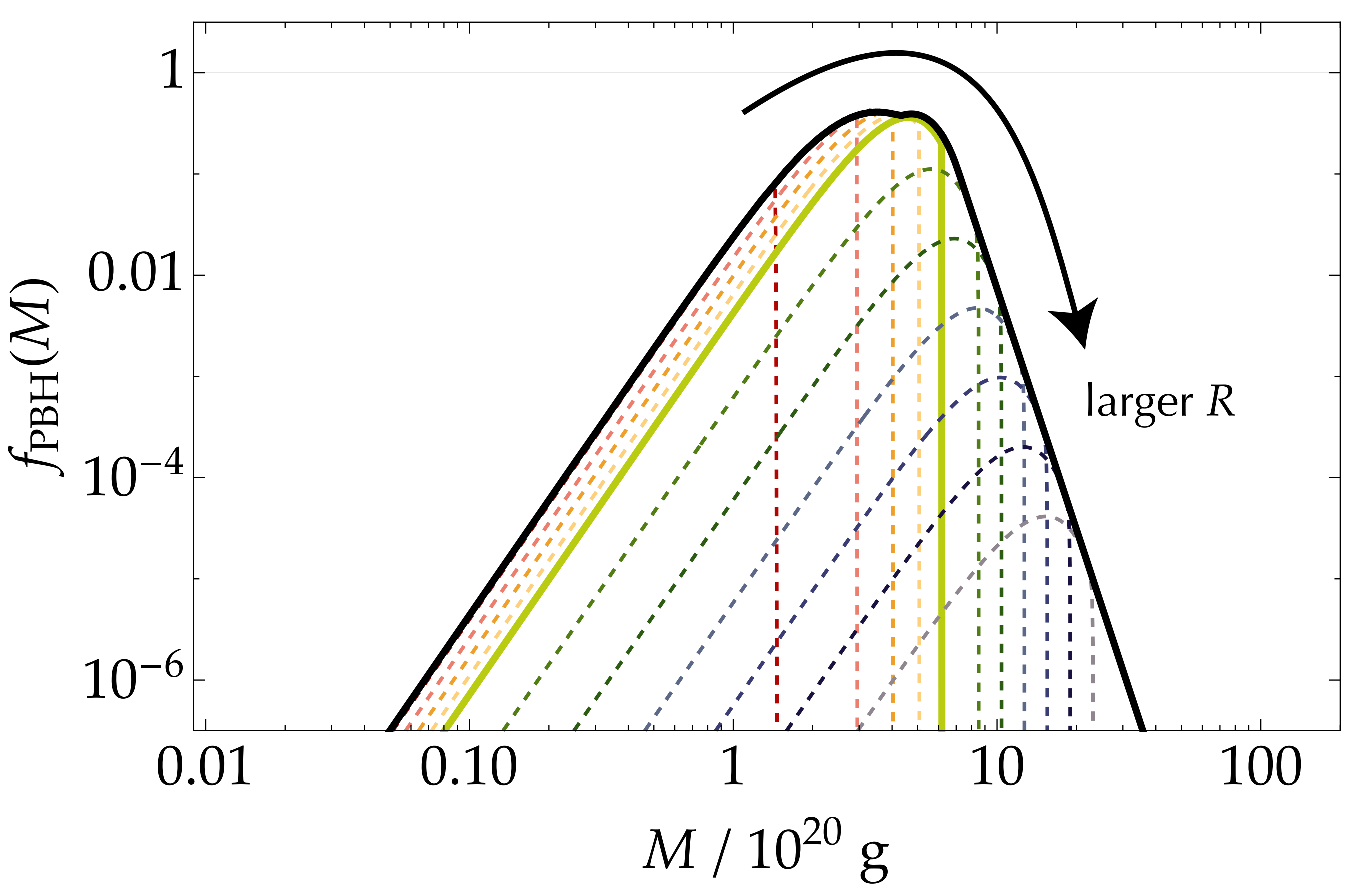
$$\mu = \frac{1}{\sqrt{6}}$$



Conclusions

- The back-reaction of vacuum quantum fluctuations on the background dynamics can be incorporated within the formalism of stochastic inflation
- This is necessary to describe regimes leading to large fluctuations, such as those yielding primordial black holes
- Quantum diffusion leads to exponential tails: non-perturbative break down of Gaussian statistics
- Most cosmological observables can be reconstructed from first-passage time analysis (power spectrum, mass functions, n-point functions?)
- Quantum diffusion makes the CMB probe the whole potential: models leading to PBHs are constrained by the CMB, even if those two sets of scales are well separated

Thank you for your attention